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Ramanujan graphs from simplicial complexes with few blockers

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Abstract
We present a construction method for Ramanujan graphs from simplicial complexes with one or two *blockers*, using the corresponding multivariable functions, where *blockers* mean minimal elements of the complement of a simplicial complex. First, we completely find all the binary linear codes obtained from simplicial complexes with one or two blockers, producing exactly 23 families of few-weight binary linear codes and 4 families of self-orthogonal codes. Furthermore, their weight distributions are explicitly computed by using the corresponding multivariable functions. Then we construct 14 families of nonbipartite Ramanujan graphs from the few-weight binary linear codes found; their eigenvalues are computed by using the weight distributions of the corresponding codes. We emphasize that it is the first time that the simplicial complexes with a few blockers are used for construction of few-weight linear codes and the corresponding Ramanujan graphs. Moreover, we verify that our families of Ramanujan graphs are different from the previous families in terms of eigenvalues and regularity.

Keywords Ramanujan graph · Simplicial complex · Blocker · Few-weight linear code · Cayley graph

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1 Introduction

Ramanujan graphs have drawn much attention due to the fact that they are "good" expanders achieving the *spectral bound*. In fact, expanders (or expander graphs) have a great deal of applications in various areas such as coding theory, cryptography, computer science, network design, and even in pure mathematics (for instance, refer to [7–9, 17, 19, 27]). In addition, the *zig–zag graph product* is used for construction of expanders [24]; this construction yields constant-degree expanders of arbitrary size. In this construction, the role of base expanders is very important; thus, we point out that Ramanujan graphs can play as base expanders. Recently, there has been a great deal of developments on explicit construction of Ramanujan graphs; for example, we refer to [2, 18, 20, 21, 23, 28]. Lubotzky et al. [20, 21] gave a construction of Ramanujan graphs from projective linear groups over the finite fields of prime order. Furthermore, Batra et al. [2] firstly constructed local bipartite Ramanujan graphs, using the result in [23].

On the other hand, there have been many studies on construction of Ramanujan graphs arising from codes; for example, we can refer to [3, 4, 15, 16, 22]. In [3, 4], Ramanujan graphs were constructed from 2-quasi-perfect Lee codes over the finite field $\mathbb{F}_p$, where p is a prime with $p \equiv \pm 5 \pmod{12}$. Mesnager et al. [22] generalized the results in [3, 4] to the case of every odd prime p; they showed that the Cayley graphs from some families of 2-quasi-perfect Lee codes over $\mathbb{F}_p$ are Ramanujan graphs, using Kloosterman sums. Recently, in [15], the connection between linear codes over $\mathbb{F}_p$ and Ramanujan graphs were characterized, and using this characterization, they constructed some families of Ramanujan graphs with three or four eigenvalues from few-weight linear codes over $\mathbb{F}_p$.

There has been much progress on constructing few-weight linear codes by using simplicial complexes (for example, [5, 6, 10, 12, 14, 25, 29]). In particular, they [5] suggested a method for computing the weight distributions of binary linear codes associated with simplicial complexes by using the corresponding multivariable functions. We note that the eigenvalues of a Ramanujan graph from a linear code C are explicitly determined by the weight distribution of C. Motivated by these facts, we focus on studying a construction method for Ramanujan graphs from simplicial complexes, using the corresponding multivariable functions.

In this paper, we find many families of Ramanujan graphs, which are constructed from simplicial complexes with one or two *blockers*, where *blockers* mean minimal elements of the complement of a simplicial complex. As a method, we use the corresponding multivariable functions for computation of the weight distributions of linear codes whose defining sets are the simplicial complexes. First, we completely classify *all* the binary linear codes obtained from simplicial complexes with one or two blockers, producing exactly 23 families of few-weight binary linear codes and 4 families of self-orthogonal codes. Furthermore, their weight distributions are explicitly computed by using the corresponding multivariable functions. Then we construct 14 families of nonbipartite Ramanujan graphs from the few-weight binary linear codes obtained. In addition, we explicitly compute the eigenvalues and the multiplicities of the Ramanujan graphs, using the weight distributions of the corresponding codes; computations are confirmed by Magma software. The Ramanujan graphs are summarized in Tables 3 through 5.

We emphasize that it is the first time that the simplicial complexes with a few blockers are used for constructing few-weight linear codes and the corresponding Ramanujan graphs. Moreover, we find that some of the families of Ramanujan graphs obtained are new, compared to the previous results [3, 4, 15, 16, 22] in terms of their regularity and eigenvalues.

We explain the significance of using simplicial complexes with one or two blockers. In fact, there is another representation of simplicial complexes using their maximal elements; in this case, it is possible to construct Ramanujan graphs from this representation. However, our focus in this paper is to use simplicial complexes with one or two blockers. We point out that even though the simplicial complex Δ has one or two blockers, the set of maximal elements of Δ can have various large cardinality (refer to Examples 2.3 and 2.4); for example, there is a simplicial complex Δ which has one blocker, but the number of its maximal elements is 9. In general, it is a hard task to handle the case when the set of maximal elements of Δ has large cardinality; the computation process can be very complicated. The advantage of using blockers is that our method is much more efficient when the cardinality of the set of the blockers of Δ is relatively much smaller than the cardinality of the number of maximal elements of Δ.

This paper is organized as follows. In Sect. 2, we introduce some definitions and previous results. In Sect. 3, we find all the binary linear codes obtained from simplicial complexes with one or two blockers and compute their weight distributions. Then, in Sect. 4, from the binary linear codes obtained in Sect. 3, we find some families of nonbipartite Ramanujan graphs, and we also compute their eigenvalues and multiplicities. In Sect. 5, we compare our results to the other results on Ramanujan graphs obtained from codes.

2 Preliminaries

For a vector $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{F}_2^n$, the *support* of $\mathbf{c}$ is defined by $\mathrm{Supp}(\mathbf{c}) = \{1 \leq i \leq n \mid c_i \neq 0\}$. Throughout this paper, we identify $\mathrm{Supp}(\mathbf{c})$ with $\mathbf{c}$ for convenience. The *(Hamming) weight* $w(\mathbf{c})$ is defined by $w(\mathbf{c}) = |\mathbf{c}|$. An $[n, k, d]$ *binary linear code* means a subspace of $\mathbb{F}_2^n$ having dimension k and minimum weight $d = \min\{w(\mathbf{c}) \mid \mathbf{0} \neq \mathbf{c} \in C\}$. An $[n, k, d]$ binary linear code is called *optimal* if there is no binary linear code of minimum weight greater than d for fixed length n and dimension k. For a binary linear code C of length n, we can obtain another binary linear code $C^\perp$, called the *dual code* of C, and it is defined by $C^\perp = \{\mathbf{x} \in \mathbb{F}_2^n \mid \mathbf{x} \cdot \mathbf{c} \text{ for all } \mathbf{c} \in C\}$, where $\cdot$ implies the classical inner product in $\mathbb{F}_2^n$. A binary linear code C is said to be *self-orthogonal* if $C \subseteq C^\perp$, and *projective* if the minimum weight of $C^\perp$ is at least 3. The following lemma provides a simple sufficient condition for identifying self-orthogonal binary linear codes.

Lemma 2.1 [11, Theorem 1.4.8] *Let C be a binary linear code. If every codewords of C has weight divisible by four, then C is self-orthogonal.*

Let A_i denote the number of codewords with weight i in a binary linear code C of length n ($1 \leq i \leq n$), and the *weight enumerator* of C is given by

$$A(X) = 1 + A_1 X + A_2 X^2 + \cdots + A_n X^n.$$

There is significant interest in binary linear codes with few nonzero weights, called *few-weight code*.

For a subset $D \subseteq \mathbb{F}_2^m$, we can define a linear code C_D as follows:

$$C_D = \{c_D(\mathbf{u}) := (\mathbf{u} \cdot \mathbf{d})_{\mathbf{d} \in D} \mid \mathbf{u} \in \mathbb{F}_2^m\}.$$

Then such a subset D is called the *defining set* of C_D. In this paper, we consider linear codes whose defining sets are simplicial complexes. A subset Δ of $\mathbb{F}_2^m$ is called a *simplicial complex* if for any $\mathbf{u} \in \Delta$, $\mathbf{v} \preceq \mathbf{u}$ implies that $\mathbf{v} \in \Delta$, where $\preceq$ denotes the partial order

between vectors, i.e. $\mathbf{v} = (v_1, v_2, \ldots, v_m) \preceq \mathbf{u} = (u_1, u_2, \ldots, u_m)$ means $\mathbf{v} \subseteq \mathbf{u}$ as their support sets. Then a *maximal element* of Δ and a *minimal element* of $\mathbb{F}_2^m \setminus \Delta$ can be defined, respectively; in detail, an element $\mathbf{m}$ of Δ is called a *maximal element* of Δ if there is no element $\mathbf{u} \in \Delta$ such that $\mathbf{m} \preceq \mathbf{u}$. Let $\mathcal{F}$ denote the set of maximal elements in the simplicial complex Δ. An element $\mathbf{b}$ of Δ is called a *minimal element* of $\mathbb{F}_2^m \setminus \Delta$ if there is no element $\mathbf{u} \in \mathbb{F}_2^m \setminus \Delta$ such that $\mathbf{u} \preceq \mathbf{b}$. Let $\mathcal{M}$ denote the set of minimal elements $\mathbf{b}$ of $\mathbb{F}_2^m \setminus \Delta$ according to the partial order $\preceq$. The elements of $\mathcal{M}$ are called the *blockers* of Δ.

We explain the relationship between the set $\mathcal{M}$ of blockers and the set $\mathcal{F}$ of maximal elements for a given simplicial complex Δ, using the following three examples. In Example 2.2, for a given set $\mathcal{M}$ of blockers of Δ, we present how to find the set $\mathcal{F}$ of maximal elements of Δ. Example 2.3 treats the one blocker case; we observe that even though Δ_i has one blocker for $i = 1, 2, 3$ ($|\mathcal{M}_i| = 1$ for $i = 1, 2, 3$, respectively), the corresponding set $\mathcal{F}_i$ of maximal elements have various cardinalities ($|\mathcal{F}_i| = 7, 8, 9$ for $i = 1, 2, 3$, respectively). Moreover, in Example 2.4, we consider the two blocker case; when Δ_i has two blockers for $i = 4, 5, 6$ ($|\mathcal{M}_i| = 2$ for $i = 4, 5, 6$, respectively), the corresponding set $\mathcal{F}_i$ have various cardinalities ($|\mathcal{F}_i| = 7, 16, 17$ for $i = 4, 5, 6$, respectively).

