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RESEARCH ARTICLE

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Simply connected positive Sasakian 5-manifolds

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Abstract

We investigate closed simply connected 5-manifolds capable of hosting positive Sasakian structures. We present a conjectural comprehensive list of such manifolds.

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1 | INTRODUCTION

Closed simply connected 5-manifolds are completely classified by Barden and Smale [2, 32]. In particular, closed simply connected spin 5-manifolds are described in [32] as follows: For a positive integer m, up to diffeomorphisms, there is a unique closed simply connected spin 5-manifold M_m with $\mathrm{H}_2(M_m, \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$. Furthermore, a closed simply connected spin 5-manifold M

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is of the form

$$M = kM_\infty \# M_{m_1} \# \dots \# M_{m_r},$$

where kM_∞ is the k-fold connected sum of $S^2 \times S^3$ for a non-negative integer k and m_i is a positive integer greater than 1 with m_i dividing m_{i+1} [32, Theorem A]. Meanwhile, closed simply connected non–spin 5-manifolds are described in [2]. For instance, the non-trivial S^3-bundle over S^2 is a closed simply connected non-spin 5-manifold. We will denote by X_∞ this 5-manifold. Every closed simply connected non-spin 5-manifold with trivial third integral Stiefel–Whitney class is of the form

$$X_\infty \# M,$$

where M is a closed simply connected spin 5-manifold [2, Lemma 1.2 and Theorem 2.3].

A Riemannian manifold (M, g) is called Sasakian if the cone metric $r^2 g + dr^2$ defines a Kähler metric on $M \times \mathbb{R}^+$. It follows from [31, Theorem 6] that every compact Sasakian manifold can accommodate a quasi-regular Sasakian structure. A quasi-regular Sasakian structure on a manifold M can be written as the unit circle subbundle $M \to (S, \Delta)$ of a holomorphic Seifert $\mathbb{C}^*$-bundle over a complex algebraic orbifold (S, Δ), where the boundary divisor $\Delta = \sum \left(1 - \frac{1}{m_i}\right) R_i$ describes the branch divisors R_is of the orbifold with their ramification indices m_is. The Sasakian manifold M is positive if the first basic Chern class of the characteristic foliation [6, Theorem/Definition 7.5.17] can be represented by a positive definite (1,1)-form. It follows from [6, Proposition 7.5.23] that M is positive if $-(K_S + \Delta)$ is ample. It has been demonstrated that a positive Sasakian manifold permits a Sasakian structure with positive Ricci curvature [9, Theorem A]. In this paper, motivated by [6, Open problem 10.2.3], we investigate which closed simply connected 5-manifolds can accommodate positive Sasakian structures.

Closed simply connected 5-manifolds have necessary topological conditions in the second integral homology groups to carry positive Sasakian structures. The following topological properties of closed simply connected positive Sasakian 5-manifolds are verified in [20, Theorems 1.4, 1.6, 1.8] and [23, Theorem 7].

Theorem 1.1. *Let M be a closed simply connected 5-manifold that admits a positive Sasakian structure.*

(1) *The torsion part of $\mathrm{H}_2(M, \mathbb{Z})$ is one of the following:*
   (a) $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$, *where m is a positive integer;*
   (b) $(\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z})^{\oplus 2}$;
   (c) $(\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})^{\oplus 2}$;
   (d) $(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^{\oplus 2}$, $(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^{\oplus 3}$, $(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^{\oplus 4}$;
   (e) $(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})^{\oplus n}$, *where $n \geqslant 1$.*
(2) *If* $\mathrm{H}_2(M, \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ *for a positive integer m*, then *m is not divisible by 30.*
(3) *If* $\mathrm{H}_2(M, \mathbb{Z})_{\mathrm{tor}} = (\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z})^{\oplus 2}$ *or* $(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^{\oplus 4}$, *then* $\mathrm{rank}(\mathrm{H}_2(M, \mathbb{Z})) = 0$.
(4) *If* $\mathrm{H}_2(M, \mathbb{Z})_{\mathrm{tor}} = (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})^{\oplus 2}$, *then* $\mathrm{rank}(\mathrm{H}_2(M, \mathbb{Z})) \leqslant 1$.
(5) *If* $\mathrm{H}_2(M, \mathbb{Z})_{\mathrm{tor}} = (\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^{\oplus 3}$, *then* $\mathrm{rank}(\mathrm{H}_2(M, \mathbb{Z})) \leqslant 1$.[1]
(6) *If* $\mathrm{H}_2(M, \mathbb{Z})_{\mathrm{tor}} = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ *for an integer $m \geqslant 12$, then* $\mathrm{rank}(\mathrm{H}_2(M, \mathbb{Z})) \leqslant 8$.

[1] This bound is obtained from a personal note of János Kollár given to the second author in 2022.

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It then follows from [2, Lemma 1.2 and Theorem 2.3] and [14, Corollary in Section 3.2] that a closed simply connected 5-manifold M that allows a positive Sasakian structure is diffeomorphic to either

$$(k+1)M_\infty \# N \quad \text{or} \quad X_\infty \# k M_\infty \# N,$$

where N is a simply connected rational homology 5-sphere whose second integral homology group is one of the torsion groups in Theorem 1.1. The former manifold is spin, but the latter is non-spin.

A Sasakian n-dimensional manifold is called Sasaki–Einstein if its metric g satisfies the Einstein condition, that is, $\mathrm{Ric}_g = \lambda g$ for some constant λ. For Sasakian manifolds, the Einstein constant λ is given as $\lambda = n - 1$ [3, Theorem A], [30, Theorem 2.1]. A closed simply connected manifold that accommodates a Sasaki–Einstein structure must be spin [26, Proposition 4.2]. Therefore, in order to find simply connected Sasaki–Einstein manifolds, we have only to consider spin manifolds.

For simply connected rational homology 5-spheres, we have the following complete description for positive Sasakian and Sasaki–Einstein structures ([5, Theorem 2], [10, Theorem A], [20, Theorems 1.4 and 1.6, Example 8.6, Proof 9.6], [23, Example 19], [33, Theorem 1.7]).

Theorem 1.2. *A simply connected rational homology 5-sphere can accommodate a positive Sasakian structure if and only if it is one of the rational homology 5-spheres*

$$M_m, nM_2, 2M_3, 3M_3, 4M_3, 2M_4, 2M_5,$$

where n is a positive integer and m is a positive integer not divisible by 30. Furthermore, every positive Sasakian simply connected rational homology 5-sphere admits a Sasaki–Einstein structure.

On the other hand, it is known that every closed simply connected spin 5-manifold with no torsion in its second homology group allows a Sasaki–Einstein structure [22].

Theorem 1.3. *For every non-negative integer s, the closed simply connected 5-manifold*

$$sM_\infty$$

allows a Sasaki–Einstein structure.

By varying the selection of classes in the second integral cohomology groups of del Pezzo orbifolds corresponding to Seifert circle bundles in [22], it becomes possible to craft positive Sasakian structures on non-spin counterparts.

Theorem A. *For every non-negative integer s,* the closed simply connected 5-manifold

$$X_\infty \# sM_\infty$$

admits a positive Sasakian structure.

For cases involving mixed types, Theorem 1.1 (1)(a) and (6) proposes the following assertion, which has been corroborated in studies such as [8, 10, 20, 23], and [17]. Due to the advancement

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in K-stability theory, it is now feasible to establish this assertion for $2 \leqslant k \leqslant 7$ in a straightforward and unified manner (see Section 6).

Theorem 1.4. *For $0 \leqslant k \leqslant 7$ and $m \geqslant 2$, the closed simply connected spin 5-manifold*

$$(k+1)M_\infty \# M_m$$

permits a Sasaki–Einstein structure.

For the corresponding non-spin cases, the following assertion will be verified in this paper.

Theorem B. *For $0 \leqslant k \leqslant 7$ and $m \geqslant 2$, the closed simply connected 5-manifold*

$$X_\infty \# kM_\infty \# M_m$$

allows a positive Sasakian structure.

In cases with small second Betti numbers, one can find a few of closed simply connected Sasaki–Einstein 5-manifolds that involve $2M_4$, $3M_3$, or $2M_3$.

Theorem C. *The closed simply connected spin 5-manifolds*

$$M_\infty \# 2M_4, \quad M_\infty \# 3M_3, \quad M_\infty \# 2M_3, \quad 2M_\infty \# 2M_3$$

admit Sasaki–Einstein structures.

The assertion has been verified for the first three 5-manifolds in [10, Theorem A] and [23, Theorem 7]. In this paper, we prove that it is also true for the 5-manifold $2M_\infty \# 2M_3$. In fact, the first two 5-manifolds are the only spin cases involving $2M_4$ or $3M_3$ with positive second Betti numbers (see Theorem 1.1). It is also expected that the last two 5-manifolds are the only spin cases involving $2M_3$ with positive second Betti numbers (see Conjecture 1.6).

We also prove existence of a positive Sasakian structure on the corresponding non-spin cases.

Theorem D. *The closed simply connected 5-manifolds*

$$X_\infty \# 2M_4, \quad X_\infty \# 3M_3, \quad X_\infty \# 2M_3, \quad X_\infty \# M_\infty \# 2M_3$$

accommodates a positive Sasakian structure.

As in the spin cases, the first two 5-manifolds are the only non-spin cases involving $2M_4$ or $3M_3$ with positive second Betti numbers.

The most intriguing aspect of studying positive Sasakian 5-manifolds lies in the cases where these manifolds carry the series nM_2. From the viewpoint of Sasaki–Einstein structures, the challenge arises from the scarcity of high-genus curves along the boundaries of log del Pezzo surfaces with orbifold Kähler–Einstein metrics, presenting a significant obstacle in addressing the following conjecture at hand.

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Conjecture 1.5. *For integers* $0 \leqslant r \leqslant 8$ *and* $n \geqslant 1$*, the closed simply connected spin 5-manifold*

$$rM_\infty \# nM_2$$

allows a Sasaki–Einstein structure.

The conjecture has been verified for $r = 0, 1, 2$ [18, 20, 33]. Currently, there is no insight into verifying the conjecture for larger values of r. However, from the view point of positive Sasakian structures, we obtain the following affirmative assertion.

Theorem E. *For integers* $r \geqslant 0$*,* $n \geqslant 0$*, the closed simply connected 5-manifolds*

$$rM_\infty \# nM_2, \quad X_\infty \# rM_\infty \# nM_2$$

admit positive Sasakian structures.

