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Continuous Orbit Equivalence on Self-Similar Graph Actions

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Abstract: For self-similar graph actions, we show that isomorphic inverse semigroups associated to a self-similar graph action are a complete invariant for the continuous orbit equivalence of inverse semigroup actions on infinite path spaces.

Keywords: self-similar graph action; continuous orbit equivalence; inverse semigroup

MSC: 37B10; 46L55; 37A55

1. Introduction

Following the groundbreaking result of Giordano, Putnam, and Skau concerning orbit equivalence on Cantor minimal systems [1], Matsumoto [2] introduced the continuous orbit equivalence of one-sided subshifts of finite type. Recently, Matsumoto's concept has been generalized to many different cases. Among these many generalizations, our interests lie in group actions and inverse semigroup actions on Cantor sets, defined by Li [3] and Cordeiro and Beuter [4], respectively.

Li [3] showed that the continuous orbit equivalence of graphs is equivalent to the continuous orbit equivalence of the actions of groups generated by the edge sets of graphs to the infinite path spaces of graphs. Furthermore, Cordeiro and Beuter [4] showed that the continuous orbit equivalence of graphs is equivalent to the continuous orbit equivalence of the actions of inverse semigroups that are naturally associated with graphs on the infinite path spaces of graphs. Combining these two results for graphs with mild restrictions, it follows that the continuous orbit equivalence of group actions is equivalent to that of inverse semigroup actions. For self-similar group cases, the author obtained positive results for group actions and inverse semigroup actions [5]. Thus, it is natural to ask whether we can extend the aforementioned equivalences between group actions and inverse semigroup actions to self-similar graph actions, which provided the motivation for the present study.

In this paper, we show that for self-similar graph actions and their inverse semigroups, the inverse semigroups are isomorphic if and only if their actions are continuous orbit equivalent (Theorem 2). For relations between group actions and inverse semigroup actions, we show that the transformation groupoid of the group action is a subgroupoid of the groupoid of germs of inverse semigroup actions. However, in reality, there is a significant obstacle to further extensions. We discuss some reasons behind this difficulty and directions for future steps.

2. Self-Similar Graph Actions

We review the properties of self-similar graph actions defined by Exel and Pardo [6]. All the material in this section is taken from works [6–8].

Suppose that $E = (E^0, E^1, d, r)$ is a directed graph, where E^0 is the set of vertices, E^1 is the set of edges, d is the domain map, and r is the range map. A directed graph E is called *finite* if E^0 and E^1 are finite sets. For a natural number n, a path of length n in E is a finite sequence of the form

$$a = a_1 a_2 \dots a_n,$$

where $a_i \in E^1$ and $r(a_i) = d(a_{i+1})$ for every i. The domain and range of a are defined by $d(a) = d(a_1)$ and $r(a) = r(a_n)$, respectively. A vertex $v \in E^0$ is considered as a path of length 0, with $d(v) = r(v) = v$. If a and b are paths such that $r(a) = d(b)$, then we denote the path obtained by juxtaposing a and b by ab. We use $|a|$ to denote the length of a. For every non-negative integer n, we denote the set of paths of length n in E by E^n and define $E^* = \cup_{n \geq 0} E^n$. The set of right-infinite paths of the form $a_1 a_2 \dots$ is denoted by E^ω.

The product topology of a discrete set E is defined on E^ω. The cylinder set $Z(a)$ for each $a \in E^*$ is defined by

$$Z(a) = \{\xi \in E^\omega \colon \xi = x_1 x_2 \cdots \text{ such that } x_1 \dots x_{|a|} = a\}.$$

Then, the collection of all such cylinder sets forms a basis for the product topology on E^ω.

An automorphism of a directed graph E is a bijective map $\tau \colon E^0 \dot\cup E^1 \to E^0 \dot\cup E^1$, such that $\tau(E^i) \subseteq E^i$ for $i = 0, 1$, $d \circ \tau = \tau \circ d$, and $r \circ \tau = \tau \circ r$ on E^1. An action of a group G on a directed graph E is a group homomorphism from G to the group of all automorphisms of E. To simplify the notation, we will adopt the shorthand notation

$$\tau_g(\xi) = g(\xi).$$

A *self-similar graph action* (G, E, ϕ) consists of a directed graph E and an action of a group G on E by automorphisms, with a *one-cocycle* $\phi \colon G \times E^1 \to G$, such that for all $g, h \in G$ and $e \in E^1$,