Example 2.2 Let $\Delta \subseteq \mathbb{F}_2^5$ be a simplicial complex. Let $\mathcal{F}$ be the set of maximal elements and $\mathcal{M}$ be the set of blockers of Δ. If $\mathcal{M} = \{(1, 1, 0, 0, 0)\}$, we have then

$$\mathbb{F}_2^5 \setminus \Delta = \left\{ \begin{array}{l} (1,1,0,0,0), (1,1,1,0,0), (1,1,0,1,0), (1,1,0,0,1), \\ (1,1,1,1,0), (1,1,1,0,1), (1,1,0,1,1), (1,1,1,1,1) \end{array} \right\}$$

and so

$$\Delta = \left\{ \begin{array}{l} (0,0,0,0,0), (1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1), \\ (1,0,1,0,0), (1,0,0,1,0), (1,0,0,0,1), (0,1,1,0,0), (0,1,0,1,0), (0,1,0,0,1), \\ (0,0,1,1,0), (0,0,1,0,1), (0,0,0,1,1), (1,0,1,1,0), (1,0,1,0,1), (1,0,0,1,1), \\ (0,1,1,1,0), (0,1,1,0,1), (0,1,0,1,1), (0,0,1,1,1), (1,0,1,1,1), (0,1,1,1,1) \end{array} \right\}.$$

Thus, the set $\mathcal{F}$ of maximal elements of Δ is given by

$$\mathcal{F} = \{(1,0,1,1,1), (0,1,1,1,1)\}.$$

Example 2.3 Let $\Delta_i \subseteq \mathbb{F}_2^{10}$ be a simplicial complex for $i = 1, 2, 3$. Let $\mathcal{F}_i$ be the set of maximal elements and $\mathcal{M}_i$ be the set of blockers of Δ_i. For the following given $\mathcal{M}_i$ with $|\mathcal{M}_i| = 1$ ($i = 1, 2, 3$), we obtain $\mathcal{F}_i$ as in Example 2.2, respectively.

(i) If $\mathcal{M}_1 = \{(1, 1, 1, 1, 1, 1, 1, 0, 0, 0)\}$, then

$$\mathcal{F}_1 = \left\{ \begin{array}{ll} (0,1,1,1,1,1,1,1,1,1), & (1,0,1,1,1,1,1,1,1,1), \\ (1,1,0,1,1,1,1,1,1,1), & (1,1,1,0,1,1,1,1,1,1), \\ (1,1,1,1,0,1,1,1,1,1), & (1,1,1,1,1,0,1,1,1,1), \\ (1,1,1,1,1,1,0,1,1,1) & \end{array} \right\}.$$

(ii) If $\mathcal{M}_2 = \{(1, 1, 1, 1, 1, 1, 1, 1, 0, 0)\}$, then

$$\mathcal{F}_2 = \left\{ \begin{array}{ll} (0,1,1,1,1,1,1,1,1,1), & (1,0,1,1,1,1,1,1,1,1), \\ (1,1,0,1,1,1,1,1,1,1), & (1,1,1,0,1,1,1,1,1,1), \\ (1,1,1,1,0,1,1,1,1,1), & (1,1,1,1,1,0,1,1,1,1), \\ (1,1,1,1,1,1,0,1,1,1), & (1,1,1,1,1,1,1,0,1,1) \end{array} \right\}.$$

(iii) If $\mathcal{M}_3 = \{(1,1,1,1,1,1,1,1,1,0)\}$, then

$$\mathcal{F}_3 = \left\{ \begin{array}{ll} (0,1,1,1,1,1,1,1,1,1), & (1,0,1,1,1,1,1,1,1,1), \\ (1,1,0,1,1,1,1,1,1,1), & (1,1,1,0,1,1,1,1,1,1), \\ (1,1,1,1,0,1,1,1,1,1), & (1,1,1,1,1,0,1,1,1,1), \\ (1,1,1,1,1,1,0,1,1,1), & (1,1,1,1,1,1,1,0,1,1), \\ (1,1,1,1,1,1,1,1,0,1) & \end{array} \right\}.$$

We explicitly compute the weight distributions of corresponding linear codes $C_{\Delta_1^*}$, $C_{\Delta_2^*}$, and $C_{\Delta_3^*}$ in Theorem 3.1, where their parameters are $[1015, 10, 504]$, $[1019, 10, 508]$, and $[1021, 10, 510]$, respectively. Moreover, we obtain nonbipartite Ramanujan graphs $\mathrm{Cay}(\mathbb{F}_2^{10}, \Delta_1^*)$, $\mathrm{Cay}(\mathbb{F}_2^{10}, \Delta_2^*)$, and $\mathrm{Cay}(\mathbb{F}_2^{10}, \Delta_3^*)$ in Theorem 4.1, respectively.

Example 2.4 Let $\Delta_i \subseteq \mathbb{F}_2^9$ be a simplicial complex for $i = 4, 5, 6$. Let $\mathcal{F}_i$ be the set of maximal elements and $\mathcal{M}_i$ be the set of blockers of Δ_i. For the following given $\mathcal{M}_i$ with $|\mathcal{M}_i| = 2$ $(i = 4, 5, 6)$, we obtain $\mathcal{F}_i$, respectively.

(i) If $\mathcal{M}_4 = \{\mathbf{b}_1 := (0,1,1,1,1,1,1,1,0), \mathbf{b}_2 := (1,1,1,1,1,0,0,0,0)\}$, then

$$\mathcal{F}_4 = \left\{ \begin{array}{ll} (0,1,1,1,1,1,0,1,1,1), & (1,1,1,1,0,1,1,1,1,1), \\ (0,1,1,1,1,1,1,1,0,1), & (0,1,1,1,1,1,1,0,1,1), \\ (1,1,1,0,1,1,1,1,1,1), & (1,0,1,1,1,1,1,1,1,1), \\ (1,1,0,1,1,1,1,1,1,1) & \end{array} \right\}.$$

Then the corresponding code $C_{\Delta_4^*}$ has parameters $[381, 9, 126]$ and its weight distribution is given in No. 21 of Table 2. Moreover, we construct a nonbipartitle Ramanujan graph $\mathrm{Cay}(\mathbb{F}_2^9, \Delta_4^*)$ in No. 12 of Table 5.

(ii) If $\mathcal{M}_5 = \{\mathbf{b}_1 := (1,1,1,1,0,0,0,0,0), \mathbf{b}_2 := (0,0,0,0,1,1,1,1,0)\}$, then

$$\mathcal{F}_5 = \left\{ \begin{array}{ll} (1,1,1,0,1,0,1,1,1), & (1,0,1,1,1,1,0,1,1), \\ (1,1,0,1,0,1,1,1,1), & (1,0,1,1,0,1,1,1,1), \\ (1,1,1,0,1,1,1,0,1), & (1,1,0,1,1,0,1,1,1), \\ (1,1,1,0,0,1,1,1,1), & (1,0,1,1,1,1,1,0,1), \\ (0,1,1,1,1,1,1,0,1), & (1,1,1,0,1,1,0,1,1), \\ (0,1,1,1,1,0,1,1,1), & (1,1,0,1,1,1,1,0,1), \\ (0,1,1,1,0,1,1,1,1), & (1,1,0,1,1,1,0,1,1), \\ (1,0,1,1,1,0,1,1,1), & (0,1,1,1,1,1,0,1,1), \end{array} \right\}.$$

Then the corresponding code C_{Δ_5} is a $[449, 9, 66]$ code with weight distribution as given in No. 6 of Table 1, and a nonbipartitle Ramanujan graph $\mathrm{Cay}(\mathbb{F}_2^9, \Delta_4^*)$ in No. 5 of Table 4 is constructed.

(iii) If $\mathcal{M}_6 = \{\mathbf{b}_1 := (0,0,0,0,1,1,1,1,1), \mathbf{b}_2 := (1,1,1,1,1,0,0,0,0)\}$, then

$$\mathcal{F}_6 = \left\{ \begin{array}{ll} (1,1,1,0,1,0,1,1,1), & (0,1,1,1,1,1,1,1,0), \\ (1,0,1,1,1,1,0,1,1), & (1,1,1,0,1,1,1,0,1), \\ (1,1,0,1,1,0,1,1,1), & (1,1,0,1,1,1,1,1,0), \\ (1,0,1,1,1,1,1,0,1), & (0,1,1,1,1,1,1,0,1), \\ (1,1,1,0,1,1,0,1,1), & (0,1,1,1,1,0,1,1,1), \\ (1,1,0,1,1,1,1,0,1), & (1,0,1,1,1,1,1,1,0), \\ (1,1,1,0,1,1,1,1,0), & (1,1,0,1,1,1,0,1,1), \\ (1,0,1,1,1,0,1,1,1), & (0,1,1,1,1,1,0,1,1), \\ (1,1,1,1,0,1,1,1,1) & \end{array} \right\}.$$

The the corresponding code C_{Δ_6} has parameters $[480, 9, 225]$ and weight distribution as given in No. 8 of Table 2 in Theorem 3.2. We also obtain a nonbipartitle Ramanujan graph $\mathrm{Cay}(\mathbb{F}_2^9, \Delta_4^*)$ in No. 6 of Table 5 in Theorem 4.2.

Remark 2.5 In fact, the concept of the blocker of Δ is closely related to Alexander dual of Δ [1]. For a simplicial complex $\Delta \subseteq \mathbb{F}_2^m$, define $\Delta^* := \{\mathbf{1}+\mathbf{b} \mid \mathbf{b} \in \mathbb{F}_2^m \setminus \Delta\}$. Then $(\Delta^*)^* = \Delta$ and Δ^* is again a simplicial complex, which we call the *Alexander dual* of Δ. The maximal elements of Δ^* are of the form $\mathbf{1}+\mathbf{b}$ for each blocker $\mathbf{b}$ of Δ, while the blockers of Δ^* are of the form $\mathbf{1}+\mathbf{m}$ for each maximal element $\mathbf{m}$ of Δ.

Another method for constructing linear codes is to use multivariable functions. Let $\mathcal{H}$ be an m-variable function of the form

$$\mathcal{H}(x_1, x_2, \ldots, x_m) = \sum_{\mathbf{y}\in\mathbb{F}_2^m} a_{\mathbf{y}} x_1^{y_1} x_2^{y_2} \cdot x_m^{y_m} \in \mathbb{Z}[x_1, x_2, \ldots, x_m],$$

where $a_{\mathbf{y}} \in \{0, 1\}$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m)$, and let $P_{\mathcal{H}}$ be a set consisting of $a_{\mathbf{y}}$-times $\mathbf{y} \in \mathbb{F}_2^m$. Then we get a linear code $C_{\mathcal{H}}$ associated with $\mathcal{H}$ defined by

$$C_{\mathcal{H}} := C_{P_{\mathcal{H}}} = \{c_{\mathcal{H}}(\mathbf{u}) := (\mathbf{u}\cdot\mathbf{y})_{\mathbf{y}\in P_{\mathcal{H}}} \mid \mathbf{u} \in \mathbb{F}_2^m\}. \tag{1}$$

For a linear code $C_{\mathcal{H}}$ associated with $\mathcal{H}$, Hyun et al. [13] found their parameters by evaluating $\mathcal{H}$.

Lemma 2.6 [13, Lemma 3.2] *Let $\mathcal{H}$ be an m-variable function and $C_{\mathcal{H}}$ a binary linear code associated with $\mathcal{H}$ as in (1). Then the length of $C_{\mathcal{H}}$ is equal to $\mathcal{H}(1, 1, \ldots, 1)$, and the dimension of $C_{\mathcal{H}}$ is at most m. Moreover, the Hamming weight $w(c_{\mathcal{H}}(\mathbf{u}))$ of a codeword $c_{\mathcal{H}}(\mathbf{u})$ is given as follows:*

$$w(c_{\mathcal{H}}(\mathbf{u})) = \frac{1}{2}\left(\mathcal{H}(1, 1, \ldots, 1) - \mathcal{H}\left((-1)^{u_1}, (-1)^{u_2}, \ldots, (-1)^{u_m}\right)\right),$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_m)$.