The following conjecture represents a significant step towards completing the comprehensive catalog of closed simply connected 5-manifolds capable of hosting Sasaki–Einstein structures, in conjunction with Conjecture 1.5. Moreover, it advances towards the exhaustive list of closed simply connected 5-manifolds suitable for positive Sasakian structures. In essence, the conjecture posits that no further closed simply connected 5-manifolds beyond those already listed thus far are capable of supporting positive Sasakian structures.

Conjecture 1.6. *Let* M *be a closed simply connected 5-manifold that allows a positive Sasakian structure.*

(1) *If* $\mathrm{H}_2(M, \mathbb{Z})_{\mathrm{tor}} = (\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^{\oplus 2}$*, then* $\mathrm{rank}\,(\mathrm{H}_2(M, \mathbb{Z})) \leqslant 2$.
(2) *If* $\mathrm{rank}\,(\mathrm{H}_2(M, \mathbb{Z})) \geqslant 9$*, then*

$$\mathrm{H}_2(M, \mathbb{Z})_{\mathrm{tor}} = (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})^{\oplus n}$$

for some non-negative integer n. If in addition M admits a Sasaki–Einstein structure, then the torsion is trivial, that is, $n = 0$.

2 | LOG DEL PEZZO SURFACES

In this section, we construct log del Pezzo surfaces that yield orbifolds with required Seifert circle bundles.

2.1 | Surface $T_s^\flat$

Let $\bar{C}_1^\flat$ and $\bar{C}_2^\flat$ be general curves of bidegrees (0,1) and (2,1), respectively, in $\mathbb{P}^1 \times \mathbb{P}^1$. For an integer $s \geqslant 5$, choose s distinct points $q_1, \ldots, q_s$ in $\bar{C}_2^\flat \setminus \bar{C}_1^\flat$. Let $T_s^\flat$ be the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at these s points. Denote by $C_1^\flat$ and $C_2^\flat$ the proper transforms of $\bar{C}_1^\flat$ and $\bar{C}_2^\flat$, respectively. Let $G_i^\flat$ be the exceptional

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curve over the point q_i. Choose two distinct odd prime numbers $g_1^\flat$ and $g_2^\flat$ such that $2g_2^\flat > (s-4)g_1^\flat$. It follows from [22, Lemma 25] that the pair

$$\left(T_s^\flat, \left(1-\frac{1}{g_1^\flat}\right)C_1^\flat + \left(1-\frac{1}{g_2^\flat}\right)C_2^\flat\right) \tag{2.1}$$

is a log del Pezzo surface. In [22], this log del Pezzo surfaces with carefully chosen s points is utilized to construct a Sasaki–Einstein structure on the 5-manifold $(s+1)M_\infty$.

2.2 | Surface S_k

For a non-negative integer $0 \leqslant k \leqslant 7$, let S_k be the blowup of $\mathbb{P}^2$ along distinct $k+1$ points $p_0, p_1, \dots, p_k$ in general position. Denote by A_i the exceptional curve of S_k over the point p_i. Choose a general member D_k in the anticanonical linear system of S_k. It is an elliptic curve. For a given positive integer $m \geqslant 2$, the pair

$$\left(S_k, \left(1-\frac{1}{m}\right)D_k\right) \tag{2.2}$$

is a log del Pezzo surface. The Picard rank of S_k is $k+2$.

2.3 | Surface T^0

Consider a quasi-smooth hypersurface of degree 6 in $\mathbb{P}(1,1,3,5)$. With a quasi-homogeneous coordinate system $[x:y:z:w]$, we may assume that it is given by the quasi-homogeneous equation

$$a_6(x,y) - z^2 + wy = 0,$$

where $a_6(x,y)$ is a homogeneous polynomial of degree 6 in x and y. Denote by T^0 the defined surface. Let C_w be the section of T^0 by the equation $w=0$. It is a smooth hyperelliptic curve of genus 2. The pair

$$\left(T^0, \frac{3}{4}C_w\right) \tag{2.3}$$

is a log del Pezzo surface. The surface T^0 has a single cyclic quotient singularity, which is of type $\frac{1}{5}(1,3)$ at $[0:0:0:1]$. The hyperplane section of T^0 by $y=0$ consists of two irreducible components

$$B_+^0 := (y=0, z-\lambda x^3 = 0),$$

$$B_-^0 := (y=0, z+\lambda x^3 = 0),$$

where λ is a certain non-zero constant. These are non-Cartier divisors non-proportional to each other. They intersect only at the singular point of T^0. Each of the two Weil divisors is a generator of the local divisor class group of T^0 at the singular point.

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The orbifold (2.3) can be regarded as a quasi-smooth hypersurface of degree 24 in $\mathbb{P}(4,4,12,5)$ as orbifolds. The link given by this hypersurface of degree 24 provides a Sasaki–Einstein structure on $M_\infty \# 2M_4$ [10, Theorem A].

2.4 | Surface T^1

Let T^1 be a quasi-smooth hypersurface of degree 8 in $\mathbb{P}(1,1,7,4)$. As before, using the quasi-homogeneous coordinate system $[x : y : z : w]$, we may assume that it is given by the quasi-homogeneous equation

$$a_8(x,y) - w^2 + zy = 0,$$

where $a_8(x,y)$ is a homogeneous polynomial of degree 8 in x and y. Denote by C_z the section of T^1 by the equation $z = 0$. The curve C_z is a smooth hyperelliptic curve of genus 3. The pair

$$\left(T^1, \frac{2}{3}C_z\right) \tag{2.4}$$

is a log del Pezzo surface. It contains a unique singular point at $[0 : 0 : 1 : 0]$, which is a cyclic quotient singularity of type $\frac{1}{7}(1,4)$.

When intersected with the hyperplane $y = 0$, the resulting section splits into two irreducible components:

$$B^1_+ := (y = 0, w - \mu x^4 = 0),$$

$$B^1_- := (y = 0, w + \mu x^4 = 0),$$

where μ represents a non-zero constant. These components only intersect at the singular point of T^1. They serve as a generator of the local divisor class group of T^1 at the singular point.

The orbifold (2.4) can be viewed as a quasi-smooth hypersurface of degree 24 within $\mathbb{P}(3,3,7,12)$ as orbifolds. Its link establishes a Sasaki–Einstein structure on $M_\infty \# 3M_3$ [10, Theorem A].

2.5 | Surface T^2

Let T^2 be the Hirzebruch surface $\mathbb{F}_2$. Denote by E_2 and F_2 the negative section and a fibre, respectively. We choose a general member Z_2 in the linear system $|2E_2 + 5F_2|$. It is a smooth curve of genus 2. The pair

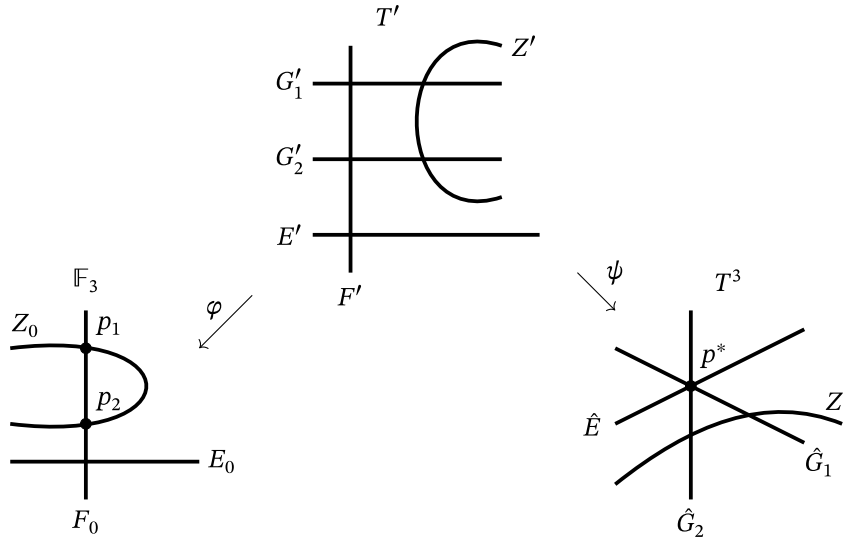
$$\left(T^2, \frac{2}{3}Z_2 + \frac{1}{2}E_2\right) \tag{2.5}$$

is a log del Pezzo surface.

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2.6 | Surface T^3

Consider the Hirzebruch surface $\mathbb{F}_3$ with the negative section E_0 and a fibre F_0. Take a smooth curve $Z_0 \in |2E_0 + 6F_0|$ such that Z_0 and F_0 intersect transversally at two points p_1, p_2. The genus formula gives that $g(Z_0) = 2$. Let $\varphi : T' \to \mathbb{F}_3$ be the blowup at p_1 and p_2 with the exceptional curve G_1' over p_1 and the exceptional curve G_2' over p_2. Let F', E', Z' be the proper transforms of F_0, E_0, Z_0 by φ. Since $F'^2 = -2$, we obtain a birational morphism $\psi : T' \to T^3$ by contracting F'. The surface T^3 has only one singular point that is a node at $p^* := \psi(F')$. The blowup φ and the contraction ψ can be summarized as follows:



where $\hat{G}_i = \psi(G_i')$ for $i = 1, 2$, $Z = \psi(Z')$, and $\hat{E} = \psi(E')$.

The $\mathbb{Q}$-divisor

$$-\left(K_{T^3} + \frac{2}{3}Z + \frac{1}{2}\hat{E}\right) = \frac{2}{3}\left(\hat{G}_1 + \hat{G}_2\right) + \frac{1}{6}\hat{E}$$

is ample, and hence the pair

$$\left(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E}\right) \tag{2.6}$$

defines a log del Pezzo surface with a single node p^*.

Before proceeding, we verify that $T^3 \setminus \hat{E}$ is simply connected as this property will be needed later.