$$\phi(gh, e) = \phi(g, h(e))\phi(h, e).$$

As Exel and Pardo showed in ([6] Proposition 2.4), the G action and one-cocycle extend to E^* via the inductive formula

$$g(e_1 e_2) = g(e_1)\phi(g, e_1)(e_2)$$

for $e_1, e_2 \in E^1$ with $r(e_1) = d(e_2)$, so that we have

$$g(ab) = g(a)\phi(g, a)(b)$$

for every $a, b \in E^*$ with $r(a) = d(b)$. In addition, the G action induces an action of G on E^ω given by

$$g(x_1 x_2 \cdots) = g(x_1 \cdots x_n)\phi(g, x_1 \cdots x_n)(x_{n+1} \cdots).$$

It is not difficult to obtain the following properties of restrictions. For $g, h \in G$ and $a, b \in E^*$ with $r(a) = d(b)$,

$$\phi(g, ab) = \phi(\phi(g, a), b), \phi(gh, a) = \phi(g, h(a))\phi(h, a), \phi(g, a)^{-1} = \phi(g^{-1}, ga).$$

A self-similar graph action (G, E, ϕ) is called pseudo-free if for any $(g, e) \in G \times E^1$ satisfying $ge = e$ and $\phi(g, e) = 1$, it holds that $g = 1$. We say that the G action on E^ω is topologically free if $\{\xi \in E^\omega \colon g(\xi) \neq \xi\}$ is dense in E^ω for every $1 \neq g \in G$. A path $a = a_1 \cdots a_n$ in E^* is said to have an entry if there is at least one $i \in \{1, \dots, n\}$ such that $r^{-1}(d(a_i))$ has more than one element.

Assumption 1. *In this study, we assume the following,*

1. *Every group is a finitely generated countable group,*

2. *every graph is a connected finite directed graph with the properties that every circuit has an entry and $d^{-1}(v)$ and $r^{-1}(v)$ are nonempty sets for every vertex,*
3. *every finite path has an entry,*
4. *our self-similar graph action (G, X, ϕ) is pseudo-free, and*
5. *the G action on X^ω is topologically free.*

Inverse Semigroups and Groupoids

Suppose that (G, E, ϕ) is a self-similar graph action. An inverse semigroup $S_{G,E}$ of (G, E, ϕ) is defined as follows ([6] §4),

$$S_{G,E} = \{(a, g, b) \in E^* \times G \times E^* \colon r(a) = g(r(b))\} \cup \{0\},$$

with the binary operation defined by

$$(a, g, b)(c, h, d) = \begin{cases} (ag(e), \phi(g, e)h, d) & \text{if } c = be, \\ (a, g\phi(h^{-1}, e)^{-1}, dh^{-1}(e)) & \text{if } b = ce, \\ 0 & \text{otherwise,} \end{cases}$$

and the unitary adjoint operation defined by

$$(a, g, b)^* = (b, g^{-1}, a).$$

Then, $S_{G,E}$ is an inverse semigroup containing zero, whose idempotent semi-lattice is given by

$$\mathcal{E}_{G,E} = \{(a, 1, a) \colon a \in E^*\} \cup \{0\}.$$

The inverse semigroup $S_{G,E}$ has a canonical action on E^ω in terms of partial homeomorphisms: The domain of (a, g, b) is

$$Z(b) = \{b\xi \in E^\omega \colon \xi \in E^\omega\}$$

and

$$(a, g, b) \colon b\xi \mapsto ag(\xi).$$

We note that $0 \in S_{G,E}$ is the empty map.

Now, we briefly describe groupoids of germs of inverse semigroups and groups. Let X be a locally compact Hausdorff space and S an inverse semigroup (or a group) acting on X. Then, consider a set $\Omega = \{(s, x) \in S \times X \colon x \in \mathrm{Dom}(s)\}$, and define an equivalence relation on Ω by $(s, x) \sim (t, y)$ if and only if $x = y$ and s and t coincide on a neighborhood of x. The equivalence class of every $(s, x) \in \Omega$ is called the germ of s at x, denoted by $[s, x]$. The set of every germ is a groupoid, called the groupoid of germs of S on X, whose operations are given by

$$d([s, x]) = x, r([s, x]) = sx, [s, x]^{-1} = [s^*, sx] \text{ (or } [s^{-1}, sx] \text{ when } S \text{ is a group).}$$

The unit space is identified with X. When two germs $[s, x]$ and $[t, y]$ satisfy $x = ty$, their product is defined as

$$[s, x][t, y] = [st, y].$$

A basis for a topology is given by the collection of sets of the form

$$[s, U] = \{[s, x] \colon s \in S, U \subset X \text{ is open}, x \in U \subset \mathrm{Dom}(s)\}.$$

The groupoid of germs of the $S_{G,E}$ action on E^ω is denoted by $C_{G,E}$, and called the *Cuntz–Pimsner groupoid* of (G, E, ϕ).