Moreover, in [5], Hyun et al. introduced a multivariable function that corresponds to a linear code defined by a simplicial complex.

Lemma 2.7 [5, Theorem 1] *Let Δ be a simplicial complex of $\mathbb{F}_2^m$ with the family of blockers $\mathcal{M}$. Let $\mathcal{H}_\Delta$ be an m-variable function defined by*

$$\mathcal{H}_\Delta(x_1, x_2, \ldots, x_m) = \sum_{T\subseteq\mathcal{M}} (-1)^{|T|} \prod_{j\in I_T} x_j \prod_{i\notin I_T} (1 + x_i),$$

where $I_T = \cup_{\mathbf{u}\in T} \mathrm{Supp}(\mathbf{u})$ and $\prod_{i\in\emptyset}(1+x_i) = 1$. Then C_Δ is associated to the multivariable function $\mathcal{H}_\Delta$, i.e., $C_\Delta = C_{\mathcal{H}_\Delta}$.

We call the function $\mathcal{H}_\Delta$ in Lemma 2.7 by a *multivariable function associated with a simplicial complex* Δ.

Example 2.8 Let $\Delta \subseteq \mathbb{F}_2^4$ be a simplicial complex with two blockers $\mathbf{b}_1 = (1, 1, 0, 0)$ and $\mathbf{b}_2 = (0, 0, 1, 0)$. Then $\mathbb{F}_2^4 \setminus \Delta$ is given by

$$\left\{\begin{array}{l}(1,1,0,0),(1,1,1,0),(1,1,0,1),(1,1,1,1),(0,0,1,0),\\(1,0,1,0),(0,1,1,0),(0,0,1,1),(1,0,1,1),(0,1,1,1)\end{array}\right\}$$

and $\Delta = \{(0,0,0,0), (1,0,0,0), (0,1,0,0), (0,0,0,1), (1,0,0,1), (0,1,0,1)\}$. Then the linear code C_Δ associated with Δ has a generator matrix

$$\begin{bmatrix} 0&1&0&0&1&0 \\ 0&0&1&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&1&1&1 \end{bmatrix};$$

so, a corresponding multivariable function is given as $\mathcal{H}_\Delta(x_1, x_2, x_3, x_4) = 1 + x_1 + x_2 + x_4 + x_1x_4 + x_2x_4$. In this case, $\mathcal{H}_\Delta(x_1, x_2, x_3, x_4) = (1 + x_1)(1 + x_2)(1 + x_3)(1 + x_4) - x_1x_2(1 + x_3)(1 + x_4) - (1 + x_1)(1 + x_2)x_3(1 + x_4) + x_1x_2x_3(1 + x_4)$ as shown in Lemma 2.7.

A *graph* Γ consists of an ordered set V and a family of sets E, where E consists of sets containing one or two vertices of V. The elements of V are called *vertices* and of E are *edges*. For two vertices $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}$ is *adjacent* to $\mathbf{v}$ if $\{\mathbf{u}, \mathbf{v}\} \in E$ and such two vertices are called *neighbors*. Two graphs $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$ are *isomorphic* if there exists a one-to-one mapping from V_1 onto V_2 which preserves adjacency and nonadjacency. A graph $\Gamma = (V, E)$ is said to be *bipartite* if V can be partitioned into two sets V_1 and V_2 such that $E \cap (V_1 \times V_1) = E \cap (V_2 \times V_2) = \emptyset$; otherwise it is called *nonbipartite*. In particular, an edge $\{\mathbf{u}, \mathbf{u}\}$ is called *loop* when $\mathbf{u} = \mathbf{v}$, and the *degree* of a vertex $\mathbf{v}$ in Γ is the number of edges incident on v plus twice the number of self-loop. If every vertex is of the same degree K, the graph Γ is called K*-regular*. A *walk* in graph Γ is a sequence of vertices $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_\ell$ such that $\{\mathbf{u}_{j-1}, \mathbf{u}_j\} \in E$ for every $1 \le j \le \ell$, and the *length of the walk* is ℓ. If between every pair of vertices there is a walk, a graph Γ is called *connected*. For a given graph Γ, a matrix A_Γ can be determined and called *adjacency matrix* of G. The matrix is defined as follows:

$$\text{the (i, j)-th entry of } A_\Gamma := A_\Gamma[i, j] = \begin{cases} 1 & \text{if } \{u_i, u_j\} \in E; \\ 0 & \text{if } \{u_i, u_j\} \notin E, \end{cases}$$

where u_i denote the i-th vertices of Γ. The *eigenvalues* and the *eigenvectors* of a graph Γ is the eigenvalues and the eigenvectors of the adjacency matrix of Γ. When a graph is connected and regular, whether the graph is bipartite can be determined by its eigenvalues as shown in the following criterion.

Lemma 2.9 [7, Proposition 1.1.4] *Let Γ be a connected K-regular graph. The graph Γ is a bipartite graph if and only if $-K$ is an eigenvalue of Γ.*

For a subset $S \subseteq V$ with $S = -S$, the corresponding *Cayley graph* $\mathrm{Cay}(V, S)$ has V as its vertex set and two vertices u and v are adjacent if and only if $u - v \in S$. It is known that the $\mathrm{Cay}(V, S)$ is connected if and only if S generates V, also if $\mathbf{0} \notin S$ then the Cayley graph is loopless, and if $\mathbf{0} \in S$ then each of vertices of the Cayley graph has a loop.

Let C be an $[n, k]$ binary linear code and let D be a defining set of C, i.e. $C = C_D$. Then we have a Cayley graph $\mathrm{Cay}(\mathbb{F}_2^k, D)$ associated with D. There has been a study on the relationship between the eigenvalues and their multiplicities of the Cayley graph $\mathrm{Cay}(\mathbb{F}_2^k, D)$ and the weight distributions of the linear code C_D in [26, Theorem 1] (in [26], the Cayley graph associated with a linear code C appears to be defined slightly in a different way, but essentially it is the same as ours).

Lemma 2.10 [26, Theorem 1] *Let C be an $[n, k, d]$ binary linear code with the weight enumerator $A(X) = \sum_{i=1}^{n} A_i X^i$ and D be a defining set of C. Then for $0 \le i \le n$, eigenvalues of $\mathrm{Cay}(\mathbb{F}_2^m, D)$ are $n - 2i$ and their corresponding multiplicities are A_i.*

There is an important graph, called a *Ramanujan graph*, which is a finite connected K-regular graph with optimal spectral gap, that is, for every nontrivial eigenvalue μ of Γ, one has $|\mu| \leq 2\sqrt{K-1}$. We have the following explicit criterion for the Cayley graph $\mathrm{Cay}(\mathbb{F}_2^k, D^*)$ to be a Ramanujan graph in terms of the bounds on the weights of C_D.

Lemma 2.11 [15, Theorem 1.1] *Let D be a subset of $\mathbb{F}_2^m \setminus \{\mathbf{0}\}$. Then the Cayley graph $\mathrm{Cay}(\mathbb{F}_2^m, D)$ is a Ramanujan graph with $t+1$ distinct eigenvalues if and only if the associated code C_D is a binary linear code of length $n = |D|$ with exactly t distinct nonzero weights, and satisfies*

$$\frac{n}{2} - \sqrt{n-1} \leq \mathrm{wt}(c_D(\mathbf{u})) \leq \frac{n}{2} + \sqrt{n-1}$$

for all $\mathbf{u} \in \mathbb{F}_2^m \setminus \{\mathbf{0}\}$.

3 All the linear codes obtained from simplicial complexes with few blockers

In this section, we completely find all the linear codes obtained from simplicial complexes with one or two blockers; they produce 23 families of few-weight codes and 4 families of self-orthogonal codes. We also compute their weight distributions explicitly; the computations are confirmed by Magma software.

Theorem 3.1 *Let m be a positive integer with $m \geq 2$. Let $\Delta_1 \subseteq \mathbb{F}_2^m$ be a simplicial complex with exactly one blocker $\mathbf{b}$ and $\Delta_1^* := \Delta_1 \setminus \{\mathbf{0}\}$. Then the code $C_{\Delta_1^*}$ is a $[2^m - 2^{m-|\mathbf{b}|} - 1, m, 2^{m-1} - 2^{m-|\mathbf{b}|}]$ projective 3-weight binary linear code, and its weight enumerator is given as follows:*

$$1 + 2^{|\mathbf{b}|} X^{2^{m-1}-2^{m-|\mathbf{b}|}} + (2^m - 2^{|\mathbf{b}|}) X^{2^{m-1}-2^{m-|\mathbf{b}|-1}} + (2^{|\mathbf{b}|} - 1) X^{2^{m-1}}.$$

In particular, the code $C_{\Delta_1^*}$ is self-orthogonal when $|\mathbf{b}| \leq m-3$.

Proof We mainly use Lemmas 2.6 and 2.7 for the proof. By Lemma 2.7, a multivariable function $\mathcal{H}_{\Delta_1^*}$ corresponding to $C_{\Delta_1^*}$ is given as

$$\mathcal{H}_{\Delta_1^*}(x_1, x_2, \ldots, x_m) = \prod_{i=1}^{m}(1+x_i) - \prod_{j\in\mathbf{b}} x_j \prod_{i\notin\mathbf{b}}(1+x_i) - 1.$$

Then by Lemma 2.6, the length of $C_{\Delta_1^*}$ is equal to $\mathcal{H}_{\Delta_1^*}(1, 1, \ldots, 1) = 2^m - 2^{m-|\mathbf{b}|} - 1$, and the weight of a codeword $c_{\mathcal{H}_{\Delta_1^*}}(\mathbf{u})$ of $C_{\Delta_1^*}$ for $\mathbf{u} = (u_1, u_2, \ldots, u_m) \in \mathbb{F}_2^m$ is given as

follows:

$$\begin{aligned} w(c_{\mathcal{H}_{\Delta_1^*}}(\mathbf{u})) &= \frac{1}{2}\left(\mathcal{H}_{\Delta_1^*}(1,1,\ldots,1) - \mathcal{H}_{\Delta_1^*}((-1)^{u_1},(-1)^{u_2},\ldots,(-1)^{u_m})\right) \\ &= 2^m - 2^{m-|\mathbf{b}|} - \prod_{i=1}^{m}(1+(-1)^{u_i}) + (-1)^{\sum_{j\in\mathbf{b}} u_j}\prod_{i\notin\mathbf{b}}(1+(-1)^{u_i}) \\ &= \begin{cases} 0 & \text{if } \mathbf{u}=\mathbf{0}; \\ 2^{m-1}-2^{m-|\mathbf{b}|-1} & \text{if } \mathbf{u}\neq\mathbf{0}, \mathbf{u}\not\preceq\mathbf{b}; \\ 2^{m-1}-2^{m-|\mathbf{b}|-1}+(-1)^{w(\mathbf{u})}2^{m-|\mathbf{b}|-1} & \text{if } \mathbf{u}\neq\mathbf{0}, \mathbf{u}\preceq\mathbf{b} \end{cases} \\ &= \begin{cases} 0 & \text{if } \mathbf{u}=\mathbf{0}; \\ 2^{m-1}-2^{m-|\mathbf{b}|-1} & \text{if } \mathbf{u}\neq\mathbf{0}, \mathbf{u}\not\preceq\mathbf{b}; \\ 2^{m-1} & \text{if } \mathbf{u}\neq\mathbf{0}, \mathbf{u}\preceq\mathbf{b}, |\mathbf{u}| \text{ is even}; \\ 2^{m-1}-2^{m-|\mathbf{b}|} & \text{if } \mathbf{u}\neq\mathbf{0}, \mathbf{u}\preceq\mathbf{b}, |\mathbf{u}| \text{ is odd}. \end{cases} \end{aligned}$$

