Weak Continuous Characters on Dual Algebras*

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ABSTRACT. In this note we study properties of the set of weak*-continuous characters on a given singly generated dual algebra of operators on Hilbert space. This topic is pertinent to the invariant subspace problem. Our results unify some known examples, and enable us to show that for a certain class of such algebras, the set of such characters is empty.

1. INTRODUCTION

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$ we write, as usual, $\sigma(T)$, $\sigma_p(T)$, and $\sigma_e(T)$ for the spectrum, point spectrum, and essential (Calkin) spectrum of T, respectively, $r(T)$ for the spectral radius of T, and $W(T)$ for the numerical range of T. We denote the kernel and range of T, as usual, by $\ker(T)$ and $\operatorname{ran}(T)$. We also denote by $\mathbb{D}$ the open unit disc $\{\zeta : |\zeta| < 1\}$ in the complex plane $\mathbb{C}$, set $\mathbb{T} := \partial\mathbb{D}$, and write $H^\infty(\mathbb{D})$, as usual, for the Banach algebra of bounded holomorphic functions on $\mathbb{D}$. If $K \neq \emptyset$ is a compact set in $\mathbb{C}$ then the (closed) convex hull of K will be denoted by $\operatorname{conh}(K)$ and the outer boundary of K (i.e., $\partial(\mathbb{C} \setminus K)$), by $\partial^\infty K$. The unbounded component of $\mathbb{C} \setminus K$ will be written as $\operatorname{unbd}(\mathbb{C} \setminus K)$. Furthermore, if T is as above and $\lambda_0 \in \mathbb{C}$, we write $T - \lambda_0$ for the operator $T - \lambda_0 1_{\mathcal{H}}$ when no confusion can result. It is well known (cf. [13, p.40]) that $\mathcal{L}(\mathcal{H})$ is the dual space of the ideal and Banach space $C_1(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ of trace-class operators under the bilinear functional

$$\langle T, L\rangle = \mathrm{tr}(TL), \quad T \in \mathcal{L}(\mathcal{H}),\ L \in C_1(\mathcal{H}). \tag{1.1}$$

A subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ that contains $1_{\mathcal{H}}$ and is closed in the weak* topology on $\mathcal{L}(\mathcal{H})$ is called a *dual algebra*, and the dual algebra generated by a single operator T in $\mathcal{L}(\mathcal{H})$ is denoted by $\mathcal{A}_T$. (Clearly $\mathcal{A}_T$ is the weak* closure of the algebra of all polynomials $p(T)$.) It follows from general principles (cf., e.g., [4]) that if $\mathcal{A}$ is a dual algebra, then $\mathcal{A}$ can be identified with the dual space of the quotient space $Q_{\mathcal{A}} = C_1(\mathcal{H})/{}^{\perp}\mathcal{A}$, where ${}^{\perp}\mathcal{A}$ is the preannihilator of $\mathcal{A}$ in $C_1(\mathcal{H})$, under the pairing

$$\langle T, [L]\rangle = \mathrm{tr}(TL), \quad T \in \mathcal{A},\ [L] \in Q_{\mathcal{A}},$$

where, of course, $[L]$ is the coset in $Q_{\mathcal{A}}$ containing the operator $L \in C_1(\mathcal{H})$. In particular, if x and y are nonzero vectors in $\mathcal{H}$, then the rank-one operator $x \otimes y$, defined by

$$(x \otimes y)(u) = (u, y)x, \quad u \in \mathcal{H},$$

belongs to $C_1(\mathcal{H})$, so $[x \otimes y]$ denotes the image of $x \otimes y$ in the quotient space $Q_{\mathcal{A}}$. For brevity we write Q_T for the predual $Q_{\mathcal{A}_T}$.

Recall that a *weak*-continuous character* on a dual algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is by definition, a multiplicative linear functional $\varphi \in \mathcal{A}^*$ that is weak* continuous and satisfies $\varphi(1_{\mathcal{H}}) = 1$.