For each $i = 1, 2$, let U_i' be a tubular neighborhood of G_i' in T'. Each U_i' is homeomorphic to the normal bundle over the smooth rational curve G_i'. We may regard $U_i' \cap F'$ as a fibre of the normal bundle. Then $U_i' \setminus F'$ is homeomorphic to $\mathbb{C}^2$. Meanwhile, $T' \setminus (F' \cup G_1' \cup G_2')$ is isomorphic to $\mathbb{F}_3 \setminus F_0$, and hence it is simply connected. It then follows from Seifert–van Kampen theorem that

$$T' \setminus (F' \cup G_i') = \left(T' \setminus (F' \cup G_1' \cup G_2')\right) \cup \left(U_j' \setminus F'\right),$$

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where $\{i, j\} = \{1, 2\}$ is simply connected. Furthermore, Seifert–van Kampen theorem also implies

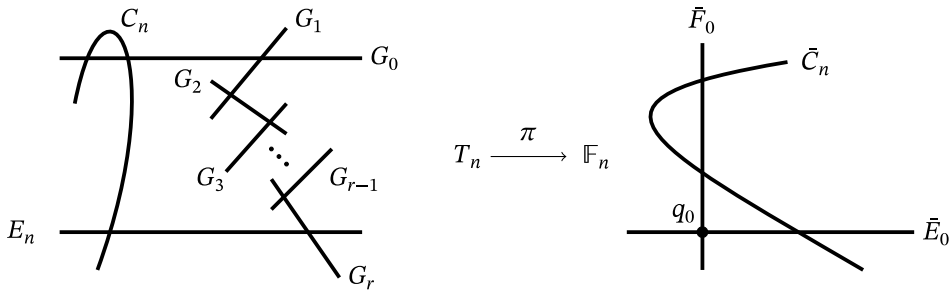
$$T' \setminus F' = \left(T' \setminus (F' \cup G_1')\right) \cup \left(T' \setminus (F' \cup G_2')\right)$$

is simply connected. Therefore, $\pi_1(T^3 \setminus \{p^*\}) = 0$. Since $T' \setminus (E' \cup F' \cup G_1' \cup G_2')$ is also simply connected, the same argument verifies that

$$\pi_1(T' \setminus (E' \cup F')) = \pi_1(T^3 \setminus \hat{E}) = 0.$$

2.7 | Surface T_n

We consider the Hirzebruch surface $\mathbb{F}_n$ with its negative section $\bar{E}_0$. Let $\bar{C}_n$ be a smooth curve on $\mathbb{F}_n$ such that $\bar{C}_n \in |2\bar{E}_0 + (2n+1)\bar{F}_0|$, where $\bar{F}_0$ is a fibre. The curve $\bar{C}_n$ meets the negative section $\bar{E}_0$ at a single point. We choose a point q_0 on $\bar{E}_0$ other than the intersection point of $\bar{E}_0$ and $\bar{C}_n$. We blow up $\mathbb{F}_n$ at q_0. Its exceptional curve meets the proper transform of $\bar{E}_0$ at a single point. We blow up this point. Again, the exceptional curve of the second blowup meets the proper transform of $\bar{E}_0$ at a single point. We repeat this process r times. Let $\pi : T_n \to \mathbb{F}_n$ be the composition of the r blowups. Denote by C_n, E_n, and G_0 the proper transforms of the smooth curve $\bar{C}_n$, the negative section $\bar{E}_0$, and the fibre passing through q_0 by π, respectively. Also denote by G_i the proper transform of the exceptional curve of the ith blowup, $1 \leqslant i \leqslant r$. The Picard rank of T_n is $r + 2$ and the genus of C_n is n.



We choose $(1 + r)$ prime numbers in an arithmetic progression for g_is with $g_0 > g_1 > \cdots > g_r$. We may further assume that g_r is sufficiently large. This choice is guaranteed by Green–Tao Theorem [15]. We consider the pair

$$(T_n, \Delta_n), \tag{2.7}$$

where

$$\Delta_n = \frac{1}{2}C_n + \left(1 - \frac{1}{g_r}\right)E_n + \sum_{i=0}^{r}\left(1 - \frac{1}{g_i}\right)G_i.$$

We have

$$-(K_{T_n} + \Delta_n) \sim_{\mathbb{Q}} \frac{1}{g_r}E_n + \sum_{i=0}^{r}\frac{g_i + 2}{2g_i}G_i.$$

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It then follows that

$$-(K_{T_n}+\Delta_n)\cdot E_n = \frac{g_r+2}{2g_r} - \frac{n+r}{g_r} > 0,$$
$$-(K_{T_n}+\Delta_n)\cdot G_0 = \frac{g_1+2}{2g_1} - \frac{g_0+2}{2g_0} = \frac{1}{g_1} - \frac{1}{g_0} > 0,$$
$$-(K_{T_n}+\Delta_n)\cdot G_r = \frac{1}{g_r} + \frac{g_{r-1}+2}{2g_{r-1}} - \frac{g_r+2}{2g_r} = \frac{1}{g_{r-1}} > 0,$$
$$-(K_{T_n}+\Delta_n)\cdot G_i = \frac{g_{i+1}+2}{2g_{i+1}} + \frac{g_{i-1}+2}{2g_{i-1}} - \frac{g_i+2}{g_i} = \frac{1}{g_{i+1}} + \frac{1}{g_{i-1}} - \frac{2}{g_i} > 0,$$

where $1 \leqslant i \leqslant r-1$.

Consequently, (T_n, Δ_n) is a log del Pezzo surface.

3 | SEIFERT BUNDLES

Let X be a complex surface with cyclic quotient singularities and a $\mathbb{Q}$-divisor $\Delta = \sum\left(1-\frac{1}{m_i}\right)R_i$ with integers $m_i > 1$ such that the pair (X,Δ) is Kawamata log terminal. This describes a complex algebraic orbifold $\mathcal{X} = (X,\Delta)$ (see [19, Section 4]). By [20, Theorem 2.4], there is a one-to-one correspondence between Seifert $\mathbb{C}^*$-bundles $f : Y \to (X,\Delta)$ and the following data:

(1) for each prime divisor R_i an integer $0 \leqslant b_i < m_i$, relatively prime to m_i,
(2) a linear equivalence class of Weil divisors B on X.

Since $\mathbb{C}^* \cong S^1 \times \mathbb{R}$, where S^1 is the unit circle in $\mathbb{C}^*$, the Seifert $\mathbb{C}^*$-bundle Y contains a real hypersurface L with a circle action. We denote $f : L \to (X,\Delta)$ as the corresponding Seifert circle bundle (in short, Seifert bundle).

The first Chern class $c_1(L/X)$ of the Seifert bundle $L \to (X,\Delta)$ is defined as the class

$$B + \sum \frac{b_i}{m_i} R_i$$

in $\mathrm{H}^2(X,\mathbb{Q})$. This $\mathbb{Q}$-divisor lifts to an orbidivisor on $\mathcal{X}$ that defines an integral class in the second orbifold cohomology [6, Proposition 4.4.13 and Theorem 4.3.14]. If the first Chern class is ample, then we may choose a Kälher form ω such that $c_1(L/X) = [\omega]$.

All the Seifert bundles chosen below have their first Chern classes represented by ample divisors on log del Pezzo surfaces presented in Section 2. They all define integral classes in the orbifold cohomologies. It then follows from [6, Theorem 7.5.2 and Proposition 7.5.23] that all the Seifert bundles in the following are able to accommodate positive Sasakian structures.

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3.1 | Seifert bundle $N_s^\flat$

Let $h_s^\flat : N_s^\flat \to \left(T_s^\flat, \left(1-\frac{1}{g_1^\flat}\right)C_1^\flat + \left(1-\frac{1}{g_2^\flat}\right)C_2^\flat\right)$ be the Seifert bundle over the orbifold $\left(T_s^\flat, \left(1-\frac{1}{g_1^\flat}\right)C_1^\flat + \left(1-\frac{1}{g_2^\flat}\right)C_2^\flat\right)$ that corresponds to the $\mathbb{Q}$-divisor

$$\frac{1}{g_1^\flat}C_1^\flat + \frac{1}{g_2^\flat}C_2^\flat$$

and the class given by $C_1^\flat$ in $\mathrm{H}_2(T_s^\flat, \mathbb{Z})$. The first Chern class $c_1(N_s^\flat/T_s^\flat)$ is represented as

$$C_1^\flat + \frac{1}{g_1^\flat}C_1^\flat + \frac{1}{g_2^\flat}C_2^\flat$$

in $\mathrm{H}^2(T_s^\flat, \mathbb{Q})$. Since the divisor $C_1^\flat$ is nef and

$$C_1^\flat + \frac{1}{g_1^\flat}C_1^\flat + \frac{1}{g_2^\flat}C_2^\flat = -\left(K_{T_s^\flat} + \left(1-\frac{1}{g_1^\flat}\right)C_1^\flat + \left(1-\frac{1}{g_2^\flat}\right)C_2^\flat\right) + C_1^\flat,$$

the class $c_1(N_s^\flat/T_s^\flat)$ is represented by an ample $\mathbb{Q}$-divisor.

3.2 | Seifert bundle L_k

We construct the Seifert bundle $f_k : L_k \to \left(S_k, \left(1-\frac{1}{m}\right)D_k\right)$ that corresponds to the $\mathbb{Q}$-divisor

$$\frac{1}{m}D_k$$

and the class H in $\mathrm{H}_2(S_k, \mathbb{Z})$ given by the pull-back of a line on $\mathbb{P}^2$ through the blowup. The first Chern class $c_1(L_k/S_k)$ of the Seifert bundle $f_k : L_k \to \left(S_k, \left(1-\frac{1}{m}\right)D_k\right)$ is defined as the class

$$H + \frac{1}{m}D_k$$

in $\mathrm{H}^2(S_k, \mathbb{Q})$. Since the divisor H is nef, the $\mathbb{Q}$-divisor $H + \frac{1}{m}D_k$ is ample.

3.3 | Seifert bundle N^0

There is a Seifert bundle $h^0 : N^0 \to \left(T^0, \frac{3}{4}C_w\right)$ that corresponds to the $\mathbb{Q}$-divisor

$$\frac{1}{4}C_w$$

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and the class given by $4B^0_- + 3B^0_+$ in $\mathrm{H}_2(T^0, \mathbb{Z})$. Its first Chern class $c_1(N^0/T^0)$ is given by

$$4B^0_- + 3B^0_+ + \frac{1}{4}C_w$$

in $\mathrm{H}^2(T^0, \mathbb{Q})$. Since the Weil divisor B^0_- generates the local divisor class group of T^0 at $[0 : 0 : 0 : 1]$, the Seifert bundle N^0 is smooth [20, Proposition 3.6]. The $\mathbb{Q}$-Cartier divisor $4B^0_- + 3B^0_+$ is nef, and hence the first Chern class $c_1(N^0/T^0)$ is represented by an ample $\mathbb{Q}$-divisor.