Then C_{Δ_1} is a $[2^m - 2^{m-|\mathbf{b}|} - 1, m, 2^{m-1} - 2^{m-|\mathbf{b}|}]$ projective 3-weight binary linear code, and its parameters and weight enumerator are obtained as desired. Moreover we find that their codewords are divisible by four when $|\mathbf{b}| \leq m-3$, and in this case, C_{Δ_1} is self-orthogonal by Lemma 2.1. □

Theorem 3.2 *Let m be a positive integer with $m \geq 4$. Let $\Delta_2 \subseteq \mathbb{F}_2^m$ be a simplicial complex with exactly two blockers $\mathbf{b}_1$ and $\mathbf{b}_2$ and $\Delta_2^* := \Delta_2 \setminus \{\mathbf{0}\}$. Then the code $C_{\Delta_2^*}$ is a projective binary linear code, and the parameters and the weight distributions of $C_{\Delta_2^*}$ are given in Tables 1 and 2. In particular, the code $C_{\Delta_2^*}$ is self-orthogonal for the following cases:*

(i) *$C_{\Delta_2^*}$ in No. 5 with $|\mathbf{b}_1| \leq m-4$;*
(ii) *$C_{\Delta_2^*}$ in No. 6 with $|\mathbf{b}_1| \leq \frac{m-3}{2}$;*
(iii) *$C_{\Delta_2^*}$ in No. 7 with $|\mathbf{b}_1| + |\mathbf{b}_2| \leq m-3$.*

Proof We prove the results for the families No. 2 and No. 18, and the results for all the other families are proved in a similar manner. By Lemma 2.7, the multivariable function $\mathcal{H}_{\Delta_2^*}(x_1, x_2, \ldots, x_m)$ corresponding to the family No. 2 is given as follows:

$$\prod_{i=1}^{m}(1+x_j) - \prod_{j\in\mathbf{b}_1} x_j \prod_{i\notin\mathbf{b}_1}(1+x_i) - \prod_{j\in\mathbf{b}_2} x_j \prod_{i\notin\mathbf{b}_2}(1+x_i) + \prod_{j\in\mathbf{b}_1\cup\mathbf{b}_2} x_j \prod_{i\notin\mathbf{b}_1\cup\mathbf{b}_2}(1+x_i) - 1. \quad (2)$$

Since $\mathbf{b}_1 \cup \mathbf{b}_2 = [m]$, Eq. (2) is equal to

$$\prod_{i=1}^{m}(1+x_j) - \prod_{j\in\mathbf{b}_1} x_j \prod_{i\notin\mathbf{b}_1}(1+x_i) - \prod_{j\in\mathbf{b}_2} x_j \prod_{i\notin\mathbf{b}_2}(1+x_i) + \prod_{j=1}^{m} x_j - 1.$$

By Lemma 2.6, the length of $C_{\Delta_2^*}$ is equal to $\mathcal{H}_{\Delta_2^*}(1,1,\ldots,1) = 2^m - 2^{m/2+1}$ and the weights of codewords $c_{\mathcal{H}_{\Delta_2^*}}(\mathbf{u})$ of $C_{\Delta_2^*}$ are obtained as

Table 1 The parameters and the weight distributions of linear codes $C_{\Delta_2^*}$ for simplicial complexes Δ_2^* having two blockers $\mathbf{b}_1$ and $\mathbf{b}_2$ with $\mathbf{b}_1 \cap \mathbf{b}_2 = \emptyset$

No.	Conditions	Parameters $[n, k, d]$	Weight distribution	
			Weight	Frequency
1	$\lvert\mathbf{b}_1\rvert = m-1$, $\lvert\mathbf{b}_2\rvert = 1$	$[2^{m-1}-1, m-1, 2^{m-2}-1]$	$2^{m-2}-1$ 2^{m-2}	2^{m-2} $2^{m-2}-1$
2	$m \geq 4$: even, $\lvert\mathbf{b}_1\rvert = \lvert\mathbf{b}_2\rvert = m/2$	$n = 2^m - 2^{\frac{m}{2}+1}$, $k = m$, $d = 2^{m-1} - 3\cdot 2^{\frac{m}{2}-1} + 1$,	$2^{m-1} - 3\cdot 2^{\frac{m}{2}-1} + 1$ $2^{m-1} - 2^{\frac{m}{2}}$ $2^{m-1} - 2^{\frac{m}{2}} + 1$ $2^{m-1} - 2^{\frac{m}{2}-1}$	$2^{\frac{m}{2}}$ $2^{m-1} - 2^{\frac{m}{2}} + 1$ $2^{m-1} - 2^{\frac{m}{2}}$ $2^{\frac{m}{2}} - 2$
3	$\mathbf{b}_1 \cup \mathbf{b}_2 = [m]$, $\lvert\mathbf{b}_1\rvert > \lvert\mathbf{b}_2\rvert > 1$	$n = 2^m - 2^{\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_1\rvert}$, $k = m$, $d = 2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_1\rvert-1} + 1$	$2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_1\rvert-1} + 1$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_1\rvert} + 1$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} + 1$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$	$2^{m-\lvert\mathbf{b}_1\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert-1}$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} + 1$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert-1} - 1$ $2^{m-\lvert\mathbf{b}_1\rvert-1} - 1$
4	$\mathbf{b}_1 \cup \mathbf{b}_2 \neq [m]$, $\lvert\mathbf{b}_1\rvert = \lvert\mathbf{b}_2\rvert = 1$	$[2^{m-2}-1, m-2, 2^{m-3}]$	2^{m-3}	$2^{m-2}-1$
5	$\mathbf{b}_1 \cup \mathbf{b}_2 \neq [m]$, $\lvert\mathbf{b}_1\rvert > 1$, $\lvert\mathbf{b}_2\rvert = 1$	$[2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 1, m-1, 2^{m-2} - 2^{m-\lvert\mathbf{b}_1\rvert-1}]$	$2^{m-2} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$ $2^{m-2} - 2^{m-\lvert\mathbf{b}_1\rvert-2}$ 2^{m-2}	$2^{\lvert\mathbf{b}_1\rvert-1}$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert}$ $2^{\lvert\mathbf{b}_1\rvert-1} - 1$
6	$\mathbf{b}_1 \cup \mathbf{b}_2 \neq [m]$, $\lvert\mathbf{b}_1\rvert = \lvert\mathbf{b}_2\rvert > 1$	$n = 2^m - 2^{m-\lvert\mathbf{b}_1\rvert+1} + 2^{m-2\lvert\mathbf{b}_1\rvert} - 1$, $k = m$, $d = 2^{m-1} - 3\cdot 2^{m-\lvert\mathbf{b}_1\rvert-1} + 2^{m-2\lvert\mathbf{b}_1\rvert}$	$2^{m-1} - 3\cdot 2^{m-\lvert\mathbf{b}_1\rvert-1} + 2^{m-2\lvert\mathbf{b}_1\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} + 2^{m-2\lvert\mathbf{b}_1\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} + 2^{m-2\lvert\mathbf{b}_1\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$	$2^{\lvert\mathbf{b}_1\rvert}$ $2^{2\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert} + 1$ $2^m - 2^{2\lvert\mathbf{b}_1\rvert}$ $2^{2\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert}$ $2^{\lvert\mathbf{b}_1\rvert} - 2$
7	$\mathbf{b}_1 \cup \mathbf{b}_2 \neq [m]$, $\lvert\mathbf{b}_1\rvert > \lvert\mathbf{b}_2\rvert > 1$	$n = 2^m - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert} - 1$, $k = m$, $d = 2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert}$	$2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert-1} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$	$2^{\lvert\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_2\rvert-1} + 1$ $2^m - 2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert}$ $2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert-1} - 1$ $2^{\lvert\mathbf{b}_2\rvert-1} - 1$

Table 2 The parameters and the weight distributions of linear codes C_{Δ_2} for simplicial complexes Δ_2 having two blockers $\mathbf{b}_1$ and $\mathbf{b}_2$ with $\mathbf{b}_1 \cap \mathbf{b}_2 \neq \emptyset$

No.	Conditions	Parameters $[n, k, d]$	Weight distribution	
			Weight	Frequency
8	$m:$ odd, $\mathbf{b}_1 \cup \mathbf{b}_2 = [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert = 1$, $\lvert\mathbf{b}_1\rvert = \lvert\mathbf{b}_2\rvert = \frac{m+1}{2}$	$[2^m - 2^{\frac{m+1}{2}}, m, 2^{m-1} - 2^{\frac{m+1}{2}} + 1]$	$2^{m-1} - 2^{\frac{m+1}{2}} + 1$ $2^{m-1} - 3 \cdot 2^{\frac{m-3}{2}} + 1$ $2^{m-1} - 2^{\frac{m-1}{2}}$ $2^{m-1} - 2^{\frac{m-1}{2}} + 1$ $2^{m-1} - 2^{\frac{m-3}{2}}$	1 $2^{\frac{m+1}{2}} - 2$ $2^{m-1} - 2^{\frac{m+1}{2}} + 1$ $2^{m-1} - 2^{\frac{m+1}{2}} + 1$ $2^{\frac{m+1}{2}} - 2$
9	$\mathbf{b}_1 \cup \mathbf{b}_2 = [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert = 1$, $\lvert\mathbf{b}_1\rvert = m-1$, $\lvert\mathbf{b}_2\rvert = 2$	$[2^m - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{\lvert\mathbf{b}_1\rvert-1}, m, 2^{m-2} - 1]$	$2^{m-2} - 1$ 2^{m-2} $3 \cdot 2^{m-3} - 1$ $3 \cdot 2^{m-2}$ $2^{m-1} - 1$	1 1 $2^{m-1} - 2$ $2^{m-1} - 2$ 1
10	$\mathbf{b}_1 \cup \mathbf{b}_2 = [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert = 1$, $m-1 > \lvert\mathbf{b}_1\rvert > \lvert\mathbf{b}_2\rvert > 1$	$[2^m - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{\lvert\mathbf{b}_1\rvert-1}, m, 2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_1\rvert} + 1]$	$2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_1\rvert} + 1$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} + 1$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-2} - 2^{m-\lvert\mathbf{b}_1\rvert} + 1$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-2} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-2} - 2^{m-\lvert\mathbf{b}_1\rvert-1} + 1$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-2}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$	1 $2^{m-\lvert\mathbf{b}_1\rvert} - 1$ $2^{\lvert\mathbf{b}_1\rvert-1} - 1$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_1\rvert} + 1$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_1\rvert} + 1$ $2^{\lvert\mathbf{b}_1\rvert-1} - 1$ $2^{m-\lvert\mathbf{b}_1\rvert} - 1$
11	$\mathbf{b}_1 \cup \mathbf{b}_2 = [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert = m-2$, $\lvert\mathbf{b}_1\rvert = \lvert\mathbf{b}_2\rvert = m-1$	$[2^m - 4, m, 2^{m-1} - 3]$	$2^{m-1} - 3$ $2^{m-1} - 2$ $2^{m-1} - 1$ 2^{m-1}	2^{m-3} $3 \cdot 2^{m-3}$ $3 \cdot 2^{m-3}$ $2^{m-3} - 1$