The theory of dual algebras, which was begun by Scott Brown in [5], is now about twenty-five years old, and has been very successful in producing new invariant-subspace results and structure theorems for various classes of operators (cf., e.g., [1], [5], [6], [7], [8], [3], [9], and [11]). In particular, one may summarize Brown's brilliant idea that frequently yields a nontrivial invariant subspace for a given operator T in $\mathcal{L}(\mathcal{H})$ in two steps, as follows:

(B_1) Identify $\mathcal{A}_T$ as being isometrically weak*-homeomorphic to some known dual algebra (usually a function algebra), and thereby establish the existence of a weak* continuous character on $\mathcal{A}_T$ (i.e., a nonzero linear functional φ on $\mathcal{A}_T$ such that $\varphi \in Q_T$ and φ is multiplicative on $\mathcal{A}_T$), and

(B_2) Show that the weak* continuous character φ on $\mathcal{A}_T$ from (B_1) has the form $\varphi = [x \otimes y]_{Q_T}$ for some x and y in $\mathcal{H}$ (which is usually done by showing that every $[L] \in Q_T$ has this "rank-one" form).

The reader can easily check, using only the definitions and the fact that

$$\mathrm{tr}([x \otimes y]) = (x, y), \quad x, y \in \mathcal{H},$$

that if a weak* continuous character φ on $\mathcal{A}_T$ has been identified and written as $\varphi = [x \otimes y]_{Q_T}$, then the subspace

$$(\mathcal{A}_T x)^- \ominus (\{\ker \varphi\} x)^- \subset \mathcal{H}$$

is one dimensional, and thus one of the subspaces $(\mathcal{A}_T x)^-$ or $(\{\ker \varphi\}x)^-$ is a nontrivial invariant subspace for T. Hence it would seem that all one has to do is to follow Brown's prescription (B_1) and (B_2) to find a nontrivial invariant subspace for every operator in $\mathcal{L}(\mathcal{H})$! Unfortunately, things are not so simple, because it was seen fairly early in the development of the theory that some singly-generated dual algebras carry *no* weak* continuous character. (The present authors learned of the first such examples some years ago from personal communications with Scott Brown and David Larson.)

The purpose of the present note is to study the properties of the collection of weak* continuous characters on a dual algebra $\mathcal{A}_T$. This enables us to generalize some results of Cassier (Theorem 2.1 (e), (g) and (i) below), who has made a penetrating study of this topic [9], [10], [11] in his study of uniform dual algebras.

2. Some Known Results

In what follows we shall only be concerned with singly generated dual algebras, i.e., those of form $\mathcal{A}_T$ for some T in $\mathcal{L}(\mathcal{H})$. If we denote the maximal ideal space of such a dual algebra by $\operatorname{Max}(\mathcal{A}_T)$, then obviously the set $C_w(\mathcal{A}_T)$ of weak* continuous characters on $\mathcal{A}_T$ can be identified with

$$\operatorname{Max}(\mathcal{A}_T) \cap Q_T,$$

which we call the *weak* character space* of $\mathcal{A}_T$. As noted above, $C_w(\mathcal{A}_T)$ may be empty (for examples, see Section 3), but it is easy to see that in any case, $C_w(\mathcal{A}_T)$ is a weakly closed (and thus norm closed) subset of Q_T. Note that if $\varphi \in C_w(\mathcal{A}_T)$, then φ is completely determined by its value $\varphi(T) = \lambda_\varphi$ (since clearly $\varphi(p(T)) = p(\lambda_\varphi)$ for every polynomial p, and every A in $\mathcal{A}_T$ is a weak* limit of a net $\{p_\mu(T)\}$ of polynomials, from which one gets $\varphi(A) = \lim p_\mu(\lambda_\varphi)$). Thus there is a 1-1 mapping $\varphi \to \varphi(T) = \lambda_\varphi$ of $C_w(\mathcal{A}_T)$ onto the set

$$\sigma^*(T) := \{\lambda_\varphi \in \mathbb{C} \mid \varphi \in C_w(\mathcal{A}_T)\},$$