3.4 | Seifert bundle N^1

Let $h^1 : N^1 \to \left(T^1, \frac{2}{3}C_z\right)$ be the Seifert bundle over the orbifold $\left(T^1, \frac{2}{3}C_z\right)$ that corresponds to the $\mathbb{Q}$-divisor

$$\frac{1}{3}C_z$$

and the class given by $5B^1_- + 4B^1_+$ in $\mathrm{H}_2(T^1, \mathbb{Z})$. Its first Chern class is represented as

$$5B^1_- + 4B^1_+ + \frac{1}{3}C_z$$

in $\mathrm{H}^2(T^1, \mathbb{Q})$. As in the previous case, the Weil divisor B^1_- generates the local divisor class group of T^1 at $[0 : 0 : 1 : 0]$, and hence the Seifert bundle N^1 is smooth. The $\mathbb{Q}$-Cartier divisor $5B^1_- + 4B^1_+$ is nef, and hence the first Chern class $c_1(N^1/T^1)$ is positive.

3.5 | Seifert bundle N^2

Let $h^2 : N^2 \to \left(T^2, \frac{2}{3}Z_2 + \frac{1}{2}E_2\right)$ denote the Seifert bundle over the orbifold $\left(T^2, \frac{2}{3}Z_2 + \frac{1}{2}E_2\right)$, associated to the $\mathbb{Q}$-divisor

$$\frac{1}{3}Z_2 + \frac{1}{2}E_2$$

and the class given by $-E_2 - 3F_2$ in $\mathrm{H}_2(T^2, \mathbb{Z})$. Its first Chern class is given as

$$-E_2 + \frac{1}{3}Z_2 + \frac{1}{2}E_2$$

in $\mathrm{H}^2(T^2, \mathbb{Q})$. Since F_2 is nef, the first Chern class

$$-E_2 + \frac{1}{3}Z_2 + \frac{1}{2}E_2 = -\left(K_{T^2} + \frac{2}{3}Z_2 + \frac{1}{2}E_2\right) + F_2$$

is represented by an ample $\mathbb{Q}$-divisor.

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3.6 | Seifert bundle N_j^3

Let B^0 (resp. B^1) be the class given by the integral divisor $3\hat{G}_1 + 3\hat{G}_2$ (resp. $4\hat{G}_1 + 3\hat{G}_2$) in $\mathrm{H}_2(T^3, \mathbb{Z})$. Consider the Seifert bundle over the orbifold $\left(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E}\right)$ associated to the $\mathbb{Q}$-divisor

$$\frac{1}{3}Z + \frac{1}{2}\hat{E}$$

and the class given by B^j in $\mathrm{H}_2(T^3, \mathbb{Z})$. Denote by $h_j^3 : N_j^3 \to \left(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E}\right)$ the Seifert bundle. Its first Chern class is

$$B^j + \frac{1}{3}Z + \frac{1}{2}\hat{E}$$

in $\mathrm{H}^2(T^3, \mathbb{Q})$. Since the Weil divisor $\hat{E}$ has self-intersection number $-\frac{5}{2}$ and T^3 has one singular point, $\hat{E}$ generates the local class group of T^3, and hence the Seifert bundle N_j^3 is smooth. We have

$$4\hat{G}_1 + 3\hat{G}_2 + \frac{1}{3}Z + \frac{1}{2}\hat{E} = -7\left(K_{T^3} + \frac{2}{3}Z + \frac{1}{2}\hat{E}\right) + \hat{G}_1$$

and $-6\left(K_{T^3} + \frac{2}{3}Z + \frac{1}{2}\hat{E}\right) + \hat{G}_1$ is nef. Therefore, the first Chern class $c_1(N_j^3/T^3)$ is represented by an ample $\mathbb{Q}$-divisor.

3.7 | Seifert bundle N_j

Let B_0 (resp. B_1) be the class given by the integral divisor $-(E_n + \sum(n+i)G_i)$ (resp. $-(E_n + \sum(n+i-1)G_i)$) in $\mathrm{H}_2(T_n, \mathbb{Z})$. For each $j = 0, 1$, there exists a Seifert bundle $h_j : N_j \to (T, \Delta_n)$ that corresponds to the $\mathbb{Q}$-divisor

$$\frac{1}{2}C_n + \frac{1}{g_r}E_n + \sum_{i=0}^{r}\frac{1}{g_i}G_i$$

and the class B_j in $\mathrm{H}_2(T_n, \mathbb{Z})$. The first Chern class $c_1(N_j/T_n)$ of the Seifert bundle $h_j : N_j \to (T_n, \Delta_n)$ is defined as the class

$$B_j + \frac{1}{2}C_n + \frac{1}{g_r}E_n + \sum_{i=0}^{r}\frac{1}{g_i}G_i$$

in $\mathrm{H}^2(T_n, \mathbb{Q})$. Note that

$$c_1(N_0/T_n) = -(K_{T_n} + \Delta_n), \quad c_1(N_1/T_n) = c_1(N_0/T_n) + \sum G_i.$$

Since $\sum G_i$ is nef, both the first Chern classes are represented by ample $\mathbb{Q}$-divisors.

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4 | TOPOLOGIES OF THE LOG DEL PEZZO SURFACES AND THE SEIFERT CIRCLE BUNDLES

We are now to identify the topological types of the Seifert circle bundles defined in the previous section. They are entirely dictated by their second integral homology groups and their second Stiefel–Whitney classes if they are simply connected.

For a given Seifert bundle $L \to (X, \Delta)$, we obtain an exact sequence

$$\pi_1(\mathbb{C}^*) \to \pi_1(L) \to \pi_1^{\mathrm{orb}}(X, \Delta) \to 0 \tag{4.1}$$

from [20], where $\pi_1^{\mathrm{orb}}(X, \Delta)$ is the orbifold fundamental group of the orbifold (X, Δ).

Claim 4.2. The orbifold fundamental groups of the orbifolds in Section 2 are trivial.

We apply [29, Proposition 3.27] to the group homomorphisms

$$\begin{aligned}
&\pi_1(S_k \setminus D_k) \to \pi_1(S_k),\\
&\pi_1(T_n \setminus (C_n \cup E_n \cup G_0 \cup \cdots \cup G_r)) \to \pi_1(T_n \setminus (E_n \cup G_0 \cup \cdots \cup G_r)),\\
&\pi_1\big(T^2 \setminus (Z_2 \cup E_2)\big) \to \pi_1\big(T^2 \setminus E_2\big),\\
&\pi_1\big(T^3 \setminus (Z \cup \hat{E})\big) \to \pi_1\big(T^3 \setminus \hat{E}\big).
\end{aligned}$$

The surface S_k is simply connected because it is obtained by blowing up $\mathbb{P}^2$. The open subset $T_n \setminus (E_n \cup G_0 \cup \cdots \cup G_r)$ of T_n is simply connected since it is isomorphic to $\mathbb{C}^2$. The subset $T^2 \setminus E_2$ of T^2 is also simply connected because it is isomorphic to $\mathbb{C}^1 \times \mathbb{P}^1$. As explained in Section 2, the open subset $T^3 \setminus \hat{E}$ is simply connected.

It then follows from the positivity $D_k^2 = 8 - k > 0$, $C_n^2 = 4n + 4 > 0$, $Z_2^2 = 12$, $Z^2 = 10$ that $\pi_1(S_k \setminus D_k)$, $\pi_1(T_n \setminus (C_n \cup E_n \cup G_0 \cup \cdots \cup G_r))$, $\pi_1\left(T^2 \setminus (Z_2 \cup E_2)\right)$, and $\pi_1\left(T^3 \setminus (Z \cup \hat{E})\right)$ are abelian.

The orbifold fundamental groups $\pi_1^{\mathrm{orb}}(S_k, \left(1 - \frac{1}{m}\right) D_k)$, $\pi_1^{\mathrm{orb}}(T_n, \Delta_n)$, $\pi_1^{\mathrm{orb}}(T^2, \frac{2}{3}Z_2 + \frac{1}{2}E_2)$, and $\pi_1^{\mathrm{orb}}(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E})$ are given by quotients of the groups $\pi_1(S_k \setminus D_k)$, $\pi_1(T_n \setminus (C_n \cup E_n \cup G_0 \cup \cdots \cup G_r))$, $\pi_1(T^2 \setminus (Z_2 \cup E_2))$, $\pi_1(T^3 \setminus (Z \cup \hat{E}))$, respectively [16], and hence they are also abelian. Therefore, their abelianisations $\mathrm{H}_1^{\mathrm{orb}}(S_k, \left(1 - \frac{1}{m}\right) D_k)$, $\mathrm{H}_1^{\mathrm{orb}}(T_n, \Delta_n)$, $\mathrm{H}_1^{\mathrm{orb}}(T^2, \frac{2}{3}Z_2 + \frac{1}{2}E_2)$, and $\mathrm{H}_1^{\mathrm{orb}}(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E})$ coincide with $\pi_1^{\mathrm{orb}}(S_k, \left(1 - \frac{1}{m}\right) C_k)$, $\pi_1^{\mathrm{orb}}(T_n, \Delta_n)$, $\pi_1^{\mathrm{orb}}(T^2, \frac{2}{3}Z_2 + \frac{1}{2}E_2)$, and $\pi_1^{\mathrm{orb}}(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E})$, respectively.

The surfaces T^2 and $T^3 \setminus \{p^*\}$ are simply connected. Furthermore, both S_k and T_n are also simply connected since they are obtained by successive blowups of simply connected surfaces $\mathbb{P}^2$ and $\mathbb{F}_n$, respectively.