Table 2 continued

No.	Conditions	Parameters $[n, k, d]$	Weight distribution	
			Weight	Frequency
12	$\mathbf{b}_1 \cup \mathbf{b}_2 = [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert > 1$, $\lvert\mathbf{b}_1\rvert = \lvert\mathbf{b}_2\rvert < m-1$	$[2^m - 2^{m-\lvert\mathbf{b}_1\rvert+1}, m, 2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert+1} + 1]$	$2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert+1} + 1$ $2^{m-1} - 3 \cdot 2^{m-\lvert\mathbf{b}_1\rvert-1} + 1$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} + 1$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$ 2^{m-1}	$2^{2\lvert\mathbf{b}_1\rvert-m-1}$ $2^{\lvert\mathbf{b}_1\rvert} - 2^{2\lvert\mathbf{b}_1\rvert-m}$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert} + 2^{2\lvert\mathbf{b}_1\rvert-m-1}$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert} + 2^{2\lvert\mathbf{b}_1\rvert-m-1}$ $2^{\lvert\mathbf{b}_1\rvert} - 2^{2\lvert\mathbf{b}_1\rvert-m}$ $2^{2\lvert\mathbf{b}_1\rvert-m-1} - 1$
13	$\mathbf{b}_1 \cup \mathbf{b}_2 = [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert > 1$, $\lvert\mathbf{b}_1\rvert = m-1, \lvert\mathbf{b}_2\rvert < m-1$	$[2^m - 2^{m-\lvert\mathbf{b}_2\rvert} - 2, m, 2^{m-1} - 2^{m-\lvert\mathbf{b}_2\rvert} - 1]$	$2^{m-1} - 2^{m-\lvert\mathbf{b}_2\rvert} - 1$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1} - 1$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 1$ 2^{m-1}	$2^{\lvert\mathbf{b}_2\rvert-2}$ $2^{\lvert\mathbf{b}_2\rvert-2}$ $2^{m-1} - 2^{\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{\lvert\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_2\rvert-2}$ $2^{\lvert\mathbf{b}_2\rvert-2} - 1$
14	$\mathbf{b}_1 \cup \mathbf{b}_2 = [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert > 1$, $m-1 > \lvert\mathbf{b}_1\rvert > \lvert\mathbf{b}_2\rvert$	$[2^m - 2^{m-\lvert\mathbf{b}_2\rvert} - 2^{m-\lvert\mathbf{b}_1\rvert}, m, 2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert} + 1]$	$2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert-1} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$	$2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-m-1}$ $2^{\lvert\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-m-1}$ $2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-m-1}$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_2\rvert-1} + 2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-m-1}$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_2\rvert-1} + 2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-m-1}$ $2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-m-1}$ $2^{\lvert\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-m-1}$ $2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-m-1} - 1$

Table 2 continued

No.	Conditions	Parameters $[n, k, d]$	Weight distribution	
			Weight	Frequency
15	$\mathbf{b}_1 \cup \mathbf{b}_2 \neq [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert = 1$, $\lvert\mathbf{b}_1\rvert = \lvert\mathbf{b}_2\rvert = 2$	$[2^{m-1} + 2^{m-3} - 1, m, 2^{m-3}]$	2^{m-3} 2^{m-2} $5 \cdot 2^{m-4}$ $3 \cdot 2^{m-3}$	1 3 $2^m - 8$ 3
16	$\mathbf{b}_1 \cup \mathbf{b}_2 \neq [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert = 1$, $\lvert\mathbf{b}_1\rvert = \lvert\mathbf{b}_2\rvert > 2$	$n = 2^m - 2^{m-\lvert\mathbf{b}_1\rvert+1} + 2^{m-2\lvert\mathbf{b}_1\rvert+1} - 1$, $k = m$, $d = 2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert+1} + 2^{m-2\lvert\mathbf{b}_1\rvert+1}$	$2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert+1} + 2^{m-2\lvert\mathbf{b}_1\rvert+1}$ $2^{m-1} - 3 \cdot 2^{m-\lvert\mathbf{b}_1\rvert-1} + 2^{m-2\lvert\mathbf{b}_1\rvert+1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} + 2^{m-2\lvert\mathbf{b}_1\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} + 2^{m-2\lvert\mathbf{b}_1\rvert+1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$	1 $2^{\lvert\mathbf{b}_1\rvert} - 2$ $2^{2\lvert\mathbf{b}_1\rvert-2} - 2^{\lvert\mathbf{b}_1\rvert} + 1$ $2^m - 2^{2\lvert\mathbf{b}_1\rvert-1}$ $2^{2\lvert\mathbf{b}_1\rvert-2} - 2^{\lvert\mathbf{b}_1\rvert} + 1$ $2^{\lvert\mathbf{b}_1\rvert} - 2$
17	$\mathbf{b}_1 \cup \mathbf{b}_2 \neq [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert = 1$, $\lvert\mathbf{b}_1\rvert > \lvert\mathbf{b}_2\rvert = 2$	$[3 \cdot 2^{m-2} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 1, m, 2^{m-2} - 2^{m-\lvert\mathbf{b}_1\rvert-1}]$	$2^{m-2} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$ 2^{m-2} $3 \cdot 2^{m-3} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$ $3 \cdot 2^{m-3} - 2^{m-\lvert\mathbf{b}_1\rvert-2}$ $3 \cdot 2^{m-3}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$	1 1 $2^{\lvert\mathbf{b}_1\rvert} - 2$ $2^m - 2^{\lvert\mathbf{b}_1\rvert+1}$ $2^{\lvert\mathbf{b}_1\rvert} - 2$ 1
18	$\mathbf{b}_1 \cup \mathbf{b}_2 \neq [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert = 1$, $\lvert\mathbf{b}_1\rvert > \lvert\mathbf{b}_2\rvert > 2$	$n = 2^m - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+1} - 1$, $k = m$, $d = 2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+1}$	$2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert-1} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$	1 $2^{\lvert\mathbf{b}_2\rvert-1} - 1$ $2^{\lvert\mathbf{b}_1\rvert-1} - 1$ $2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-2} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_2\rvert-1} + 1$ $2^m - 2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-2} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_2\rvert-1} + 1$ $2^{\lvert\mathbf{b}_1\rvert-1} - 1$ $2^{\lvert\mathbf{b}_2\rvert-1} - 1$

Table 2 continued

No.	Conditions	Parameters $[n,k,d]$	Weight distribution	
			Weight	Frequency
19	$\mathbf{b}_1 \cup \mathbf{b}_2 \neq [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert > 1$, $\lvert\mathbf{b}_1\rvert = \lvert\mathbf{b}_2\rvert$, $\lvert\mathbf{b}_1 \setminus \mathbf{b}_2\rvert = 1$	$[2^m - 3\cdot 2^{m-\lvert\mathbf{b}_1\rvert-1}, m, 2^{m-1} - 3\cdot 2^{m-\lvert\mathbf{b}_1\rvert-1}]$	$2^{m-1} - 3\cdot 2^{m-\lvert\mathbf{b}_1\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert}$ $2^{m-1} - 3\cdot 2^{m-\lvert\mathbf{b}_1\rvert-2}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$ 2^{m-1}	$2^{\lvert\mathbf{b}_1\rvert-2}$ $3\cdot 2^{\lvert\mathbf{b}_1\rvert-2}$ $2^m - 2^{\lvert\mathbf{b}_1\rvert+1}$ $3\cdot 2^{\lvert\mathbf{b}_1\rvert-2}$ $2^{\lvert\mathbf{b}_1\rvert-2} - 1$
20	$\mathbf{b}_1 \cup \mathbf{b}_2 \neq [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert > 1$, $\lvert\mathbf{b}_1\rvert = \lvert\mathbf{b}_2\rvert$, $\lvert\mathbf{b}_1 \setminus \mathbf{b}_2\rvert > 1$	$n = 2^m - 2^{m-\lvert\mathbf{b}_1\rvert+1} + 2^{m-2\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert} - 1$, $k = m$, $d = 2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert+1} + 2^{m-2\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$	$2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert+1} + 2^{m-2\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$ $2^{m-1} - 3\cdot 2^{m-\lvert\mathbf{b}_1\rvert-1} + 2^{m-2\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} + 2^{m-2\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} + 2^{m-2\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$ 2^{m-1}	$2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert} - 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$ $2^{2\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert} + 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^m - 2^{2\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$ $2^{2\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert} + 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert} - 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$ $2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1} - 1$
21	$\mathbf{b}_1 \cup \mathbf{b}_2 \neq [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert > 1$, $\lvert\mathbf{b}_1\rvert > \lvert\mathbf{b}_2\rvert$, $\lvert\mathbf{b}_2 \setminus \mathbf{b}_1\rvert = 1$	$n = 2^m - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert} - 1$, $k = m$, $d = 2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert}$	$2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-2} - 2^{m-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$ 2^{m-1}	$2^{\lvert\mathbf{b}_2\rvert-2}$ $2^{\lvert\mathbf{b}_2\rvert-2}$ $2^{\lvert\mathbf{b}_1\rvert} - 2^{\lvert\mathbf{b}_2\rvert-1}$ $2^m - 2^{\lvert\mathbf{b}_1\rvert+1}$ $2^{\lvert\mathbf{b}_1\rvert} - 2^{\lvert\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_2\rvert-2}$ $2^{\lvert\mathbf{b}_2\rvert-2} - 1$
22	$\mathbf{b}_1 \cup \mathbf{b}_2 \neq [m]$, $\lvert\mathbf{b}_1 \cap \mathbf{b}_2\rvert > 1$, $\lvert\mathbf{b}_1\rvert > \lvert\mathbf{b}_2\rvert$, $\lvert\mathbf{b}_2 \setminus \mathbf{b}_1\rvert > 1$	$n = 2^m - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert} - 1$ $k = m$ $d = 2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$	$2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert-1} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{m-\lvert\mathbf{b}_1\rvert-1}$ 2^{m-1}	$2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_2\rvert-1} + 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^m - 2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$ $2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_2\rvert-1} + 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1} - 1$