We recall that an inverse semigroup S has an order relation defined by

$$s \le t \iff s = ts^*s \text{ for all } s,t \in S.$$

An inverse semigroup S is said to be E^*-unitary (or 0-E-unitary in [9]) if for any $s \in S$ and a nonzero idempotent e of S, $e \le s$ implies that s is idempotent ([9] Chapter 9). When S is an E^*-unitary inverse semigroup acting on a topological space, the action is called *topologically free* if for every non-idempotent element $s \in S$, the set of fixed points for s has an empty interior ([10] Proposition 4.4).

Remark 1. *Under our Assumptions, the following properties hold.*

1. *The inverse semigroup $S_{G,E}$ is E^*-unitary, by ([6] Proposition 5.8).*
2. *The $S_{G,E}$-action on E^ω is topologically free by ([6] Corollary 14.13).*
3. *In [6], Exel and Pardo considered a different notion of groupoids of germs; however, fortunately, their definition and the one given above coincide. See ([6] §14) for further details.*
4. *The groupoid of germs $\mathcal{C}_{G,E}$ is Hausdorff by ([6] Proposition 12.1), essentially principal by ([10] Theorem 4.7), and trivially étale and locally compact with a second countable unit space.*

Orbit Equivalence of Self-Similar Graph Actions

We review the notions of orbit equivalence for group actions and inverse semigroup actions. See [3,4] for further details.

We consider two self-similar graph actions, (G, E, ϕ) and (H, F, ψ), satisfying our assumptions, with corresponding right-infinite path spaces E^ω and F^ω and inverse semigroups $S_{G,E}$ and $S_{H,F}$, respectively. For the next definition, we let

$$S_{G,E} * E^\omega = \{(s,\xi) \colon s \in S_{G,E}, \xi \in E^\omega, \xi \in \mathrm{Dom}\,(\mathrm{s})\}.$$

Definition 1 ([4] Definition 8.1)**.** *The partial dynamical systems $(S_{G,E}, E^\omega)$ and $(S_{H,F}, F^\omega)$ are said to be continuously orbit equivalent if there exist a homeomorphism*

$$f \colon E^\omega \to F^\omega$$

and continuous maps

$$a \colon S_{G,E} * E^\omega \to S_{H,F} \text{ and } b \colon S_{H,F} * F^\omega \to S_{G,E}$$

such that

$$f(s(\xi)) = a(s,\xi) f(\xi) \text{ and}$$
$$f^{-1}(t(\eta)) = b(t,\eta) f^{-1}(\eta)$$

for all $s \in S_{G,E}$, $\xi \in \mathrm{Dom}(s)$, $t \in S_{H,F}$, and $\eta \in \mathrm{Dom}(t)$.

Remark 2. *If the semigroup actions on right-infinite path spaces are topologically free, then the continuous maps a and b are uniquely determined. See ([3] Remark 2.7) for a complete explanation.*

Theorem 1 ([4] Theorem 8.15)**.** *Suppose that (G, E, ϕ) and (H, F, ψ) are self-similar graph actions with corresponding partial dynamical systems $(S_{G,E}, E^\omega)$ and $(S_{H,F}, F^\omega)$ and Cuntz–Pimsner groupoids $\mathcal{C}_{G,E}$ and $\mathcal{C}_{H,F}$, respectively. Then, the following statements are equivalent.*

1. *The partial dynamical systems $(S_{G,E}, E^\omega)$ and $(S_{H,F}, F^\omega)$ are continuously orbit equivalent.*
2. *The Cuntz–Pimsner groupoids $\mathcal{C}_{G,E}$ and $\mathcal{C}_{H,F}$ are topologically isomorphic.*
3. *There is a diagonal-preserving isomorphism between the groupoid algebras $C^*(\mathcal{C}_{G,E})$ and $C^*(\mathcal{C}_{H,F})$.*

3. Continuous Orbit Equivalences

In this section, we improve Theorem 1 by showing that isomorphic inverse semigroups of self-similar graph actions are also a complete invariant of the topological orbit equivalence of inverse semigroup actions. For this purpose, we will prove that isomorphic inverse semigroups implies isomorphic Cuntz-Pimsner groupoids (Proposition 1) and that continuous orbit equivalence induces continuous orbit equivalence (Proposition 2).