Due to [20, Proposition 4.8], for the triviality of $\pi_1^{\mathrm{orb}}(S_k, \left(1 - \frac{1}{m}\right) D_k)$, $\pi_1^{\mathrm{orb}}(T_n, \Delta_n)$, $\pi_1^{\mathrm{orb}}(T^2, \frac{2}{3}Z_2 + \frac{1}{2}E_2)$, and $\pi_1^{\mathrm{orb}}(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E})$, it is enough to show that the group homomorphism

$$\mathrm{H}^2(S_k, \mathbb{Z}) \to \mathbb{Z}/m\mathbb{Z} \tag{4.3}$$

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given by

$$D \mapsto D \cdot D_k,$$

the group homomorphism

$$\mathrm{H}^2(T_n, \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/g_r\mathbb{Z} \oplus \bigoplus_{i=0}^{r} \mathbb{Z}/g_i\mathbb{Z} \tag{4.4}$$

given by

$$D \mapsto (D \cdot C_n, D \cdot E_n, D \cdot G_0, \dots, D \cdot G_r),$$

the group homomorphism

$$\mathrm{H}^2(T^2, \mathbb{Z}) \to \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \tag{4.5}$$

given by

$$D \mapsto (D \cdot Z_2, D \cdot E_2),$$

and the group homomorphism

$$\mathrm{H}^2(T^3, \mathbb{Z}) \to \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \tag{4.6}$$

given by

$$D \mapsto (D \cdot Z, D \cdot \hat{E})$$

are all surjective. Because $A_k \cdot D_k = 1$, the first group homomorphism is surjective. Since $g_0, \dots, g_r, 2$ are all distinct prime numbers, one can easily generate the finite abelian group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/g_r\mathbb{Z} \oplus \bigoplus_{i=0}^{r} \mathbb{Z}/g_i\mathbb{Z}$ with the images of the second group homomorphism

$$C_n \mapsto (0, 1, 2, 0, \dots, 0), \quad E_n \mapsto (1, -n - r, 0, \dots, 0, 1), \quad \sum_{i=0}^{r} G_i \mapsto (0, 1, 0, 0, 0 \cdots, 0),$$

$$\sum_{i=0}^{k} G_i \mapsto (\underbrace{0, \dots, 0}_{(k+2) \text{ times}}, -1, 1, 0, \dots, 0) \text{ for } k = 0, \dots, r-1.$$

Therefore, the second group homomorphism is also surjective. The third and the last ones are also surjective because $F_2 \cdot Z_2 = (\hat{G}_1 + \hat{G}_2) \cdot Z = 2$ and $F_2 \cdot E_2 = (\hat{G}_1 + \hat{G}_2) \cdot \hat{E} = 1$.

Consequently, $\mathrm{H}_1^{\mathrm{orb}}(S_k, \left(1 - \frac{1}{m}\right) D_k) = 0$, $\mathrm{H}_1^{\mathrm{orb}}(T_n, \Delta_n) = 0$, $\mathrm{H}_1^{\mathrm{orb}}(T^2, \frac{2}{3}Z_2 + \frac{1}{2}E_2) = 0$, and $\mathrm{H}_1^{\mathrm{orb}}(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E}) = 0$.

It follows from [10] that there is a simply connected Seifert circle bundle over each of the orbifolds

$$\left(T^0, \frac{3}{4}C_w\right), \quad \left(T^1, \frac{2}{3}C_w\right).$$

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Therefore, the orbifold fundamental groups $\pi_1^{\mathrm{orb}}\left(T^0, \frac{3}{4}C_w\right)$ and $\pi_1^{\mathrm{orb}}\left(T^1, \frac{2}{3}C_w\right)$ are all trivial by (4.2). The triviality of the orbifold fundamental group $\pi_1^{\mathrm{orb}}\left(T_s^\flat, \left(1-\frac{1}{g_1^\flat}\right)C_1^\flat + \left(1-\frac{1}{g_2^\flat}\right)C_2^\flat\right)$ immediately follows from [22, Lemma 25].

Claim 4.7. The 5-manifolds L_k, N_j, N^0, N^1, N^2, N_j^3, and $N_s^\flat$ are simply connected.

From the exact sequence (4.1) and Claim 4.2, it follows that $\pi_1(L_k)$, $\pi_1(N_j)$, $\pi_1(N^0)$, $\pi_1(N^1)$, $\pi_1(N^2)$, $\pi_1(N_j^3)$, and $\pi_1(N_s^\flat)$ are cyclic groups. According to [9, Theorem A], all our 5-manifolds permit positive Ricci curvatures. Thus, by Bonnet–Myer's Theorem, their fundamental groups are finite cyclic groups. Consequently, $\mathrm{H}_1(L_k,\mathbb{Q})=0$, $\mathrm{H}_1(N_j,\mathbb{Q})=0$, $\mathrm{H}_1(N^0,\mathbb{Q})=0$, $\mathrm{H}_1(N^1,\mathbb{Q})=0$, $\mathrm{H}_1(N^2,\mathbb{Q})=0$, $\mathrm{H}_1(N_j^3,\mathbb{Q})=0$, and $\mathrm{H}_1(N_s^\flat,\mathbb{Q})=0$.

In the Picard group of S_k, the divisor classes given $H, A_0, \dots, A_k$ are linearly independent over $\mathbb{Z}$, where H is the pull-back of a line on $\mathbb{P}^2$ by the blowup. Therefore, the class

$$m\cdot c_1(L_k/S_k) = (m+3)H - (A_0+\cdots+A_k)$$

is not divisible in $\mathrm{Pic}(S_k)$.

Similarly, in the Picard group of T_n, the divisors $E_n, G_0, \dots, G_r$ form a $\mathbb{Z}$-basis. The least common multiple of $2, g_0, \dots, g_r$ is $2g_0g_1\cdots g_r$. Consequently,

$$\begin{aligned}2g_0g_1\cdots g_r\cdot c_1(N_0/T_n) = {}& 2g_0\cdots g_{r-1}E_n + (g_0+2)g_1g_2\cdots g_rG_0 + g_0(g_1+2)g_2\cdots g_rG_1 + \cdots + \\ & + g_0g_1g_2\cdots g_{r-2}(g_{r-1}+2)g_rG_{r-1} + g_0g_1g_2\cdots g_{r-1}(g_r+2)G_r,\end{aligned}$$

$$\begin{aligned}2g_0g_1\cdots g_r\cdot c_1(N_1/T_n) = {}& 2g_0\cdots g_{r-1}E_n + (g_0+4)g_1g_2\cdots g_rG_0 + g_0(g_1+4)g_2\cdots g_rG_1 + \\ & + \cdots + g_0g_1g_2\cdots g_{r-2}(g_{r-1}+4)g_rG_{r-1} + g_0g_1g_2\cdots g_{r-1}(g_r+4)G_r\end{aligned}$$

are primitive in $\mathrm{Pic}(T_n)$.

The Cartier divisors $5B_-^0$ and $5B_+^0$ are $\mathbb{Z}$-independent in $\mathrm{Pic}(T^0)$. Therefore,

$$20\cdot c_1(N^0/T^0) = 80B_-^0 + 60B_+^0 + 5C_w = 105B_-^0 + 85B_+^0$$

is not divisible in $\mathrm{Pic}(T^0)$.

Similarly, the Cartier divisors $7B_-^1$ and $7B_+^1$ are $\mathbb{Z}$-independent in $\mathrm{Pic}(T^1)$. Hence,

$$21\cdot c_1(N^1/T^1) = 21(5B_-^1 + 4B_+^1) + 7C_z = 154B_-^1 + 133B_+^1$$

is not divisible in $\mathrm{Pic}(T^1)$.

For $\mathrm{Pic}(T^2)$, the divisors E_2 and F_2 form a $\mathbb{Z}$-basis, implying that

$$6\cdot c_1(N^2/T^2) = -6E_2 + 2Z_2 + 3E_2 = E_2 + 10F_2$$

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is primitive in $\mathrm{Pic}(T^2)$. In $\mathrm{Pic}(T^3)$, the Cartier divisors $2\hat{G}_1$, $2\hat{G}_2$, and $2\hat{E}$ constitute a $\mathbb{Z}$-basis. Hence,

$$\begin{aligned} 12\cdot c_1(N_0^3/T^3) &= 36(\hat{G}_1+\hat{G}_2)+4Z+6\hat{E}\\ &= 36(\hat{G}_1+\hat{G}_2)+4(2\hat{E}+6(\hat{G}_1+\hat{G}_1))+6\hat{E}\\ &= 60(\hat{G}_1+\hat{G}_2)+14\hat{E},\\ 12\cdot c_1(N_1^3/T^3) &= 12\hat{G}_1+36(\hat{G}_1+\hat{G}_2)+4Z+6\hat{E}\\ &= 48\hat{G}_1+36\hat{G}_2+4(2\hat{E}+6(\hat{G}_1+\hat{G}_1))+6\hat{E}\\ &= 72\hat{G}_1+60\hat{G}_2+14\hat{E} \end{aligned}$$

are not divisible in $\mathrm{Pic}(T^3)$.

In addition, considering the pull-backs $H_1^\flat$ and $H_2^\flat$ of the curves of bidegrees (1,0) and (0,1) on $\mathbb{P}^1\times\mathbb{P}^1$, respectively, via the blowup, along with the exceptional divisors $G_1^\flat,\dots,G_s^\flat$, which form a $\mathbb{Z}$-basis for $\mathrm{Pic}(T_s^\flat)$, we find that

$$\begin{aligned} g_1^\flat g_2^\flat\cdot c_1(N_s^\flat/T_s^\flat) &= g_2^\flat(g_1^\flat+1)C_1^\flat+g_1^\flat C_2^\flat\\ &= 2g_1^\flat H_1^\flat+(g_1^\flat+g_2^\flat+g_1^\flat g_2^\flat)H_2^\flat-g_1^\flat(G_1^\flat+\cdots+G_s^\flat) \end{aligned}$$

is primitive in $\mathrm{Pic}(T_s^*)$.

Consequently, by [20, Theorem 5.7], it follows that $\mathrm{H}^4(L_k,\mathbb{Z})=0$, $\mathrm{H}^4(N_j,\mathbb{Z})=0$, $\mathrm{H}^4(N^0,\mathbb{Z})=0$, $\mathrm{H}^4(N^1,\mathbb{Z})=0$, $\mathrm{H}^4(N^2,\mathbb{Z})=0$, $\mathrm{H}^4(N_j^3,\mathbb{Z})=0$, and $\mathrm{H}^4(N_s^\flat,\mathbb{Z})=0$, thereby verifying Claim 4.7.

Furthermore, [20, Theorem 5.7] implies

$$\begin{aligned} &\mathrm{H}_2(L_k,\mathbb{Z})=\mathbb{Z}^{k+1}\oplus(\mathbb{Z}/m\mathbb{Z})^{\oplus 2},\\ &\mathrm{H}_2(N_j,\mathbb{Z})=\mathbb{Z}^{r+1}\oplus(\mathbb{Z}/2\mathbb{Z})^{\oplus 2n},\\ &\mathrm{H}_2(N^0,\mathbb{Z})=\mathbb{Z}\oplus(\mathbb{Z}/4\mathbb{Z})^{\oplus 4},\\ &\mathrm{H}_2(N^1,\mathbb{Z})=\mathbb{Z}\oplus(\mathbb{Z}/3\mathbb{Z})^{\oplus 6},\\ &\mathrm{H}_2(N^2,\mathbb{Z})=\mathbb{Z}\oplus(\mathbb{Z}/3\mathbb{Z})^{\oplus 4},\\ &\mathrm{H}_2(N_j^3,\mathbb{Z})=\mathbb{Z}^2\oplus(\mathbb{Z}/3\mathbb{Z})^{\oplus 4},\\ &\mathrm{H}_2(N_s^\flat,\mathbb{Z})=\mathbb{Z}^{s+1}. \end{aligned} \tag{4.8}$$

Claim 4.9. The second Stiefel–Whitney classes of L_k, N_1, N^0, N^1, N_1^3, and $N_s^\flat$ are all non-trivial, while the second Stiefel–Whitney classes of N_0 and N_0^3 are trivial.