$$
\begin{aligned}
&w(c_{\mathcal{H}_{\Delta_2^*}}(\mathbf{u}))\\
&= \frac{1}{2}\left(\mathcal{H}_{\Delta_2^*}(1,1,\ldots,1) - \mathcal{H}_{\Delta_2^*}((-1)^{u_1},(-1)^{u_2},\ldots,(-1)^{u_m})\right)\\
&= \frac{1}{2}\left(2^m - 2^{m/2+1} - \prod_{i=1}^{m}(1+(-1)^{u_j}) + (-1)^{\sum_{j\in\mathbf{b}_1} u_j}\prod_{i\notin\mathbf{b}_1}(1+(-1)^{u_i})\right.\\
&\qquad\left. + (-1)^{\sum_{j\in\mathbf{b}_2} u_j}\prod_{i\notin\mathbf{b}_2}(1+(-1)^{u_i}) - (-1)^{\sum_{j\in\mathbf{b}_1} u_j+\sum_{\ell\in\mathbf{b}_2} u_\ell} + 1\right)\\
&= \begin{cases}
0 & \text{if } \mathbf{u}=\mathbf{0};\\
2^{m-1} - 2^{\frac{m}{2}} + (-1)^{\sum_{j=1}^m u_j} 2^{\frac{m}{2}-1} - \frac{1}{2}((-1)^{\sum_{j=1}^m u_j} - 1) & \text{if } \mathbf{u}\neq\mathbf{0}, \mathbf{u}\preceq\mathbf{b}_1;\\
2^{m-1} - 2^{\frac{m}{2}} + (-1)^{\sum_{j=1}^m u_j} 2^{\frac{m}{2}-1} - \frac{1}{2}((-1)^{\sum_{j=1}^m u_j} - 1) & \text{if } \mathbf{u}\neq\mathbf{0}, \mathbf{u}\preceq\mathbf{b}_2;\\
2^{m-1} - 2^{\frac{m}{2}} - \frac{1}{2}((-1)^{\sum_{j=1}^m u_j} - 1) & \text{if } \mathbf{u}\not\preceq\mathbf{b}_1, \mathbf{u}\not\preceq\mathbf{b}_2
\end{cases}\\
&= \begin{cases}
0 & \text{if } \mathbf{u}=\mathbf{0};\\
2^{m-1} - 2^{\frac{m}{2}-1} & \text{if } \mathbf{u}\neq\mathbf{0}, \mathbf{u}\preceq\mathbf{b}_1 \text{ or } \mathbf{u}\preceq\mathbf{b}_2, \sum_{j=1}^m u_j = 0;\\
2^{m-1} - 3\cdot 2^{\frac{m}{2}-1} + 1 & \text{if } \mathbf{u}\neq\mathbf{0}, \mathbf{u}\preceq\mathbf{b}_1 \text{ or } \mathbf{u}\preceq\mathbf{b}_2, \sum_{j=1}^m u_j = 1;\\
2^{m-1} - 2^{\frac{m}{2}} & \text{if } \mathbf{u}\not\preceq\mathbf{b}_1, \mathbf{u}\not\preceq\mathbf{b}_2, \sum_{j=1}^m u_j = 0;\\
2^{m-1} - 2^{\frac{m}{2}} + 1 & \text{if } \mathbf{u}\not\preceq\mathbf{b}_1, \mathbf{u}\not\preceq\mathbf{b}_2, \sum_{j=1}^m u_j = 1,
\end{cases}
\end{aligned}
$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_m) \in \mathbb{F}_2^m$. Then $C_{\Delta_2^*}$ is a $[2^m - 2^{\frac{m}{2}+1}, m, 2^{m-1} - 3\cdot 2^{\frac{m}{2}-1} + 1]$ 4-weight binary linear code and their weight distribution is given as in Table 1.

Similarly, multivariable functions $\mathcal{H}_{\Delta_2^*}(x_1, x_2, \ldots, x_m)$ associated with the binary linear codes in the family No. 18 (in Table 2) are given as follows:

$$\prod_{i=1}^{m}(1+x_i) - \prod_{j\in\mathbf{b}_1} x_j \prod_{i\notin\mathbf{b}_1}(1+x_i) - \prod_{j\in\mathbf{b}_2} x_j \prod_{i\notin\mathbf{b}_2}(1+x_i) + \prod_{j\in\mathbf{b}_1\cup\mathbf{b}_2} x_j \prod_{i\notin\mathbf{b}_1\cup\mathbf{b}_2}(1+x_i) - 1.$$

Then we will obtain the parameters and the weight distributions by using Lemma 2.6. The length of $C_{\Delta_2^*}$ is equal to $2^m - 2^{m-|\mathbf{b}_1|} - 2^{m-|\mathbf{b}_2|} + 2^{m-|\mathbf{b}_1|-|\mathbf{b}_2|+1} - 1$, and the weights of codewords $c_{\mathcal{H}_{\Delta_2^*}}(\mathbf{u})$ for $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ can be calculated by

$$
\begin{aligned}
&w(c_{\mathcal{H}_{\Delta_2^*}}(\mathbf{u}))\\
&= \frac{1}{2}\left(\mathcal{H}_{\Delta_2^*}(1,1,\ldots,1) - \mathcal{H}_{\Delta_2^*}((-1)^{u_1},(-1)^{u_2},\ldots,(-1)^{u_m})\right)\\
&= \frac{1}{2}\left(2^m - 2^{m-|\mathbf{b}_1|} - 2^{m-|\mathbf{b}_2|} + 2^{m-|\mathbf{b}_1|-|\mathbf{b}_2|+1} - \prod_{i=1}^{m}(1+(-1)^{u_i}) + (-1)^{\sum_{j\in\mathbf{b}_1} u_j}\prod_{i\notin\mathbf{b}_1}(1+(-1)^{u_i})\right.\\
&\qquad\left. + (-1)^{\sum_{j\in\mathbf{b}_2} u_j}\prod_{i\notin\mathbf{b}_2}(1+(-1)^{u_i}) - (-1)^{\sum_{j\in\mathbf{b}_1\cup\mathbf{b}_2} u_j}\prod_{i\notin\mathbf{b}_1\cup\mathbf{b}_2}(1+(-1)^{u_i})\right)
\end{aligned}
$$

Table 3 The eigenvalues and the multiplicities of nonbipartite Ramanujan graphs $\mathrm{Cay}(\mathbb{F}_2^m, \Delta_1^*)$ in Theorem 4.1

Eigenvalues	Multiplicities
$2^m - 2^{m-\lvert\mathbf{b}\rvert} - 1$	1
$2^{m-\lvert\mathbf{b}\rvert} - 1$	$2^{\lvert\mathbf{b}\rvert-1}$
-1	$2^m - 2^{\lvert\mathbf{b}\rvert}$
$-2^{m-\lvert\mathbf{b}\rvert} - 1$	$2^{\lvert\mathbf{b}\rvert-1} - 1$

$$= \begin{cases} 0 & \text{if } \mathbf{u} = \mathbf{0}; \\ 2^{m-1} - 2^{m-|\mathbf{b}_1|} - 2^{m-|\mathbf{b}_2|} + 2^{m-|\mathbf{b}_1|-|\mathbf{b}_2|+1} & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{u} = \mathbf{b}_1 \cap \mathbf{b}_2; \\ 2^{m-1} - 2^{m-|\mathbf{b}_2|-1} - (1-(-1)^{\sum_{j=1}^m u_j})2^{m-|\mathbf{b}_1|-1} & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{u} \neq \mathbf{b}_1 \cap \mathbf{b}_2, \mathbf{u} \preceq \mathbf{b}_1; \\ \quad + (1-(-1)^{\sum_{j=1}^m u_j})2^{m-|\mathbf{b}_1|-|\mathbf{b}_2|} & \\ 2^{m-1} - 2^{m-|\mathbf{b}_1|-1} - (1-(-1)^{\sum_{j=1}^m u_j})2^{m-|\mathbf{b}_2|-1} & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{u} \neq \mathbf{b}_1 \cap \mathbf{b}_2, \mathbf{u} \preceq \mathbf{b}_2; \\ \quad + (1-(-1)^{\sum_{j=1}^m u_j})2^{m-|\mathbf{b}_1|-|\mathbf{b}_2|} & \\ 2^{m-1} - 2^{m-|\mathbf{b}_1|-1} - 2^{m-|\mathbf{b}_2|-1} + (1-(-1)^{\sum_{j=1}^m u_j})2^{m-|\mathbf{b}_1|-|\mathbf{b}_2|} & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{u} \neq \mathbf{b}_1 \cap \mathbf{b}_2, \\ & \quad \mathbf{u} \not\preceq \mathbf{b}_1, \mathbf{u} \not\preceq \mathbf{b}_2, \mathbf{u} \preceq \mathbf{b}_1 \cup \mathbf{b}_2; \\ 2^{m-1} - 2^{m-|\mathbf{b}_1|-1} - 2^{m-|\mathbf{b}_2|-1} + 2^{m-|\mathbf{b}_1|-|\mathbf{b}_2|} & \text{if } \mathbf{u} \not\preceq \mathbf{b}_1 \cup \mathbf{b}_2 \end{cases}$$

$$= \begin{cases} 0 & \text{if } \mathbf{u} = \mathbf{0}; \\ 2^{m-1} - 2^{m-|\mathbf{b}_1|} - 2^{m-|\mathbf{b}_2|} + 2^{m-|\mathbf{b}_1|-|\mathbf{b}_2|+1} & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{u} = \mathbf{b}_1 \cap \mathbf{b}_2, ; \\ 2^{m-1} - 2^{m-|\mathbf{b}_2|-1} & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{u} \neq \mathbf{b}_1 \cap \mathbf{b}_2, \mathbf{u} \preceq \mathbf{b}_1, \sum_{j=1}^m u_j = 0; \\ 2^{m-1} - 2^{m-|\mathbf{b}_1|} - 2^{m-|\mathbf{b}_2|-1} + 2^{m-|\mathbf{b}_1|-|\mathbf{b}_2|+1} & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{u} \neq \mathbf{b}_1 \cap \mathbf{b}_2, \mathbf{u} \preceq \mathbf{b}_1, \sum_{j=1}^m u_j = 1; \\ 2^{m-1} - 2^{m-|\mathbf{b}_1|-1} & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{u} \neq \mathbf{b}_1 \cap \mathbf{b}_2, \mathbf{u} \preceq \mathbf{b}_2, \sum_{j=1}^m u_j = 0; \\ 2^{m-1} - 2^{m-|\mathbf{b}_1|-1} - 2^{m-|\mathbf{b}_2|} + 2^{m-|\mathbf{b}_1|-|\mathbf{b}_2|+1} & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{u} \neq \mathbf{b}_1 \cap \mathbf{b}_2, \mathbf{u} \preceq \mathbf{b}_2, \sum_{j=1}^m u_j = 1; \\ 2^{m-1} - 2^{m-|\mathbf{b}_1|-1} - 2^{m-|\mathbf{b}_2|-1} & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{u} \neq \mathbf{b}_1 \cap \mathbf{b}_2, \sum_{j=1}^m u_j = 0, \\ & \quad \mathbf{u} \not\preceq \mathbf{b}_1, \mathbf{u} \not\preceq \mathbf{b}_2, \mathbf{u} \preceq \mathbf{b}_1 \cup \mathbf{b}_2; \\ 2^{m-1} - 2^{m-|\mathbf{b}_1|-1} - 2^{m-|\mathbf{b}_2|-1} + 2^{m-|\mathbf{b}_1|-|\mathbf{b}_2|+1} & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{u} \neq \mathbf{b}_1 \cap \mathbf{b}_2, \sum_{j=1}^m u_j = 1, \\ & \quad \mathbf{u} \not\preceq \mathbf{b}_1, \mathbf{u} \not\preceq \mathbf{b}_2, \mathbf{u} \preceq \mathbf{b}_1 \cup \mathbf{b}_2; \\ 2^{m-1} - 2^{m-|\mathbf{b}_1|-1} - 2^{m-|\mathbf{b}_2|-1} + 2^{m-|\mathbf{b}_1|-|\mathbf{b}_2|} & \text{if } \mathbf{u} \not\preceq \mathbf{b}_1 \cup \mathbf{b}_2. \end{cases}$$