which was introduced by Cassier [9] and called the *weak spectrum* of T. The definition $\|\lambda_\varphi - \lambda_\psi\| = \|\varphi - \psi\|_{Q_T}$ clearly turns $\sigma^*(T)$ into a complete metric space, even though, as is seen below, $\sigma^*(T)$ need not be closed as a subset of $\mathbb{C}$. There are various relations known between $\sigma^*(T)$ and $\sigma(T)$, and to review some of these, we need a bit more notation. We write $\mathcal{L}(\mathcal{A}_T)$ $[\mathcal{L}(Q_T)]$ for the algebra of all bounded linear operators on the Banach space $\mathcal{A}_T$ $[Q_T]$, $M_T[m_T]$ for the operator in $\mathcal{L}(\mathcal{A}_T)$ $[\mathcal{L}(Q_T)]$ defined by

$$M_T(A) = AT\ (= TA), \quad A \in \mathcal{A}_T,$$
$$[m_T([L]) = [LT]\ (= [TL]), [L] \in Q_T],$$

and $\sigma_{\mathcal{A}_T}(T) = \{\varphi(T) \mid \varphi \in \operatorname{Max}(\mathcal{A}_T)\} \supset \sigma(T)$ for the spectrum of T as an element of the unital Banach algebra $\mathcal{A}_T$. Recall from general principles that

$\partial\sigma_{\mathcal{A}_T}(T) \subset \partial\sigma(T)$, and thus that $\sigma_{\mathcal{A}_T}(T)$ consists of $\sigma(T)$ together with some of its holes (i.e., bounded components of $\mathbb{C} \setminus \sigma(T)$). Note also that for every $0 \neq \lambda \in \mathbb{C}$, $\mathcal{A}_T = \mathcal{A}_{T-\lambda} = \mathcal{A}_{\lambda T}$, so

$$C_w(\mathcal{A}_T) = C_w(\mathcal{A}_{T-\lambda}) = C_w(\mathcal{A}_{\lambda T}).$$

In other words, $C_w(\mathcal{A}_T)$ does not depend on which particular generator for $\mathcal{A}_T$ is singled out, but $\sigma^*(T)$ is related to $\sigma^*(T-\lambda)$ and $\sigma^*(\lambda T)$ as in (d) below.

Parts (a)–(d) of the following theorem are essentially elementary, and parts (e)–(i) were proved by Cassier in the articles cited above.

Theorem 2.1. *For every operator T in $\mathcal{L}(\mathcal{H})$, the following are valid:*

(a) $\sigma_p(T) \cup (\sigma_p(T^*))^* \subset \sigma^*(T)$,
(b) $\lambda \in \sigma^*(T) \iff \{(T - \lambda 1_{\mathcal{H}})\mathcal{A}_T\}^{-w*}$ $(= \{(M_T - \lambda 1_{\mathcal{A}_T})\mathcal{A}_T\}^{-w^*}) \neq \mathcal{A}_T$ $\iff \ker(m_T - \lambda 1_{Q_T}) \neq 0$,
(c) *for every invertible S in $\mathcal{L}(\mathcal{H})$, $\sigma^*(T) = \sigma^*(STS^{-1})$,*
(d) *for every* $0 \neq \lambda \in \mathbb{C}$, $\sigma^*(T-\lambda) = \sigma^*(T) - \lambda$ *and* $\sigma^*(\lambda T) = \lambda\sigma^*(T)$,
(e) $\sigma^*(T) \cap \{\zeta \in \mathbb{C} : |\zeta| = \|T\|\} \subset \sigma_p(T)$,
(f) $\partial\sigma^*(T) \subset \sigma(T)$, *which implies that $\sigma^*(T)$ is a subset of the union of $\sigma(T)$ with its holes (i.e., the polynomial hull of $\sigma(T)$).*
(g) *if $\lambda \in \mathbb{C} \setminus \sigma^*(T)$, then either $T - \lambda$ is not a semi-Fredholm operator or $\lambda \notin \sigma_{\mathcal{A}_T}(T)$,*
(h) *if $\mathcal{J}$ is a simple closed Jordan curve in $\mathbb{C}$ and* $\mathrm{Int}(\mathcal{J})$ *denotes the interior domain of $\mathcal{J}$ given by the Jordan curve theorem, then $\mathcal{J} \subset \sigma^*(T)^\circ \Rightarrow$* $\mathrm{Int}(\mathcal{J}) \subset \sigma^*(T)^\circ$, *and*
(i) *if $\mathcal{A}_T$ is a uniform dual algebra (i.e., the Gelfand map of $\mathcal{A}_T$ into the space* $C(\mathrm{Max}(\mathcal{A}_T))$ *of continuous functions on* $\mathrm{Max}(\mathcal{A}_T)$ *is an isometry), then $\sigma(T) \cup \sigma^*(T) = \sigma_{\mathcal{A}_T}(T)$.*