In the following, $[*]$ is the class in $\mathrm{H}^2(\,\cdot\,,\mathbb{Z}/2\mathbb{Z})$ represented by the divisor $*$ in the bracket.

By [21, Corollary 37], the computation of the second Stiefel–Whitney class of L_k yields:

$$\begin{aligned} w_2(L_k) &= f_k^*(w_2(S_k)+[C_k]+[H])\\ &= f_k^*([H]), \end{aligned} \tag{4.10}$$

where $f_k^*:\mathrm{H}^2(S_k,\mathbb{Z}/2\mathbb{Z})\to\mathrm{H}^2(L_k,\mathbb{Z}/2\mathbb{Z})$.

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For N_1, we derive its second Stiefel–Whitney class as follows:

$$\begin{aligned} w_2(N_1) &= h_1^*\left(w_2(T_n) + [C_n] + [E_n] + \sum_{i=0}^{r}[G_i] + [B_1] \right) \\ &= h_1^*\left(5[E_n] + 3\sum_{i=0}^{r} i[G_i] + (3n+4)\sum_{i=0}^{r}[G_i] - [E_n] - \sum_{i=0}^{r}(n+i-1)[G_i] \right) \\ &= h_1^*\left(4[E_n] + 2\sum_{i=0}^{r} i[G_i] + (2n+5)\sum_{i=0}^{r}[G_i] \right) \\ &= h_1^*\left(\sum_{i=0}^{r}[G_i] \right), \end{aligned} \tag{4.11}$$

where $h_1^* : \mathrm{H}^2(T_n, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2(N_1, \mathbb{Z}/2\mathbb{Z})$.

The second Stiefel–Whitney class of N^2 is expressed as:

$$\begin{aligned} w_2(N^2) &= h^{2*}\left(w_2(T^2) + [Z_2] \right) \\ &= h^{2*}([F_2]), \end{aligned} \tag{4.12}$$

where $h^{2*} : \mathrm{H}^2(T^2, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2(N^2, \mathbb{Z}/2\mathbb{Z})$.

The second Stiefel–Whitney class of $N_s^\flat$ is determined as follows:

$$\begin{aligned} w_2(N_s^\flat) &= h_s^{\flat *}\left(w_2(T_s^\flat) + [C_1^\flat] + [C_2^\flat] + [C_1^\flat] \right) \\ &= h_s^{\flat *}\left([C_1^\flat] \right), \end{aligned} \tag{4.13}$$

where $h_s^{\flat *} : \mathrm{H}^2(T_s^\flat, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2(N_s^\flat, \mathbb{Z}/2\mathbb{Z})$.

The non-triviality of the second Stiefel–Whitney classes in Equations (4.10), (4.11), (4.12), and (4.13) immediately follows from [28, Proposition 13].

In the case where the orbifolds have singular points, we apply [27, Proposition 7.1], which generalizes [28, Proposition 13]. In the following, we denote by $(T^i)^\circ$ the smooth locus of T^i and put $(N^i)^0 = (h^i)^{-1}((T^i)^\circ)$. We then obtain a Seifert bundle over a smooth orbifold

$$h^i : (N^i)^\circ \to (T^i)^\circ$$

by a restriction.

By [27, (7.2)], the second Stiefel–Whitney class of N^0 can be computed as

$$\begin{aligned} w_2(N^0) &= h^{0*}\left(w_2((T^0)^\circ) + [C_w] + 4[B_-^0] + 3[B_+^0] \right) \\ &= h^{0*}\left([C_w] + [B_+^0] \right) \\ &= h^{0*}\left(2[B_+^0] + [B_-^0] \right) \\ &= h^{0*}\left([B_-^0] \right), \end{aligned} \tag{4.14}$$

where $h^{0*} : \mathrm{H}^2((T^0)^\circ, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2((N^0)^\circ, \mathbb{Z}/2\mathbb{Z})$.

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The second Stiefel–Whitney class of N^1 can be computed as

$$\begin{aligned} w_2(N^1) &= h^{1*}\big(w_2((T^1)^\circ) + [C_z] + 5[B^1_-] + 4[B^1_+]\big) \\ &= h^{1*}\big([B^1_-]\big), \end{aligned} \tag{4.15}$$

where $h^{1*} : \mathrm{H}^2((T^1)^\circ, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2((N^1)^\circ, \mathbb{Z}/2\mathbb{Z})$.

The second Stiefel–Whitney class of N^3 can be computed as

$$\begin{aligned} w_2(N^3) &= h^{3*}\big(w_2((T^3)^\circ) + [Z] + [\hat{E}] + 4[\hat{G}_1] + 3[\hat{G}_2]\big) \\ &= h^{3*}\big([\hat{E}] + [\hat{G}_2]\big), \end{aligned} \tag{4.16}$$

where $h^{3*} : \mathrm{H}^2((T^3)^\circ, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2((N^3)^\circ, \mathbb{Z}/2\mathbb{Z})$.

The second Stiefel–Whitney classes (4.14), (4.15), (4.16) are non-zero by [27, Proposition 7.1].

Meanwhile, [7, Proposition 13] implies that the second Stiefel–Whitney classes of N_0 and N_0^3 are trivial, because the first Chern class $c_1(W_0/T_n)$ (resp. $c_1(W_0^3/T^3)$) is a positive multiple of $-(K_{T_n} + \Delta_n)$ (resp. $-(K_{T^3} + \frac{2}{3}Z + \frac{1}{2}\hat{E})$).

5 | PROOFS OF THEOREMS A, B, D, E

By Claim 4.9 the 5-manifolds L_k, N_1, N^0, N^1, N^2, N_1^3, N_s^* are non-spin but N_0 and N_0^3 are spin. As we have shown, all these 5-manifolds are simply connected and they allow positive Sasakian structures. Therefore, (4.8) implies the diffeomorphisms

$$\begin{aligned} &N_s^* \cong X_\infty \# sM_\infty \quad \text{for } s \geqslant 5, \\ &L_k \cong X_\infty \# kM_\infty \# M_m \quad \text{for } 0 \leqslant k \leqslant 7 \text{ and } m \geqslant 2, \\ &N^0 \cong X_\infty \# 2M_4, \\ &N^1 \cong X_\infty \# 3M_3, \\ &N^2 \cong X_\infty \# 2M_3, \\ &N_0^3 \cong 2M_\infty \# 2M_3, \\ &N_1^3 \cong X_\infty \# M_\infty \# 2M_3, \\ &N_0 \cong (r+1)M_\infty \# nM_2 \quad \text{for } r \geqslant 0 \text{ and } n \geqslant 1, \\ &N_1 \cong X_\infty \# rM_\infty \# nM_2 \quad \text{for } r \geqslant 0 \text{ and } n \geqslant 1. \end{aligned}$$

As shown in [6, Proposition 10.4.4], positive Sasakian structures exist on all the manifolds $X_\infty \# sM_\infty$, $0 \leqslant s \leqslant 7$ by choosing the circle bundles appropriately over smooth del Pezzo surfaces of Picard ranks $(s+2)$. Therefore, the first diffeomorphism implies Theorem A. The other diffeomorphisms complete the proofs of Theorems B, D, and E.

The diffeomorphism $N_0^3 \cong 2M_\infty \# 2M_3$ verifies that $2M_\infty \# 2M_3$ allows a positive Sasakian structure. In the following section, we prove that it also admits a Sasaki–Einstein structure.

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6 | PROOF OF THEOREM C AND REMARKS ON SASAKI–EINSTEIN STRUCTURES

In this section, we make a few remarks on Sasaki–Einstein structures on certain simply connected spin 5-manifolds and provide the proof of Theorem C.

The simply connected spin 5-manifolds $(k+1)M_\infty \# M_m$ have been proved to admit Sasaki–Einstein structures if $0 \leqslant k \leqslant 7$ and $m \geqslant 2$. Indeed, for the case $2 \leqslant k \leqslant 7$ and $m \geqslant 2$, we can provide a simple and unified proof. For the simple proof, we consider the Seifert bundle $f_k^E : L_k^E \to \left(S_k, \left(1-\frac{1}{m}\right)D_k\right)$ that corresponds to the $\mathbb{Q}$-divisor $\frac{1}{m}D_k$ and the class 0 in $\mathrm{H}_2(S_k, \mathbb{Z})$. The first Chern class $c_1(L_k^E/S_k)$ of the Seifert bundle $f_k^E : L_k^E \to \left(S_k, \left(1-\frac{1}{m}\right)D_k\right)$ is defined as the class

$$\frac{1}{m}D_k$$

in $\mathrm{H}^2(S_k, \mathbb{Q})$. Following the same argument as for L_k, we can show that L_k^E is diffeomorphic to $(k+1)M_\infty \# M_m$ and has a positive Sasakian structure. The curve D_k is the only curve whose log discrepancy with respect to (S_k, D_k) is zero. The log discrepancy of D_k on S_k is 1 and

$$\frac{1}{(-K_{S_k})^2}\int_0^1 \mathrm{vol}(-K_{S_k} - vD_k)dv = \frac{1}{3}.$$

Moreover, we have $\delta(S_k) > 1$ for $k > 2$ and $\delta(S_2) = 1$ [34]. It then follows from [1, Proposition 1.66] that $\left(S_k, \left(1-\frac{1}{m}\right)D_k\right)$ is K-stable. Therefore, the orbifold $\left(S_k, \left(1-\frac{1}{m}\right)D_k\right)$ allows an orbifold Kähler–Einstein metric [24, 25]. This implies that L_k^E allows a Sasaki–Einstein metric since the first Chern class $c_1(L_k^E/S_k)$ is a positive multiple of $-\left(K_{S_k} + \left(1-\frac{1}{m}\right)D_k\right)$ [20].

Proof of Theorem C. As aforementioned, the first three 5-manifolds $M_\infty \# 2M_4$, $M_\infty \# 3M_3$, $M_\infty \# 2M_3$ are proved to allow Sasaki–Einstein structures in [10, Theorem A] and [23, Theorem 7].