To complete the proof, we verify the self-orthogonality of the code $C_{\Delta_2^*}$ in No. 5 through No. 7; the binary linear codes $C_{\Delta_2^*}$ only have the codewords whose weights are divisible by four under the conditions (i) through (iii), respectively, so they are self-orthogonal by Lemma 2.1. Thus, the results follow as desired. □

4 Families of Ramanujan graphs from simplicial complexes with few blockers

In this section, we find fourteen families of nonbipartite Ramanujan graphs. These families are constructed from the simplicial complexes with one or two blockers given in Sect. 3. Furthermore, we explicitly compute the eigenvalues and the multiplicities of the Ramanujan graphs, using the weight distributions of the corresponding codes obtained in Sect. 3.

Theorem 4.1 *Let m be a positive integer with $m \geq 4$. Let $\Delta_1 \subseteq \mathbb{F}_2^m$ be a simplicial complex with exactly one blocker $\mathbf{b}$ with $|\mathbf{b}| \geq \frac{m}{2}$. Then the Cayley graph $\mathrm{Cay}(\mathbb{F}_2^m, \Delta_1^*)$ is a nonbipartite Ramanujan graph with the eigenvalues and the multiplicities given in Table 3.*

Proof We mainly use Lemmas 2.10 and 2.11. Combining Lemma 2.11 and Theorem 3.1, we have that $\mathrm{Cay}(\mathbb{F}_2^m, \Delta_1^*)$ is Ramanujan if and only if

$$2^{m-|\mathbf{b}|-1} - \frac{1}{2} \le \sqrt{2^m - 2^{m-|\mathbf{b}|} - 2} \text{ and } 2^{m-|\mathbf{b}|-1} + \frac{1}{2} \le \sqrt{2^m - 2^{m-|\mathbf{b}|} - 2}$$
$$\iff 2^{m-|\mathbf{b}|-1} + \frac{1}{2} \le \sqrt{2^m - 2^{m-|\mathbf{b}|} - 2}$$
$$\iff 2^{2m-2|\mathbf{b}|-2} + 2^{m-|\mathbf{b}|-1} + \frac{1}{4} \le 2^m - 2^{m-|\mathbf{b}|} - 2$$
$$\iff 2^{m-|\mathbf{b}|-1}(2^{m-|\mathbf{b}|-1} - 2^{|\mathbf{b}|+1} + 3) \le -\frac{7}{4}. \tag{3}$$

Under the assumption that $m \ge 4$ and $|\mathbf{b}| \ge \frac{m}{2}$, we have that $2^{m-|\mathbf{b}|-1} - 2^{|\mathbf{b}|+1} \le 2^{\frac{m}{2}-1} - 2^{\frac{m}{2}+1} \le -6$; thus, Eq. (3) follows. Therefore, $\mathrm{Cay}(\mathbb{F}_2^m, \Delta_1^*)$ is a Ramanujan graph. Their eigenvalues and multiplicities are computed by Lemma 2.10 as given in Table 3. Moreover, we find that the Cayley graph $\mathrm{Cay}(\mathbb{F}_2^m, \Delta_1^*)$ is nonbipartite by Lemma 2.9. Therefore, we get the result. □

Theorem 4.2 *Let m be a positive integer with $m \ge 4$. Let $\Delta_2 \subseteq \mathbb{F}_2^m$ be a simplicial complex having exactly two blockers $\mathbf{b}_1$ and $\mathbf{b}_2$ corresponding to the code families No. 1, 2, 4, 5, 6, 8, 9, 11, 12, 13, 19, 21, and 22 in Tables 1 and 2. Then $\mathrm{Cay}(\mathbb{F}_2^m, \Delta_2^*)$ is a nonbipartite Ramanujan graph under the conditions with the eigenvalues and the multiplicities; these Ramanujan graphs are listed in Tables 4 and 5.*

Proof We only prove the result for No. 5 in Table 4 since the other results can be obtained in a similar way. Combining Theorem 3.2 with Lemma 2.11, we have that $\mathrm{Cay}(\mathbb{F}_2^m, \Delta_2^*)$ is a Ramanujan graph if and only if

$$2^{m-|\mathbf{b}_1|-1} - 2^{m-2|\mathbf{b}_1|-1} + \frac{1}{2} \le \sqrt{2^m - 2^{m-|\mathbf{b}_1|+1} + 2^{m-2|\mathbf{b}_1|} - 2}$$
$$\iff 2^{\frac{m-1}{2}} - \frac{1}{2} \le \sqrt{2^m - 2^{\frac{m+3}{2}}} \iff 2^{m-1} - 2^{\frac{m-1}{2}} + \frac{1}{4} \le 2^m - 2^{\frac{m+3}{2}}$$
$$\iff 2^{m-1} - 3 \cdot 2^{\frac{m-1}{2}} \ge \frac{1}{4} \iff \left(2^{m-1} - \frac{3}{2}\right)^2 \ge \frac{10}{4};$$

the first equivalence is obtained by the assumption $|\mathbf{b}_1| = \frac{m-1}{2}$. Then the last inequality holds since $m \ge 5$; thus, $\mathrm{Cay}(\mathbb{F}_2^m, \Delta_2^*)$ is a Ramanujan graph. The eigenvalues and the multiplicity of $\mathrm{Cay}(\mathbb{F}_2^m, \Delta_2^*)$ are computed by combining Lemma 2.10 with Theorem 3.2. In addition, by Lemma 2.9, we find that $\mathrm{Cay}(\mathbb{F}_2^m, \Delta_2^*)$ is nonbipartite as desired. □

5 Comparison with other Ramanujan graphs

In this section, we compare our results (Tables 4 and 5) to the previous results (Table 6) on Ramanujan graphs which are constructed from codes. We obtain many families of nonbipartite Ramanujan graphs in Sect. 4. Now, we verify that our families are different from the previous families in Table 6 in terms of eigenvalues and regularity as follows.

(i) Compared with [15], our Ramanujan graphs (No. 2, No. 5, and No. 6–No. 13 in Tables 4 and 5) have more than 4 distinct eigenvalues. However, Ramanujan graphs (No. 4 through No. 10 in Table 6) obtained in [15] have exactly 3 or 4 distinct eigenvalues.
(ii) Compared with [3, 16, 22], our Ramanujan graphs (No. 5 and No. 11–No. 13 in Tables 4 and 5) have odd regularity while the Ramanujan graphs (No. 1 with odd prime p, No. 2 and No. 3 in Table 6) given in [3, 16, 22] have even regularity.

Table 4 Families of nonbipartite Ramanujan graphs with the eigenvalues and multiplicities (corresponding to the code families in Table 1)

No.	Code family No.	Conditions	Regularity	Eigenvalues	Multiplicities
1	1	always	$2^{m-1}-2$	$2^{m-1}-2$ 0 -2	1 2^{m-2} $2^{m-2}-1$
2	2	always	$2^m-2^{\frac{m+2}{2}}$	$2^m-2^{\frac{m+2}{2}}$ $2^{\frac{m}{2}}-2$ 0 -2 $-2^{\frac{m}{2}}$	1 $2^{\frac{m}{2}}$ $2^{m-1}-2^{\frac{m}{2}}+1$ $2^{m-1}-2^{\frac{m}{2}}$ $2^{\frac{m}{2}}-2$
3	4	always	$2^{m-2}-1$	$2^{m-2}-1$ -1	1 $2^{m-2}-1$
4	5	$\lvert\mathbf{b}_1\rvert \geq \frac{m-2}{2}$	$2^{m-1}-2^{m-\lvert\mathbf{b}_1\rvert-1}-1$	$2^{m-1}-2^{m-\lvert\mathbf{b}_1\rvert-1}-1$ $2^{m-\lvert\mathbf{b}_1\rvert-1}-1$ -1 $-2^{m-\lvert\mathbf{b}_1\rvert-1}-1$	1 $2^{\lvert\mathbf{b}_1\rvert-1}$ $2^{m-1}-2^{\lvert\mathbf{b}_1\rvert}$ $2^{\lvert\mathbf{b}_1\rvert-1}-1$
5	6	$m \geq 5 : odd,$ $\lvert\mathbf{b}_1\rvert = \frac{m-1}{2}$	$2^m-2^{\frac{m+3}{2}}+1$	$2^m-2^{\frac{m+3}{2}}+1$ $2^{\frac{m+1}{2}}-3$ 1 -1 -3 $-2^{\frac{m+1}{2}}+1$	1 $2^{\frac{m-1}{2}}$ $2^{m-2}-2^{\frac{m-1}{2}}+1$ 2^{m-1} $2^{m-2}-2^{\frac{m-1}{2}}$ $2^{\frac{m-1}{2}}-2$

Table 5 Families of nonbipartite Ramanujan graphs with the eigenvalues and multiplicities (corresponding to the code families in Table 2)