3. SOME NEW RESULTS

In this section we present some results which we believe to be new and which unify some known examples. Recall that a subspace $\mathcal{M} \subset \mathcal{H}$ is called a semi-invariant subspace for $T \in \mathcal{L}(\mathcal{H})$ if there exist invariant subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ for T with $\mathcal{N}_2 \subset \mathcal{N}_1$ such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2$. Relative to the decomposition $\mathcal{H} = \mathcal{N}_2 \oplus \mathcal{M} \oplus \mathcal{N}_1^\perp$, T has a matrix

$$T = \begin{pmatrix} * & * & * \\ 0 & T_{\mathcal{M}} & * \\ 0 & 0 & * \end{pmatrix},$$

where $T_{\mathcal{M}} \in \mathcal{L}(\mathcal{M})$ is defined by

$$T_{\mathcal{M}}x = P_{\mathcal{M}}Tx, \quad x \in \mathcal{M},$$

(with $P_{\mathcal{M}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$). The map $A \to A_{\mathcal{M}}$ defined on $\mathcal{A}_T$ is clearly a weak* continuous algebra homomorphism of $\mathcal{A}_T$ into $\mathcal{A}_{T_{\mathcal{M}}}$. Thus we obtain, by composing the appropriate maps, the following result.

Proposition 3.1. *If $T \in \mathcal{L}(\mathcal{H})$ and $T_{\mathcal{M}}$ is the compression of T to a semi-invariant subspace $\mathcal{M}$, then $\sigma^*(T_{\mathcal{M}}) \subset \sigma^*(T)$.*

Corollary 3.2. *If $T_1 \oplus T_2 \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, then $\sigma^*(T_1) \cup \sigma^*(T_2) \subset \sigma^*(T_1 \oplus T_2)$, but equality need not hold. However, if $\mathcal{A}_{T_1 \oplus T_2} = \mathcal{A}_{T_1} \oplus \mathcal{A}_{T_2}$ which happens (at least) whenever $\sigma(T_2) \subset \operatorname{unbd}(\mathbb{C} \setminus \sigma(T_1))$ (or, equivalently, $\sigma(T_1) \subset \operatorname{unbd}(\mathbb{C} \setminus \sigma(T_2))$), then $\sigma^*(T_1 \oplus T_2) = \sigma^*(T_1) \cup \sigma^*(T_2)$.*