Recall in [11] that the respective inverse semigroups $S_{G,E}$ and $S_{H,F}$ of the self-similar graph actions (G, E, ϕ) and (H, Y, ψ) are isomorphic if there exists a homeomorphism $f\colon E^{\omega} \to F^{\omega}$, such that

$$f \circ S_{G,E} \circ f^{-1} = \{f \circ \alpha \circ f^{-1} \colon \alpha \in S_{G,E}\} = S_{H,F}.$$

Proposition 1. *Suppose that (G, E, ϕ) and (H, F, ψ) are self-similar graph actions with corresponding inverse semigroups $S_{G,E}$ and $S_{H,F}$ and Cuntz–Pimsner groupoids $C_{G,E}$ and $C_{H,F}$, respectively. If $S_{G,E}$ and $S_{H,F}$ isomorphic, then $C_{G,E}$ and $C_{H,F}$ are isomorphic as topological groupoids.*

Proof. Suppose that $S_{G,E}$ is isomorphic to $S_{H,F}$ and that $f\colon E^{\omega} \to E^{\omega}$ is the homeomorphism defined above. Then, we define $\tilde{f}\colon C_{G,E} \to C_{H,F}$ by

$$[\alpha, \xi] \mapsto [f \circ \alpha \circ f^{-1}, f(\xi)].$$

First, we need to show that $\tilde{f}$ is well defined. If $[\alpha, \xi] = [\beta, \xi]$ holds in $C_{G,E}$, then there exists a neighborhood U of ξ in E^{ω} such that $\alpha|_U = \beta|_U$. We recall that f is a homeomorphism, and so $f(U)$ is a neighborhood of $f(\xi)$. Thus, we have

$$f \circ \alpha \circ f^{-1}|_{f(U)} = f \circ \alpha|_U = f \circ \beta|_U = f \circ \beta \circ f^{-1}|_{f(U)},$$

implying that $[f \circ \alpha \circ f^{-1}, f(\xi)] = [f \circ \beta \circ f^{-1}, f(\xi)]$. Therefore, $\tilde{f}$ is a well-defined map.

Conversely, if we have $[f \circ \alpha \circ f^{-1}, f(\xi)] = [f \circ \beta \circ f^{-1}, f(\xi)]$, then $f(\xi) = f(\eta)$, implying that $\xi = \eta$ as f is a homeomorphism. Therefore, there exists a neighborhood V of $f(\xi) = f(\eta)$ such that

$$f \circ \alpha|_{f^{-1}(V)} = f \circ \alpha \circ f^{-1}|_V = f \circ \beta \circ f^{-1}|_V = f \circ \beta|_{f^{-1}(V)}.$$

Again, f is a homeomorphism, implying that $f^{-1}(V)$ is a neighborhood of ξ and

$$\alpha|_{f^{-1}(V)} = \beta|_{f^{-1}(V)}.$$

Thus, $\tilde{f}$ is an injection.

For every $[\gamma, \zeta] \in C_{H,F}$, it is easy to see that $[f^{-1} \circ \gamma \circ f, f^{-1}(\zeta)] \in C_{G,E}$ and

$$\tilde{f}([f^{-1} \circ \gamma \circ f, f^{-1}(\zeta)]) = [\gamma, \zeta].$$

Therefore, $\tilde{f}$ is a surjection.