To complete the proof of Theorem C, we here demonstrate that the Seifert bundle N_0^3, which is diffeomorphic to $2M_\infty \# 2M_3$, allows a Sasaki–Einstein structure. Due to [20, Theorem 1.1], it is enough to prove that the log del Pezzo surface $\left(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E}\right)$ allows an orbifold Kähler–Einstein metric. The existence of a Kähler–Einstein metric is guaranteed by the K-stability of the log pair $\left(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E}\right)$ [24, 25].

Lemma 6.1. *The log del Pezzo surface* $\left(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E}\right)$ *is K-stable.*

Proof of Lemma 6.1. To verify the K-stability of the log pair $\left(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E}\right)$, we utilize local δ-invariant as in [18]. For simplicity, set $\Lambda := \frac{2}{3}Z + \frac{1}{2}\hat{E}$. To estimate its local δ-invariants, for a prime divisor Ω over T^3, we denote by $S_{T^3,\Lambda}(\Omega)$ the value

$$\frac{1}{(K_{T^3} + \Lambda)^2}\int_0^\infty \mathrm{vol}(\phi^*(-K_{T^3} - \Lambda) - t\Omega)dt,$$

where $\phi : \tilde{T}_3 \to T_3$ is a birational morphism such that Ω is a divisor on $\tilde{T}_3$.

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Since $Z \in |2\hat{E} + 5\hat{G}_1 + 5\hat{G}_2|$, we have

$$-(K_{T^3} + \Lambda) \equiv \frac{1}{6}\hat{E} + \frac{2}{3}\hat{G}_1 + \frac{2}{3}\hat{G}_2.$$

For each point p outside $\hat{G}_1 \cup \hat{G}_2$, there is a curve Θ such that $p \in \Theta$ and $\Theta \equiv \hat{G}_1 + \hat{G}_2$. For $0 \leqslant t \leqslant \frac{2}{3}$, the divisor

$$-(K_{T^3} + \Lambda) - t\Theta \equiv \frac{1}{6}\hat{E} + \left(\frac{2}{3} - t\right)(\hat{G}_1 + \hat{G}_2)$$

is big. It is not big for $t > \frac{2}{3}$. For $0 \leqslant t \leqslant \frac{2}{3}$, its Zariski decomposition $P_\Theta(t) + N_\Theta(t)$, where $P_\Theta(t)$ is the nef part and $N_\Theta(t)$ is the negative part, is given as follows:

$$P_\Theta(t) = \begin{cases} \frac{1}{6}\hat{E} + \left(\frac{2}{3} - t\right)(\hat{G}_1 + \hat{G}_2) & \text{for } 0 \leqslant t \leqslant \frac{1}{4} \\ \left(\frac{2}{3} - t\right)\left(\frac{2}{5}\hat{E} + \hat{G}_1 + \hat{G}_2\right) & \text{for } \frac{1}{4} \leqslant t \leqslant \frac{2}{3}, \end{cases}$$

$$N_\Theta(t) = \begin{cases} 0 & \text{for } 0 \leqslant t \leqslant \frac{1}{4} \\ \frac{2}{5}\left(t - \frac{1}{4}\right)\hat{E} & \text{for } \frac{1}{4} \leqslant t \leqslant \frac{2}{3}. \end{cases}$$

Therefore,

$$\mathrm{vol}(-(K_{T^3} + \Lambda) - t\Theta) = (P_\Theta(t))^2 = \begin{cases} \frac{11}{72} - \frac{1}{3}t & \text{for } 0 \leqslant t \leqslant \frac{1}{4} \\ \frac{2}{5}\left(\frac{2}{3} - t\right)^2 & \text{for } \frac{1}{4} \leqslant t \leqslant \frac{2}{3}, \end{cases}$$

and hence

$$S_{T^3,\Lambda}(\Theta) = \frac{72}{11}\int_0^{\frac{2}{3}} \mathrm{vol}(-(K_{T^3} + \Lambda) - t\Theta)dt = \frac{97}{396}. \tag{6.2}$$

We denote by $S(W^\Theta_{\bullet,\bullet}; p)$ the value

$$\frac{2}{(K_{T^3} + \Lambda)^2}\int_0^{\frac{2}{3}} \left\{ (P_\Theta(u) \cdot \Theta) \cdot \mathrm{ord}_p(N_\Theta(u)|_\Theta) + \int_0^\infty \mathrm{vol}(P_\Theta(u)|_\Theta - vp)dv \right\} du.$$

If $p \in \hat{E}$, we have

$$(P_\Theta(u) \cdot \Theta) \cdot \mathrm{ord}_p(N_\Theta(u)|_\Theta) + \int_0^\infty \mathrm{vol}(P_\Theta(u)|_\Theta - vp)dv$$

$$= \begin{cases} \frac{1}{72} & \text{for } 0 \leqslant u \leqslant \frac{1}{4} \\ \frac{1}{25}\left(2u^2 - u + \frac{17}{36}\right) & \text{for } \frac{1}{4} \leqslant u \leqslant \frac{2}{3}. \end{cases}$$

Therefore, $S(W^\Theta_{\bullet,\bullet}; p) = \frac{10}{99}$. The log discrepancy $A_{T^3,\Lambda}(\Theta)$ of the pair (T^3, Λ) along the divisor Θ is 1 and the log discrepancy $A_{\Theta,\Lambda|_\Theta}(p)$ of the pair $(\Theta, \Lambda|_\Theta)$ along the divisor p is $\frac{1}{2}$. We are then able

to conclude from [13, Theorem 4.8 (2) and Corollary 4.9] that

$$\delta_p\left(T^3,\Lambda\right) \geqslant \min\left\{\frac{A_{T^3,\Lambda}(\Theta)}{S_{T^3,\Lambda}(\Theta)}, \frac{A_{\Theta,\Lambda|_\Theta}(p)}{S(W^\Theta_{\bullet,\bullet};p)}\right\} = \min\left\{\frac{1}{97/396}, \frac{1/2}{10/99}\right\} > \frac{66}{65}.$$

If $p \notin \hat{E}$, then

$$\begin{aligned}(P_\Theta(u)\cdot\Theta)\cdot \mathrm{ord}_p(N_\Theta(u)|_\Theta) + \int_0^\infty \mathrm{vol}(P_\Theta(u)|_\Theta - vp)dv \\ = \begin{cases} \frac{1}{72} & \text{for } 0 \leqslant u \leqslant \frac{1}{4} \\ \frac{2}{25}\left(\frac{2}{3}-u\right)^2 & \text{for } \frac{1}{4} \leqslant u \leqslant \frac{2}{3}, \end{cases}\end{aligned}$$

and hence $S(W^\Theta_{\bullet,\bullet};p) = \frac{7}{99}$.

Suppose that Θ does not meet Z tangentially at p. The log discrepancy of the pair $(\Theta, \Lambda|_\Theta)$ along the divisor p is given as follows:

$$A_{\Theta,\Lambda|_\Theta}(p) = \begin{cases} 1 & \text{if } p \notin Z \\ \frac{1}{3} & \text{if } p \in Z. \end{cases}$$

We then obtain

$$\delta_p\left(T^3,\Lambda\right) \geqslant \min\left\{\frac{1}{97/396}, \frac{1/3}{7/99}\right\} > \frac{66}{65}.$$

We now suppose that Z and Θ intersect at p tangentially and $\delta_p\left(T^3,\Lambda\right) < \frac{66}{65}$. There is then an m-basis type divisor D such that $\left(T^3, \Lambda + \frac{66}{65}D\right)$ is not log canonical at p. Write

$$D = aZ + b\Theta + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor whose support contains neither Z nor Θ.

Note that

$$-(K_{T^3}+\Lambda) - tZ \equiv \left(\frac{1}{6}-2t\right)\hat{E} + \left(\frac{2}{3}-5t\right)(\hat{G}_1+\hat{G}_2)$$

is nef and big for $0 \leqslant t \leqslant \frac{1}{12}$, but not big for $t > \frac{1}{12}$. Thus,

$$\begin{aligned} S_{T^3,\Lambda}(Z) &= \frac{1}{(K_{T^3}+\Lambda)^2}\int_0^{\frac{1}{12}} \mathrm{vol}(-(K_{T^3}+\Lambda)-tZ)dt \\ &= \frac{72}{11}\int_0^{\frac{1}{12}} \left(10t^2 - \frac{8}{3}t + \frac{11}{72}\right)dt \\ &= \frac{7}{198}. \end{aligned} \tag{6.3}$$

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Due to [11, Theorem 2.9], it follows from (6.2) and (6.3) that we may assume

$$a < \frac{7}{180}, \quad b < \frac{27}{110}.$$

Let $\pi : \bar{T}^3 \to T^3$ be the blowup at p and $\bar{A}$ be its exceptional curve. Then, we have

$$\begin{aligned}&\pi^*\Big(K_{T^3} + \Lambda + \frac{66}{65}D\Big)\\ &= K_{T^3} + \Big(\frac{2}{3} + \frac{66}{65}a\Big)\bar{Z} + \frac{66}{65}b\bar{\Theta} + \frac{66}{65}\bar{\Omega} + c_{\bar{A}}\bar{A},\end{aligned}$$

where $\bar{Z}$, $\bar{\Theta}$, $\bar{\Omega}$ are the proper transforms of Z, Θ, Ω, and

$$c_{\bar{A}} = \frac{66}{65}(a + b + \mathrm{mult}_p\Omega) - \frac{1}{3}.$$

The pair

$$\Big(\bar{T}^3, \Big(\frac{2}{3} + \frac{66}{65}a\Big)\bar{Z} + \frac{66}{65}b\bar{\Theta} + \frac{66}{65}\bar{\Omega} + c_{\bar{A}}\bar{A}\Big)$$

is not log canonical at some point q on $\bar{A}$. Since

$$\mathrm{mult}_p\Omega \leqslant \Theta \cdot \Omega = \Theta \cdot (D - aZ - b\Theta) = \frac{1}{6} - 2a,$$

we have

$$c_{\bar{A}} \leqslant \frac{66}{65}b + \frac{32}{195} < \frac{31}{75}.$$

For each point q' lying on $\bar{A} \setminus \bar{Z}$, we have

$$\mathrm{mult}_{q'}\Big(\frac{66}{65}\bar{\Omega} + c_{\bar{A}}\bar{A}\Big) < \frac{66}{65}\Big(\frac{1}{6} - 2a\Big) + \frac{31}{75} < 1.$$

Therefore, it follows from [11, Lemma 2.1] that the point q must be the intersection point of $\bar{A}$, $\bar{Z}$, and $\bar{\Theta}$. Since $\frac{66}{65}b \leqslant 1$, the pair

$$\Big(T^3, \Big(\frac{2}{3} + \frac{66}{65}a\Big)\bar{Z} + \bar{\Theta} + \frac{66}{65}\bar{\Omega} + c_{\bar{A}}\bar{A}\Big)$$

is not log canonical at q either. Therefore,

$$1 < \bar{\Theta} \cdot \Big(\Big(\frac{2}{3} + \frac{66}{65}a\Big)\bar{Z} + \frac{66}{65}\bar{\Omega} + c_{\bar{A}}\bar{A}\Big)$$

by [11, Lemma 2.4]. On the other hand, we have

$$\bar{\Theta} \cdot \bar{\Omega} = \Theta \cdot \Omega - \mathrm{mult}_p\Omega = \frac{1}{6} - 2a - \mathrm{mult}_p\Omega.$$

This yields a contradiction

$$1 < \Big(\frac{2}{3} + \frac{66}{65}a\Big) + \frac{66}{65}\Big(\frac{1}{6} - 2a - \mathrm{mult}_p\Omega\Big) + \frac{66}{65}(a + b + \mathrm{mult}_p\Omega) - \frac{1}{3} = \frac{98}{195} + \frac{66}{65}b < \frac{733}{975}.$$

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Consequently,

$$\delta_p(T^3,\Lambda) \geqslant \frac{66}{65}$$

for every point p outside $\hat{G}_1 \cup \hat{G}_2$.