Proof. By Proposition 3.1, $\sigma^*(T_1) \cup \sigma^*(T_2) \subset \sigma^*(T_1 \oplus T_2)$. To see that equality need not hold, let T_1 be $M_{e^{i\theta}}$ (multiplication by $e^{i\theta}$) on $L^2(\{e^{i\theta} \mid 0 \le \theta \le \pi\}, \mu)$ and T_2 be $M_{e^{i\theta}}$ on $L^2(\{e^{i\theta} \mid \pi \le \theta \le 2\pi\}, \mu)$, where μ is arclength measure on $\mathbb{T} := \partial\mathbb{D}$. Then we have $\sigma^*(T_1) = \sigma^*(T_2) = \emptyset$ (see Proposition 3.13), but $T_1 \oplus T_2$ is the bilateral shift on $L^2(\mathbb{T}, \mu)$, so by Proposition 3.1 and Corollary 3.3 below, $\sigma^*(T_1 \oplus T_2) = \mathbb{D}$. To prove the last statement of the corollary, observe that $\mathcal{A}_{T_1 \oplus T_2} = \mathcal{A}_{T_1} \oplus \mathcal{A}_{T_2}$ whenever the polynomial hulls of $\sigma(T_1)$ and $\sigma(T_2)$ are disjoint (cf., e.g., [14, Th. 4.16]). Now suppose that $\mathcal{A}_{T_1 \oplus T_2} = \mathcal{A}_{T_1} \oplus \mathcal{A}_{T_2}$, and let $\lambda \in \sigma^*(T_1 \oplus T_2)$, so there exists $\varphi_\lambda \in C_w(T_1 \oplus T_2)$ such that $\varphi_\lambda(T_1 \oplus T_2) = \lambda$. We know from Theorem 2.1 (b) that $(T_1 - \lambda)\mathcal{A}_{T_1} \oplus (T_2 - \lambda)\mathcal{A}_{T_2}$ is not weak* dense in $\mathcal{A}_{T_1 \oplus T_2}$, and since $1_{\mathcal{H}} \oplus 0, 0 \oplus 1_{\mathcal{H}} \in \mathcal{A}_{T_1 \oplus T_2}$, it follows easily that $(T_i - \lambda)\mathcal{A}_{T_i}$ is not weak* dense in $\mathcal{A}_{T_i}$ for either $i = 1$ or 2, which shows that $\lambda \in \sigma^*(T_1) \cup \sigma^*(T_2)$ and completes the proof. ◻

The following corollary of Proposition 3.1 has been known for some time, but the proof is new.

Corollary 3.3. *For every absolutely continuous contraction $T \in \mathcal{L}(\mathcal{H})$ such that the Sz.-Nagy-Foias functional calculus $H^\infty(\mathbb{D}) \to \mathcal{A}_T$ is an isometry, $\sigma^*(T) = \mathbb{D}$.*

Proof. In the language of dual algebras, $T \in \mathbb{A} = \mathbb{A}_1$ (cf. [3, Ch. I], [1], [12]), and one knows that for every $\lambda \in \mathbb{D}$, T has a semi-invariant subspace $\mathcal{M}_\lambda \neq (0)$ such that $T_{\mathcal{M}_\lambda} = \lambda \cdot 1_{\mathcal{M}_\lambda}$. Thus by Proposition 3.1, $\mathbb{D} \subset \sigma^*(T)$, and since T is absolutely continuous and $\|T\| = 1$, by Theorem 2.1 (e) and (f), $\sigma^*(T) \cap \mathbb{T} = \emptyset$. ◻

The following also seems to be new.

Theorem 3.4. *Suppose $T \in \mathcal{L}(\mathcal{H})$. Then for every $\lambda \in \mathbb{C} \setminus \sigma_\ell(T)$ (the complement of the left spectrum of T), the following are equivalent:*

(a) $\lambda \in \sigma^*(T)$,
(b) $m_T - \lambda$ $(= m_T - \lambda 1_{Q_T})$ *is a Fredholm operator in* $\mathcal{L}(Q_T)$ *with index* 1,
(c) $M_T - \lambda$ $(= M_T - \lambda 1_{\mathcal{A}_T})$ *is a Fredholm operator in* $\mathcal{L}(\mathcal{A}_T)$ *with index* -1.

Proof. Suppose $\lambda \in \mathbb{C} \setminus \sigma_\ell(T)$, so $T - \lambda$ has a left inverse, say $V(T - \lambda) = 1_{\mathcal{H}}$. This obviously implies that $M_T - \lambda$ is bounded below, which, in turn, gives that

$\ker(M_T - \lambda) = (0)$ and $(M_T - \lambda)$ has closed range. (Thus $M_T - \lambda$ is a semi-Fredholm operator.) We use these facts several times in this proof. Since $(m_T)^* = M_T$, (b) and (c) are clearly equivalent. Next, suppose that $M_T - \lambda$ is a Fredholm operator of index -1. Then $(M_T - \lambda)\mathcal{A}_T \neq \mathcal{A}_T$ (and $\ker(m_T - \lambda) \neq (0)$) so $\lambda \in \sigma^*(T)$ by Theorem 2.1 (b); thus (c) implies (a).