If $[\alpha, \xi]$ and $[\beta, \eta]$ are composable in $C_{G,E}$ with

$$\xi = \beta(\eta) \text{ and } [\alpha, \xi] \cdot [\beta, \eta] = [\alpha \circ \beta, \eta],$$

then $f(\xi) = f \circ \beta(\eta)$ implies that $[f \circ \alpha \circ f^{-1}, f(\xi)]$ and $[f \circ \beta \circ f^{-1}, f(\eta)]$ are composable in $C_{H,F}$, and

$$\begin{aligned}\tilde{f}([\alpha, \xi]) \cdot \tilde{f}([\beta, \eta]) &= [f \circ \alpha \circ f^{-1}, f(\xi)] \cdot [f \circ \beta \circ f^{-1}, f(\eta)] \\ &= [f \circ \alpha \circ f^{-1} \circ f \circ \beta \circ f^{-1}, f(\eta)] \\ &= [f \circ \alpha \circ \beta \circ f^{-1}, f(\eta)] \\ &= \tilde{f}([\alpha \circ \beta, \eta]) \\ &= \tilde{f}([\alpha, \xi] \cdot [\beta, \eta]).\end{aligned}$$

Therefore, $\tilde{f}$ is a groupoid homomorphism.

To show that $\tilde{f}$ is continuous, let $[\gamma, V]$ be a base element of the germ topology on $C_{H,F}$. Because V is an open set in F^ω, $f^{-1}(V)$ is an open set in E^ω, and

$$\tilde{f}^{-1}([\gamma, V]) = [f^{-1} \circ \gamma \circ f, f^{-1}(V)]$$

is a base element of the germ topology of $C_{G,E}$. Thus, $\tilde{f}$ is a continuous map, and by the same argument so is $\tilde{f}^{-1}$. Therefore, $\tilde{f}$ is a continuous isomorphism, and $C_{G,E}$ and $C_{H,F}$ are isomorphic as topological groupoids. □

Lemma 1. *Every compact open subset of E^ω is a disjoint union of cylinder sets.*

Proof. Let U be a compact open subset of E^ω. Then, the open condition implies that U is a union of cylinder sets, say $U = \cup Z(u_a)$, because the collection of cylinder sets is a base of the topology on E^ω. Therefore, $\{Z(u_a)\}$ is an open covering of a compact set U, and there exists a finite subcover $\{Z(u_1), \ldots, Z(u_n)\}$.

If there exist u_i and u_j such that $u_j = u_i v$ for some $v \in E^*$, then $Z(u_j) \subset Z(u_i)$ implies that we can remove $Z(u_j)$ from the finite subcover $\{Z(u_1), \ldots, Z(u_n)\}$. Therefore, without loss of generality, for any two u_i and u_j we may say that $u_j \neq u_i v$ for any $v \in E^*$. Thus, the finite subcover $\{Z(u_1), \ldots, Z(u_n)\}$ is a disjoint collection. Then $U \subset \cup Z(u_i) \subset \cup Z(u_a) = U$ implies that U is a disjoint union of cylinder sets. □

Lemma 2. *Let (G, E, ϕ) and (H, F, ψ) be self-similar graph actions such that their partial dynamical systems $(S_{G,E}, E^\omega)$ and $(S_{H,F}, F^\omega)$ are continuously orbit equivalent by $f\colon E^\omega \to F^\omega$, $a\colon S_{G,E} * E^\omega \to S_{H,F}$, and $b\colon S_{H,F} * F^\omega \to S_{G,E}$. Then, for every $(u, g, v) \in S_{G,E}$ and $\xi = v\eta \in E^\omega$, there exists a neighborhood U of ξ such that a is a constant map on U.*

Proof. Let $f\colon E^\omega \to F^\omega$, $a\colon S_{G,E} * E^\omega \to S_{H,F}$, and $b\colon S_{H,F} * F^\omega \to S_{G,E}$ be as in Definition 1. For a fixed $(u, g, v) \in S_{G,E}$ and $\xi = v\eta \in E^\omega$, we can find a neighborhood U of ξ such that

$$a((u, g, v), \zeta) = a((u, g, v), \xi)$$

for every $\zeta \in U$.