No.	Code family No.	Conditions	Regularity	Eigenvalues	Multiplicities
6	8	*always*	$2^m - 2^{\frac{m+1}{2}}$	$2^m - 2^{\frac{m+1}{2}}$ $2^{\frac{m+1}{2}} - 2$ $2^{\frac{m-1}{2}} - 2$ 0 -2 $-2^{\frac{m-1}{2}}$	1 1 $2^{\frac{m+1}{2}} - 2$ $2^{m-1} - 2^{\frac{m+1}{2}} + 1$ $2^{m-1} - 2^{\frac{m+1}{2}} + 1$ $2^{\frac{m+1}{2}} - 2$
7	9	$m = 4, 5$	$3 \cdot 2^{m-2} - 2$	$3 \cdot 2^{m-2} - 2$ 2^{m-2} $2^{m-2} - 2$ 0 -2 -2^{m-2}	1 1 1 $2^{m-1} - 2$ $2^{m-1} - 2$ 1
8	11	*always*	$2^m - 4$	$2^m - 4$ 2 0 -2 -4	1 2^{m-3} $3 \cdot 2^{m-3}$ $3 \cdot 2^{m-3}$ $2^{m-3} - 1$
9	12	$m \geq 6$, $\lvert\mathbf{b}_1\rvert \geq \frac{m+1}{2}$	$2^m - 2^{m-\lvert\mathbf{b}_1\rvert+1}$	$2^m - 2^{m-\lvert\mathbf{b}_1\rvert+1}$ $2^{m-\lvert\mathbf{b}_1\rvert+1} - 2$ $2^{m-\lvert\mathbf{b}_1\rvert} - 2$ 0 -2 $-2^{m-\lvert\mathbf{b}_1\rvert}$ $-2^{m-\lvert\mathbf{b}_1\rvert+1}$	1 $2^{2\lvert\mathbf{b}_1\rvert-m-1}$ $2^{\lvert\mathbf{b}_1\rvert} - 2^{2\lvert\mathbf{b}_1\rvert-m}$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert} + 2^{2\lvert\mathbf{b}_1\rvert-m-1}$ $2^{m-1} - 2^{\lvert\mathbf{b}_1\rvert} + 2^{2\lvert\mathbf{b}_1\rvert-m-1}$ $2^{\lvert\mathbf{b}_1\rvert} - 2^{2\lvert\mathbf{b}_1\rvert-m}$ $2^{2\lvert\mathbf{b}_1\rvert-m-1} - 1$

Table 5 continued

No.	Code family No.	Conditions	Regularity	Eigenvalues	Multiplicities
10	13	$m \geq 5$, $\lvert\mathbf{b}_2\rvert \geq \frac{m-1}{2}$	$2^m - 2^{m-\lvert\mathbf{b}_2\rvert} - 2$	$2^m - 2^{m-\lvert\mathbf{b}_2\rvert} - 2$ $2^{m-\lvert\mathbf{b}_2\rvert}$ $2^{m-\lvert\mathbf{b}_2\rvert} - 2$ 0 -2 $-2^{m-\lvert\mathbf{b}_2\rvert}$ $-2^{m-\lvert\mathbf{b}_2\rvert} - 2$	1 $2^{\lvert\mathbf{b}_2\rvert-2}$ $2^{\lvert\mathbf{b}_2\rvert-2}$ $2^{m-1} - 2^{\lvert\mathbf{b}_2\rvert-1}$ $2^{m-1} - 2^{\lvert\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_2\rvert-2}$ $2^{\lvert\mathbf{b}_2\rvert-2} - 1$
11	19	$m \geq 5$, $\lvert\mathbf{b}_1\rvert \geq \frac{m}{2}$	$2^m - 3 \cdot 2^{m-\lvert\mathbf{b}_1\rvert-1} - 1$	$2^m - 3 \cdot 2^{m-\lvert\mathbf{b}_1\rvert-1} - 1$ $3 \cdot 2^{m-\lvert\mathbf{b}_1\rvert-1} - 1$ $2^{m-\lvert\mathbf{b}_1\rvert-1} - 1$ -1 $-2^{m-\lvert\mathbf{b}_1\rvert-1} - 1$ $-3 \cdot 2^{m-\lvert\mathbf{b}_1\rvert-1} - 1$	1 $2^{\lvert\mathbf{b}_1\rvert-2}$ $3 \cdot 2^{\lvert\mathbf{b}_1\rvert-2}$ $2^m - 2^{\lvert\mathbf{b}_1\rvert+1}$ $3 \cdot 2^{\lvert\mathbf{b}_1\rvert-2}$ $2^{\lvert\mathbf{b}_1\rvert-2} - 1$
12	21	$m \geq 7$, $\lvert\mathbf{b}_2\rvert \geq \frac{m+1}{2}$	$2^m - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert} - 1$	$2^m - 2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert} - 1$ $2^{m-\lvert\mathbf{b}_1\rvert-1} + 2^{m-\lvert\mathbf{b}_2\rvert} - 1$ $-2^{m-\lvert\mathbf{b}_1\rvert-1} + 2^{m-\lvert\mathbf{b}_2\rvert} - 1$ $2^{m-\lvert\mathbf{b}_1\rvert-1} - 1$ -1 $-2^{m-\lvert\mathbf{b}_1\rvert-1} - 1$ $2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert} - 1$ $-2^{m-\lvert\mathbf{b}_1\rvert-1} - 2^{m-\lvert\mathbf{b}_2\rvert} - 1$	1 $2^{\lvert\mathbf{b}_2\rvert-2}$ $2^{\lvert\mathbf{b}_2\rvert-2}$ $2^{\lvert\mathbf{b}_1\rvert} - 2^{\lvert\mathbf{b}_2\rvert-1}$ $2^m - 2^{\lvert\mathbf{b}_1\rvert+1}$ $2^{\lvert\mathbf{b}_1\rvert} - 2^{\lvert\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_2\rvert-2}$ $2^{\lvert\mathbf{b}_2\rvert-2} - 1$
13	22	$m \geq 11$, $\lvert\mathbf{b}_2\rvert \geq \frac{m+3}{2}$	$2^m - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert} - 1$	$2^m - 2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert} - 1$ $2^{m-\lvert\mathbf{b}_1\rvert} + 2^{m-\lvert\mathbf{b}_2\rvert} - 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert} - 1$ $2^{m-\lvert\mathbf{b}_2\rvert} - 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert} - 1$ $2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert} - 1$ $2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert} - 1$ -1 $-2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert} - 1$ $-2^{m-\lvert\mathbf{b}_1\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert} - 1$ $-2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert} - 1$ $-2^{m-\lvert\mathbf{b}_1\rvert} - 2^{m-\lvert\mathbf{b}_2\rvert} + 2^{m-\lvert\mathbf{b}_1\rvert-\lvert\mathbf{b}_2\rvert+\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert} - 1$	1 $2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_2\rvert-1} + 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^m - 2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert}$ $2^{\lvert\mathbf{b}_1\rvert+\lvert\mathbf{b}_2\rvert-\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_2\rvert-1} + 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\rvert-1} - 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_2\rvert-1} - 2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1}$ $2^{\lvert\mathbf{b}_1\cap\mathbf{b}_2\rvert-1} - 1$

Table 6 Previous results on Ramanujan graphs from codes

No.	Regularity	Eigenvalues	Bipartite	Conditions	References
1	$q+1$	–	–	p : a prime, $q = p^m$	[16]
2	$p+1$	–	nonbipartite	p: a prime with $p \equiv 3 \pmod 4$	[3, 4]
3	$q+1$	$q+1$, $-\mathcal{K}_q(1, N(a))$ for all $0 \neq a \in \mathbb{F}_{q^2}$	nonbipartite	p : an odd prime, $q = p^m$ $\mathcal{K}_q$: a Kloosterman sum over $\mathbb{F}_q$ N : the norm from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$	[22]
4	$2^{2m-e} - 2^m + 2^{m-e} - 1$	$2^{2m-e} - 2^m + 2^{m-e} - 1$, $2^{m-e} - 1$, $-2^m + 2^{m-e} - 1$	nonbipartite	$e \leq 2, \mu \geq 2, e \mid m$,	[15]
5	$(2^m - 1)/3$	$(2^m - 1)/3$, $\frac{-1-(-1)^{\frac{m-2}{2}} 2^{\frac{m}{2}}}{3}$, $\frac{-1+(-1)^{\frac{m-2}{2}} 2^{\frac{m+2}{2}}}{3}$	nonbipartite	$m \geq 4$: even	[15]
6	$2^{m-1} - 2^{\frac{m-2}{2}} - 1$	$2^{m-1} - 2^{\frac{m-2}{2}} - 1$, $2^{\frac{m-2}{2}} - 1$, $-2^{\frac{m-2}{2}} - 1$	nonbipartite	$m \geq 4$: even	[15]
7	$p^{m-1} \pm (-1)^{\frac{(p-1)m}{4}} (p-1) p^{\frac{m-2}{2}} - 1$	$p^{m-1} \pm (-1)^{\frac{(p-1)m}{4}} (p-1) p^{\frac{m-2}{2}} - 1$, $\pm(p-1) p^{\frac{m-2}{2}} - 1$, $\mp p^{\frac{m-2}{2}} - 1$	nonbipartite	$p \in \{3, 5\}, m \geq 4$	[15]
8	$p^{m-1} - 1$	$p^{m-1} - 1$, -1, $\pm p^{\frac{m-1}{2}} - 1$	nonbipartite	$m \geq 3$: odd	[15]

Table 6 continued

No.	Regularity	Eigenvalues	Bipartite	Conditions	References
9	$\frac{p-1}{2}(p^{m-1} \pm p^{\frac{m-2}{2}})$	$\frac{p-1}{2}(p^{m-1} \pm p^{\frac{m-2}{2}})$, $\pm\frac{1}{2}(p-1)p^{\frac{m-2}{2}}$, $\mp\frac{1}{2}(p+1)p^{\frac{m-2}{2}}$	nonbipartite	$(p, m) \neq (3, 2)$, $m \geq 2$: even, p : an odd prime	[15]
10	$\frac{p-1}{2}(p^{m-1} \pm p^{\frac{m-1}{2}})$	$\frac{p-1}{2}(p^{m-1} \pm p^{\frac{m-1}{2}})$, $\pm\frac{1}{2}(p-1)p^{\frac{m-1}{2}}$, 0, $\mp p^{\frac{m-1}{2}}$	nonbipartite	$m \geq 3$: odd, $p \in \{3, 5, 7\}$	[15]

(iii) Compared with [16] (No. 1 with $p = 2$ in Table 6), we find that our Ramanujan graphs (No. 5 and No. 11–No. 13 in Tables 4 and 5) have different regularity for large m. For example, our Ramanujan graph $\mathrm{Cay}(\mathbb{F}_2^m, \Delta_2^*)$ (No. 5 in Table 4) have regularity $2^m - 2^{\frac{m+3}{2}} + 1$, while their Ramanujan graph (No. 1 with $p = 2$ in Table 6) has regularity $2^k + 1$ for some k (for $m \geq 6$, $2^m - 2^{\frac{m+3}{2}} + 1$ cannot be of the form $2^k + 1$ since $2^{m-1} + 1 < 2^m - 2^{\frac{m+3}{2}} < 2^m$). The other cases can be verified in a similar argument as well.

Therefore, we confirm that some of our Ramanujan graphs (No. 5 and No. 11–No. 13 in Tables 4 and 5) are different from those in the previous works [3, 15, 16, 22].

6 Conclusion

We constructed 14 families of nonbipartite Ramanujan graphs by using the binary linear codes obtained from simplicial complexes with few blockers, furthermore, we explicitly computed the eigenvalues and the multiplicities of the Ramanujan graphs. We emphasize that it is the first time that the simplicial complexes with a few blockers are used for construction of few-weight linear codes and the corresponding Ramanujan graphs. Moreover, we confirm that some of the families of Ramanujan graphs produced are different from the previous Ramanujan graphs in terms of their regularity and eigenvalues. In this paper, we focused exclusively on binary linear codes; so, this framework can be extended to arbitrary finite fields. Hence, it would be desirable to use simplicial complexes with many blockers for constructing optimal linear codes over arbitrary finite fields and the corresponding Ramanujan graphs.

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Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no conflict of interest.

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