Suppose, finally, that $\lambda \in \sigma^*(T)$, and let $\varphi \in C_w(\mathcal{A}_T)$ be such that $\varphi(T) = \lambda$. Then clearly $\operatorname{ran}(M_T - \lambda) \subset \ker \varphi \neq \mathcal{A}_T$. We show that $M_T - \lambda$ has index -1 by showing that $\widetilde{\mathcal{A}} := \mathbb{C} + \operatorname{ran}(M_T - \lambda) = \mathcal{A}_T$. Clearly $\widetilde{\mathcal{A}}$ is a subalgebra of $\mathcal{A}_T$ containing $1_{\mathcal{H}}$ and T, so it suffices to show that $\widetilde{\mathcal{A}}$ is weak*-closed, or what is the same thing, that the unit ball $(\widetilde{\mathcal{A}})_1$ of $\widetilde{\mathcal{A}}$ is weak*-closed (cf., e.g., [4, Prob. 16X]). But since Q_T is separable, $((\widetilde{\mathcal{A}})_1,\ \text{weak}^*)$ is metrizable (cf., e.g., [4, Prob. 15N]), so it suffices to show that $(\widetilde{\mathcal{A}})_1$ is sequentially weak*-closed. Thus let $\{B_n = \mu_n + (T-\lambda)A_n\}_{n\in\mathbb{N}}$ be a sequence in $(\widetilde{\mathcal{A}})_1$ that is weak*-convergent, say to B_0. Then $\varphi(B_n) = \mu_n \to \varphi(B_0)$, and consequently the sequence $\{(T-\lambda)A_n\}$ is weak*-convergent. Since $A_n = V(T-\lambda)A_n$, this shows that the sequence $\{A_n\}$ is weak*-convergent to some $A_0 \in \mathcal{A}_T$, and thus $B_0 \in \mathcal{A}_T$ also. Thus (a) implies (c) and the proof is complete. □

The following corollary generalizes Theorem 2.1 (g) and (i).

Corollary 3.5. *For every T in $\mathcal{L}(\mathcal{H})$, $\sigma_{\mathcal{A}_T}(T) = \sigma_\ell(T) \cup \sigma^*(T)$.*

Proof. Suppose that $\lambda \in \sigma^*(T) \setminus \sigma_\ell(T)$. Then, by Theorem 3.4, $\lambda \in \sigma(M_T) = \sigma_{\mathcal{A}_T}(T)$, which proves (since obviously $\sigma_\ell(T) \subset \sigma_{\mathcal{A}_T}(T)$) that $\sigma_\ell(T)\cup \sigma^*(T) \subset \sigma_{\mathcal{A}_T}(T)$.

To establish the opposite inclusion, note that

$$\sigma_{\mathcal{A}_T}(T) = (\sigma_{\mathcal{A}_T}(T) \cap \sigma_\ell(T)) \cup (\sigma_{\mathcal{A}_T}(T) \setminus \sigma_\ell(T)),$$

so let us suppose that $\lambda \in \sigma_{\mathcal{A}_T}(T) \setminus \sigma_\ell(T)$. Then, as seen above, $M_T - \lambda$ is bounded below, and $\operatorname{ran}(M_T - \lambda) = \mathcal{A}_T$ is impossible since $\lambda \in \sigma(M_T) = \sigma_{\mathcal{A}_T}(T)$. Thus $\ker(m_T - \lambda) \neq 0$ and $\lambda \in \sigma^*(T)$ by Theorem 2.1 (b). Thus $\sigma_{\mathcal{A}_T}(T) \subset \sigma_\ell(T) \cup \sigma^*(T)$, so the corollary is proved. □

The following contains another new idea.