Because $Z(v)$ is a compact open set in E^ω, $f(Z(v))$ is also a compact open set in F^ω, because f is a homeomorphism. Then, Lemma 1 implies that there exist finitely many disjoint cylinder sets $Z(x_i) \in F^\omega$ such that

$$f(Z(v)) = \cup Z(x_i).$$

Therefore, there exists a unique element $x \in \{x_1 \ldots, x_n\}$ such that

$$f(\xi) = f(v\eta) \in Z(x).$$

On the other hand, we recall that $(u, g, v) \in S_{G,E}$ is a partial homeomorphism on E^ω whose domain is $Z(v)$. Therefore, $f \circ (u, g, v)(Z(v))$ is a compact open set in F^ω, and we can choose finitely many disjoint cylinder sets $Z(y_j) \in F^\omega$, such that

$$f \circ (u, g, v)(Z(v)) = \cup Z(y_j).$$

Let y be the unique element in $\{y_1 \ldots, y_m\}$ such that

$$f \circ (u, g, v)(\xi) = f \circ (u, g, v)(v\eta) = f(ug(\eta)) \in Z(y).$$

Then, continuous orbit equivalence implies that there exists a unique $h \in H$ such that

$$f \circ (u, g, v)(\xi) = a((u, g, v), \xi) f(\xi) = (y, h, x) f(\xi).$$

We define

$$U = f^{-1}(Z(x)) \cap (v, g^{-1}, u) \circ f^{-1} \circ (y, h, x)(Z(x)).$$

It is easy to observe that U is an open set because $Z(x)$ is an open set, f is a homeomorphism and $(u, g, v) = (v, g^{-1}, u)^*$ and (y, h, x) are partial homeomorphisms on their domains. Moreover, $f(\xi) \in Z(x)$ and $f \circ (u, g, v)(\xi) \in (y, h, x)(Z(x))$ imply that U is a neighborhood of ξ.

Now, we show that $a\colon S_{G,E} * E^\omega \to S_{H,F}$ is a constant map on U. For every $\zeta \in U$, we have

$$f(\zeta) \in Z(x) \text{ and } f \circ (u, g, v)(\zeta) = a((u, g, v), \zeta) f(\zeta) \in (y, h, x) Z(x).$$

Thus, we obtain

$$a((u, g, v), \zeta) = (y, h, x) = a((u, g, v), \xi)$$

for every $\zeta \in U$, and a is a constant map on U. □

We present some observations from Lemma 2.

Remark 3.

1. *Fix $(u, g, v) \in S_{G,E}$ and consider any $\xi \in Z(v)$. We let U_ξ be the neighborhood of ξ given in Lemma 2. Then, $\{U_\xi \colon \xi \in Z(v)\}$ is an open cover of $Z(v)$, and the compactness of $Z(v)$ implies that there exists a finite subcover $\{U_1, \ldots, U_n\}$.*

   If $U_i \cap U_j \neq \emptyset$, then for $\zeta \in U_i \cap U_j$, we have

   $$(y_i, h_i, x_i) f(\zeta) = f \circ (u, g, v)(\zeta) = (y_j, h_j, x_j) f(\zeta).$$

   Thus, topological freeness of the action (see Remark 1 and Remark 2) implies

   $$(y_i, h_i, x_i) = (y_j, h_j, x_j)$$

   so that we have $U_i = U_j$ from the construction of U in Lemma 2. Hence, we can remove U_j from the finite cover, and we may say that $\{U_1, \ldots, U_n\}$ is a disjoint collection.
2. *Let $Z(v) = \cup U_i$. For the fixed $(u, g, v) \in S_{G,E}$, we have families $\{(u, g, v)_{U_i}\} \subset S_{G,E}$ and $\{(y_i, h_i, x_i)\} \subset S_{H,F}$ such that for every $\zeta \in U_i$,*

   $$f \circ (u, g, v)(\zeta) = f \circ (u, g, v)_{U_i}(\zeta) = (y_i, h_i, x_i) \circ f(\zeta).$$

Proposition 2. *Let (G, E, ϕ) and (H, F, ψ) be self-similar graph actions with corresponding partial dynamical systems $(S_{G,E}, E^\omega)$ and $(S_{H,F}, F^\omega)$, respectively. If the partial dynamical systems $(S_{G,E}, E^\omega)$ and $(S_{H,F}, F^\omega)$ are continuously orbit equivalent, then the inverse semigroups $S_{G,E}$ and $S_{H,F}$ are isomorphic.*

Proof. Let $f\colon E^\omega \to F^\omega$ be as in Definition 1. Then, we will show that $f \circ S_{G,E} \circ f^{-1} = S_{H,F}$. Instead of a general element (u, g, v) in $S_{G,E}$, we consider $(u, g, v)_{U_i} \in S_{G,E}$ and $(y_i, h_i, x_i) \in S_{H,F}$. Then, Lemma 2 and Remark 3 imply that

$$f \circ (u, g, v)_{U_i} \circ f^{-1} = (y_i, h_i, x_i) \in S_{H,F}.$$

A similar argument shows that $f^{-1} \circ (y, h, x)_{V_j} \circ f = (u_j, g_j, v_j) \in S_{G,E}$. Therefore, we have that $f \circ S_{G,E} \circ f^{-1} = S_{H,F}$, and $S_{G,E}$ is isomorphic to $S_{H,F}$. □

Combining the above results, we obtain a generalization of Theorem 1.