We now consider a point p in $\hat{G}_1 \setminus \hat{G}_2$. The divisor

$$-(K_{T^3}+\Lambda) - t\hat{G}_1 \equiv \frac{1}{6}\hat{E} + \left(\frac{2}{3}-t\right)\hat{G}_1 + \frac{2}{3}\hat{G}_2$$

is big for $0 \leqslant t \leqslant \frac{2}{3}$, but not big for $t > \frac{2}{3}$. Its Zariski decomposition $P_{\hat{G}_1}(t) + N_{\hat{G}_1}(t)$, where $P_{\hat{G}_1}(t)$ is the nef part and $N_{\hat{G}_1}(t)$ is the negative part, is as follows:

$$P_{\hat{G}_1}(t) = \begin{cases} \frac{1}{6}\hat{E} + \left(\frac{2}{3}-t\right)\hat{G}_1 + \frac{2}{3}\hat{G}_2 & \text{for } 0 \leqslant t \leqslant \frac{1}{6} \\ \frac{1}{6}E + \left(\frac{2}{3}-t\right)\hat{G}_1 + \left(\frac{5}{6}-t\right)\hat{G}_2 & \text{for } \frac{1}{6} \leqslant t \leqslant \frac{1}{3} \\ \left(\frac{2}{3}-t\right)\left(\frac{1}{2}\hat{E} + \hat{G}_1 + \frac{3}{2}\hat{G}_2\right) & \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3} \end{cases}$$

$$N_{\hat{G}_1}(t) = \begin{cases} 0 & \text{for } 0 \leqslant t \leqslant \frac{1}{6} \\ \left(t-\frac{1}{6}\right)\hat{G}_2 & \text{for } \frac{1}{6} \leqslant t \leqslant \frac{1}{3} \\ \left(\frac{1}{2}t-\frac{1}{6}\right)\hat{E} + \left(\frac{3}{2}t-\frac{1}{3}\right)\hat{G}_2 & \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3}. \end{cases}$$

Then,

$$\mathrm{vol}\left(-(K_{T^3}+\Lambda) - t\hat{G}_1\right) = \left(P_{\hat{G}_1}(t)\right)^2 = \begin{cases} -\frac{1}{2}t^2 - \frac{1}{6}t + \frac{11}{72} & \text{for } 0 \leqslant t \leqslant \frac{1}{6} \\ -\frac{1}{3}t + \frac{1}{6} & \text{for } \frac{1}{6} \leqslant t \leqslant \frac{1}{3} \\ \frac{1}{2}\left(\frac{2}{3}-t\right)^2 & \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \end{cases}$$

and hence

$$S_{T^3,\Lambda}(\hat{G}_1) = \frac{5}{18}.$$

Denote by $S(W^{\hat{G}_1}_{\bullet,\bullet}; p))$ the value

$$\frac{2}{(K_{T^3}+\Lambda)^2}\int_0^{\frac{2}{3}} \left\{ (P_{\hat{G}_1}(u)\cdot\hat{G}_1)\cdot \mathrm{ord}_p(N_{\hat{G}_1}(u)|_{\hat{G}_1}) + \int_0^{\infty} \mathrm{vol}(P_{\hat{G}_1}(u)|_{\hat{G}_1} - vp)dv \right\} du.$$

Since

$$(P_{\hat{G}_1}(u)\cdot\hat{G}_1)\cdot \mathrm{ord}_p(N_{\hat{G}_1}(u)|_{\hat{G}_1}) + \int_0^{\infty} \mathrm{vol}(P_{\hat{G}_1}(u)|_{\hat{G}_1} - vp)dv$$

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$$= \begin{cases} \frac{1}{8}\left(t+\frac{1}{6}\right)^2 & \text{for } 0 \leqslant t \leqslant \frac{1}{6} \\ \frac{1}{72} & \text{for } \frac{1}{6} \leqslant t \leqslant \frac{1}{3} \\ \frac{1}{8}\left(\frac{2}{3}-t\right)^2 & \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \end{cases}$$

we obtain

$$S(W^{\hat{G}_1}_{\bullet,\bullet}; p) = \frac{3}{44}.$$

The log discrepancy of the pair $(\hat{G}_1, \Lambda|_{\hat{G}_1})$ along the divisor p is given as follows:

$$A_{\hat{G}_1,\Lambda|_{\hat{G}_1}}(p) = \begin{cases} 1 & \text{if } p \notin Z \\ \frac{1}{3} & \text{if } p \in Z. \end{cases}$$

Therefore, it follows from [13, Theorem 4.8 (2) and Corollary 4.9] that

$$\delta_p\left(T^3, \Lambda\right) \geqslant \min\left\{\frac{1}{5/18}, \frac{1/3}{3/44}\right\} > \frac{66}{65}$$

for each point $p \in \hat{G}_1 \setminus \{p^*\}$. We can verify the same inequality for a point in $\hat{G}_2 \setminus \{p^*\}$ verbatim.

The divisor

$$-(K_{T^3} + \Lambda) - t\hat{E} = \left(\frac{1}{6} - t\right)\hat{E} + \frac{2}{3}\hat{G}_1 + \frac{2}{3}\hat{G}_2$$

is nef and big for $0 \leqslant t \leqslant \frac{1}{6}$, but not big for $t > \frac{1}{6}$. Therefore,

$$\begin{aligned} S_{T^3,\Lambda}(\hat{E}) &= \frac{1}{(K_{T^3}+\Lambda)^2}\int_0^{\frac{1}{6}} \mathrm{vol}\left(-(K_{T^3}+\Lambda) - t\hat{E}\right)dt \\ &= \frac{72}{11}\int_0^{\frac{1}{6}} \left(-\frac{5}{2}t^2 - \frac{1}{2}t + \frac{11}{72}\right)dt \\ &= \frac{19}{198}. \end{aligned} \tag{6.4}$$

We suppose $\delta_{p^*}\left(T^3, \Lambda\right) < \frac{66}{65}$. There is then an m-basis type divisor D such that the pair $\left(T^3, \Lambda + \frac{66}{65}D\right)$ is not log canonical at p^*. We write

$$D = e\hat{E} + \Omega,$$

where Ω is an effective $\mathbb{Q}$ whose support does not contain $\hat{E}$. Then, we may assume $e < \frac{191}{1980}$ by [11, Theorem 2.9] and (6.4). Since Z does not pass through p^*, the pair

$$\left(T^3, \frac{1}{2}\hat{E} + \frac{66}{65}D\right)$$

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is not log canonical at p^* either. Since

$$\frac{1}{2}\hat{E} + \frac{66}{65}D = \left(\frac{1}{2} + \frac{66}{65}e\right)\hat{E} + \frac{66}{65}\Omega,$$

$\frac{1}{2} + \frac{66}{65}e < 1$, and p^* is a singularity of type $\frac{1}{2}(1,1)$, we obtain an absurd inequality

$$\frac{1}{2} < \frac{66}{65}\Omega \cdot \hat{E} = \frac{66}{65}(D - e\hat{E}) \cdot \hat{E} = \frac{66}{65}\left(\frac{1}{4} + \frac{5}{2}e\right) < \frac{389}{780}$$

from [11, Lemma 2.5]. Consequently,

$$\delta_{p^*}\left(T^3, \Lambda\right) \geqslant \frac{66}{65}.$$

In summary, we have verified that

$$\delta\left(T^3, \Lambda\right) \geqslant \frac{66}{65},$$

which implies the statement [4]. □

Consequently, the log del Pezzo surface $\left(T^3, \frac{2}{3}Z + \frac{1}{2}\hat{E}\right)$ allows an orbifold Kähler–Einstein metric and it follows that the Seifert bundle N_0^3 allows a Sasaki–Einstein structure. This completes the proof of Theorem C. □

Meanwhile, as stated in Conjecture 1.5, Sasaki–Einstein structures are expected to exist on certain simply connected spin 5-manifolds that carry the series nM_2. However, the positive Sasakian structure given to $(r+1)M_\infty \# nM_2$ by the Seifert bundle $N_0 \to (T_n, \Delta_n)$ cannot be a Sasaki–Einstein structure for the following reason. Consider the divisor

$$D := \frac{1}{g_r}E_n + \sum_{i=0}^{r} \frac{g_i + 2}{2g_i}G_i.$$

The log pair $(T_n, \Delta_n + \lambda D)$ is not log canonical along the curve G_0 if $\lambda > \frac{2}{g_0+2}$. Therefore, the α-invariant of (T_n, Δ_n) is at most $\frac{2}{g_0+2}$. Since we chose a sufficiently large g_r, the α-invariant of the log del Pezzo surface (T_n, Δ_n) is strictly less than $\frac{1}{3}$ [12, Appendix A], and hence its δ-invariant is strictly less than 1 [4]. In other words, the log pair (T_n, Δ_n) is K-unstable [4]. In particular, it cannot accommodate an orbifold Kähler–Einstein metric [24, 25]. The positive Sasakian structure of the Seifert bundle N_0 over (T_n, Δ_n) cannot be Einstein.

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