Theorem 3.6. *Suppose $T \in \mathcal{L}(\mathcal{H})$, $\lambda_0 \in \mathbb{C} \setminus (\sigma_p(T) \cup \sigma_p(T^*)^*)$, and there exist a number $K > 0$ and a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ lying in* $\operatorname{unbd}(\mathbb{C} \setminus \sigma(T))$ *such that $\lambda_n \to \lambda_0$ and*

$$\|(T-\lambda_n)^{-1}\| \leq K/|\lambda_n - \lambda_0|, \quad n \in \mathbb{N}. \tag{3.1}$$

Then $\lambda_0 \notin \sigma^*(T)$.

Proof. Since obviously $C_w(\mathcal{A}_T) = C_w(\mathcal{A}_{T-\lambda_0})$ and $\sigma^*(T) = \sigma^*(T-\lambda_0) + \lambda_0$, we may simply prove that $0 \notin \sigma^*(T-\lambda_0)$. Thus, by the harmless change of notation $T - \lambda_0 \to T$, we have $\lambda_n \to 0$, (3.1) becomes

$$\|(T-\lambda_n)^{-1}\| \le K/|\lambda_n|, \quad n \in \mathbb{N}, \tag{3.2}$$

and the problem is to show that $0 \notin \sigma^*(T)$. Furthermore, since each λ_n lies in $\operatorname{unbd}(\mathbb{C} \setminus \sigma(T))$, one knows that if $0 \notin \sigma(T)$, then, since $\operatorname{unbd}(\mathbb{C} \setminus \sigma(T))$ is open and $\lambda_n \to 0$, $0 \in \operatorname{unbd}(\mathbb{C} \setminus \sigma(T))$. But then Theorem 2.1 (f) gives that $0 \notin \sigma^*(T)$. In other words, we may assume that $0 \in \sigma(T)$. Since $\lambda_n \in \operatorname{unbd}(\mathbb{C} \setminus \sigma(T))$, $(T-\lambda_n)^{-1} \in \mathcal{A}_T$ for $n \in \mathbb{N}$. Thus, by Theorem 2.1 (b), it suffices to show that the sequence $\{T(T-\lambda_n)^{-1}\}_{n\in\mathbb{N}}$ converges weak* to $1_{\mathcal{H}}$. Since

$$T(T-\lambda_n)^{-1} = 1_{\mathcal{H}} + \lambda_n(T-\lambda_n)^{-1}, \quad n \in \mathbb{N},$$

it suffices to show that the sequence $\{\lambda_n(T-\lambda_n)^{-1}\}_{n\in\mathbb{N}}$ converges weak* to zero. Since, by hypothesis,

$$\|\lambda_n(T-\lambda_n)^{-1}\| \le K, \quad n \in \mathbb{N},$$

and the relative weak* and weak operator topologies coincide on bounded subsets of $\mathcal{L}(\mathcal{H})$, it is enough to show that the sequence $\{\lambda_n(T-\lambda_n)^{-1}\}_{n\in\mathbb{N}}$ converges to zero in the weak operator topology. Moreover, by use of the polarization identity, it suffices to show that

$$(\lambda_n(T-\lambda_n)^{-1}x, x) \to 0, \quad x \in \mathcal{H}.$$

Let x_0 be arbitrary in $\mathcal{H}$, and set

$$y_n = \lambda_n(T-\lambda_n)^{-1}x_0, \quad n \in \mathbb{N}. \tag{3.3}$$

Let $\{y_{n_k}\}_{n\in\mathbb{N}}$ be an arbitrary subsequence of $\{y_n\}_{n\in\mathbb{N}}$. One knows that it suffices to show that $\{y_{n_k}\}_{k\in\mathbb{N}}$ has a subsequence that converges weakly to zero. Thus, since $\|y_n\| \le K\|x_0\|$ for $n \in \mathbb{N}$, we can choose a subsequence $\{y_{n_{k_j}}\}_{j\in\mathbb{N}}$ of $\{y_{n_k}\}_{k\in\mathbb{N}}$ that converges weakly—say to y_0. We will complete the proof by showing that $y_0 = 0$. Since, by hypothesis, T is a quasiaffinity, it is enough to show that $Ty_0 = 0$. Moreover, since $T : \mathcal{H} \to \mathcal{H}$ is continuous when both copies of $\mathcal{H}$ are given its weak topology, we know that $\{Ty_{n_{k_j}}\}_{j\in\mathbb{N}}$ converges weakly to Ty_0. For brevity, we write $z_j = y_{n_{k_j}}$ and $\lambda^{(j)} = \lambda_{n_{k_j}}$ for $j \in \mathbb{N}$, and using (3.3) we obtain that

$$(Tz_j, Ty_0) = \lambda^{(j)}(x_0, Ty_0) + \lambda^{(j)}(z_j, Ty_0), \quad j \in \mathbb{N}. \tag{3.4}$$