Theorem 2. *Suppose that (G, E, ϕ) and (H, F, ψ) are self-similar graph actions with corresponding partial dynamical systems $(S_{G,E}, E^\omega)$ and $(S_{H,F}, F^\omega)$ and Cuntz–Pimsner groupoids $C_{G,E}$ and $C_{H,F}$, respectively. Then, the following are equivalent.*

1. *$S_{G,E}$ is isomorphic to $S_{H,F}$.*
2. *$(S_{G,E}, E^\omega)$ is continuously orbit equivalent to $(S_{H,F}, F^\omega)$.*
3. *$C_{G,E}$ is topologically isomorphic to $C_{H,F}$.*
4. *$(C^*(C_{G,E}), C(E^\omega))$ is isomorphic to $((C^*(C_{H,F}), C(F^\omega))$.*

A Remark on GROUP actions

We recall that self-similar groups and self-similar graph actions consider group actions on path spaces of graphs. We briefly examine the continuous orbit equivalence of group actions.

For a G action on the path space E^ω, the transformation groupoid is considered rather than the groupoid of germs. The transformation groupoid $T(G, E^\omega)$ of a G action on E^ω is given by the set $G \times E^\omega = \{(g, \xi)\}$, with the multiplication

$$(h, \eta) \cdot (g, \xi) = (hg, \xi) \text{ if } \eta = g(\xi).$$

Definition 2 ([3] Definition 2.5)**.** *The group actions on path spaces (G, E^ω) and (H, F^ω) are said to be continuously orbit equivalent if there exists a homeomorphism $f\colon E^\omega \to F^\omega$ and continuous maps $a\colon G \times E^\omega \to H$ and $b\colon H \times F^\omega \to G$, such that*

$$f(g(\xi)) = a(g, \xi) \circ f(\xi) \text{ and}$$
$$f^{-1}(h(\eta)) = b(h, \eta) \circ f^{-1}(\eta)$$

for all $g \in G, h \in H, \xi \in E^\omega$ and $\eta \in F^\omega$.

Theorem 3 ([3] Theorem 1.2)**.** *Suppose that (G, E, ϕ) and (H, F, ψ) are self-similar graph actions. Then, the following are equivalent.*

1. *The G action on E^ω and H action on F^ω are continuously orbit equivalent.*
2. *The transformation groupoids $T(G, E^\omega)$ and $T(H, F^\omega)$ are topologically isomorphic.*
3. *There exists a $*$-isomorphism $\phi\colon C(E^\omega) \rtimes_r G \to C(F^\omega) \rtimes_r H$ with $\phi(C(E^\omega)) = C(F^\omega)$.*

For a self-similar graph action (G, E), every $g \in G$ may be considered as

$$g = (\emptyset, g, \emptyset^*).$$

Therefore, G is a subsemigroup of the inverse semigroup $S_{G,E}$, and it would be reasonable to expect that some relations exist between group actions and inverse semigroup actions. However, it turns out that it is not easy to obtain any reasonable relations.

In the case of self-similar groups, with the aid of the *recurrent* condition, the continuous orbit equivalence of inverse semigroup actions implies the continuous orbit equivalence of group actions.

Moreover, the converse also holds when shift maps and the homeomorphism on infinite path spaces commute [5].

For self-similar graph actions, it is difficult to define a condition similar to the recurrent condition. One reason for the difficulty is that, although in self-similar groups the initial and terminal vertices of a finite path are the same, in self-similar graph actions they may not be. Although Exel and Pardo [6] introduced related concepts called G-transitivity and weak G-transitivity, these are not direct generalizations of the recurrent condition.

Still, there exists a subgroupoid relation between the transformation groupoid and the groupoid of germs of a G action.

Proposition 3. *Let (G, E) be a self-similar graph action. Then, the transformation groupoid $T(G, E^\omega)$ of the G action on E^ω is isomorphic to a subgroupoid of $C_{G,E}$.*

Proof. For any $(g, \xi) \in T(G, X)$, let a be a prefix of ξ, i.e., $\xi = a\eta$ for some $\eta \in E^\omega$. We define $\alpha\colon T(G, E^\omega) \to C_{G,E}$ by

$$(g, \xi) \mapsto [(g(a), \phi(g, a), a), \xi] \in C_{G,E}.$$

Then, we have $[(g(a), \phi(g, a), a), \xi] \in C_{G,E}$, because

$$|a| = |g(a)| \text{ and } g(\xi) = g(a\eta) = g(a)\phi(g, a)(\eta).$$

First, we need to show that α is well defined. Let b be another prefix of ξ, such that $\xi = b\zeta$ for some $\zeta \in E^\omega$, and show that

$$(g(a), \phi(g, a), a) = (g(b), \phi(g, b), b)$$

on a neighborhood of ξ. Assume that $|a| > |b|$, so that $a = bc$ for some $c \in E^*$. Then,

$$g(a) = g(bc) = g(b)\phi(g, b)(c) \text{ and } \phi(g, bc) = \phi(\phi(g, b), c)$$

implying that for every $\zeta = a\mu = bc\nu \in Z(a)$

$$\begin{aligned}
(g(b), \phi(g, b), b)(bc\nu) &= g(b) \cdot \phi(g, b)(c\nu) \\
&= g(b) \cdot \phi(g, b)(c) \cdot \phi(\phi(g, b), c)(\nu) \\
&= g(bc) \cdot \phi(g, bc)(\nu) \\
&= (g(bc), \phi(g, bc), bc)(bc\nu) \\
&= (g(a), \phi(g, a), a)(a\mu).
\end{aligned}$$

Therefore, we have that $(g(a), \phi(g, a), a) = (g(b), \phi(g, b), b)$ on $Z(a)$, and so

$$[(g(a), \phi(g, a), a), \xi] = [(g(b), \phi(g, b), b), \xi].$$

Thus, α is a well-defined map.

Let $(g, \xi), (h, \eta) \in T(G, E^\omega)$ be composable, implying that

$$\xi = h(\eta) \text{ and } (g, \xi) \cdot (h, \eta) = (gh, \eta).$$

Then, for $\eta = a\zeta$ and some $a \in E^*$ and $\zeta \in E^\omega$, we have

$$\xi = h(\eta) = h(a\zeta) = h(a)\phi(h, a)(\zeta)$$

and

$$\begin{aligned}\alpha((g,\xi)\cdot(h,\eta)) &= \alpha(gh,\eta)\\ &= [(gh(a),\phi(gh,a),a),\eta]\\ &= [(gh(a),\phi(g,h(a))\phi(h,a),a),\eta]\\ &= [(gh(a),\phi(g,h(a)),h(a)),h(a)\phi(h,a)(\zeta)]\cdot[(h(a),\phi(h,a),a),\eta]\\ &= [(g(h(a)),\phi(g,h(a)),h(a)),\xi]\cdot[(h(a),\phi(h,a),a),\eta]\\ &= \alpha(g,\xi)\cdot\alpha(h,\eta).\end{aligned}$$

Thus, α is a groupoid homomorphism.

To show that α is injective, let $\alpha(g,\xi)=\alpha(h,\eta)$, and show that $(g,\xi)=(h,\eta)$. For $\xi=a\mu$ and $\eta=b\nu$,

$$[(g(a),\phi(g,a),a),\xi]=\alpha(g,\xi)=\alpha(h,\eta)=[(h(b),\phi(h,b),b),\eta]$$

induces that $\xi=\eta$. Then, the well-definedness of α implies that we may choose $a=b$ such that for every $\mu=a\nu\in Z(a)$ it holds that

$$\begin{aligned}(g(a),\phi(g,a),a)(\mu) &= g(a)\phi(g,a)(\nu)\\ &= g(\mu)\\ &= h(\mu)\\ &= h(a)\phi(h,a)(\nu)\\ &= (h(a),\phi(h,a),a)(\mu).\end{aligned}$$

Therefore, we have that $g=h$ on $Z(a)$. Because we assumed that the G action on E^ω is topologically free, $g=h$ on $Z(a)$ implies that $g=h$ on E^ω, and α is injective. Therefore, $T(G,E^\omega)$ is isomorphic to a subgroupoid of $C_{G,E}$. □

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