Since the sequence $\{(Tz_j, Ty_0)\}_{j\in\mathbb{N}}$ converges to $\|Ty_0\|^2$ and $\lambda^{(j)} \to 0$, from (3.4) we get that $(Tz_j, Ty_0) \to 0$, so the proof is complete. ◻

The following lemma is folklore, but the authors were unable to find suitable references.

Lemma 3.7. *For every T in $\mathcal{L}(\mathcal{H})$ the following are valid:*

(a) *for every closed half-plane H in $\mathbb{C}$, H is a spectral set for T if and only if $W(T)^- \subset H$, and*

(b) *for every $\lambda_0 \in \mathbb{C} \setminus \operatorname{conh}(\sigma(T))$, there exists a number $k = k(\lambda_0)$ and a corresponding closed half-plane H in $\mathbb{C}$ such that H is a k-spectral set for T (which implies, of course, that $\operatorname{conh}(\sigma(T)) \subset H$) and such that $\lambda_0 \notin H$. Thus $\operatorname{conh}(\sigma(T))$ is the intersection of all closed half-planes H in $\mathbb{C}$ such that for some $k = k(H)$, H is a k-spectral set for T.*

Proof. To establish (a), suppose first that H is a closed half-plane such that $W(T)^- \subset H$. By a harmless normalization we may suppose that $H = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$. Write $T = L + iK$, where L and K are Hermitian. Then $L \geq 0$, so the Cayley transform $C(T)$, where

$$C(z) = (z-1)(z+1)^{-1}, \quad z \in \mathbb{C} \setminus \{-1\},$$

is an invertible contraction (cf. [17, page 167]). Thus $\mathbb{D}^-$ is a spectral set for $C(T)$ and the function $C^{-1}(z)$ maps $\mathbb{D}^-$ to H, so H is a spectral set for T. The other statement in (a) follows from the fact [16, Th. 2], that a closed convex set $K \subset \mathbb{C}$ contains $W(T)^-$ if and only if

$$\|(T-\lambda)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, K)} = \sup_{z \in K} \left| \frac{1}{z - \lambda} \right|, \quad \lambda \notin K.$$

To establish (b), suppose $\lambda_0 \notin \operatorname{conh}(\sigma(T))$. Let γ be a circle in $\mathbb{C}$ such that $\operatorname{conh}(\sigma(T)) \subset \operatorname{Int}(\gamma)$ and $\lambda_0 \in \mathbb{C} \setminus (\gamma \cup \operatorname{Int}(\gamma))$. By a harmless translation, dilation, and rotation, we may suppose that $\gamma = \mathbb{T}$ and $\lambda_0 > 1$. By Rota's theorem, T is similar to a contraction, and thus $\mathbb{D}^-$ is a k-spectral set for T for some $k \geq 1$. But obviously then, $H = \{\zeta \in \mid \operatorname{Re} \zeta \leq 1\}$ is a k-spectral set for T excluding λ_0. □

Remark 3.8. Note that (b) in the above lemma (which is included here only for completeness), unlike (a), does *not* say that every closed half-plane H in $\mathbb{C}$ such that $\operatorname{conh}(\sigma(T)) \subset H$ is a k-spectral set for T for some k. (In fact, this stronger statement is false for nonzero nilpotent operators on a 2-dimensional space.)

The following is a consequence of Theorem 3.6 and generalizes Theorem 2.1 (e).

Theorem 3.9. *For every T in $\mathcal{L}(\mathcal{H})$, the set $\partial(W(T)) \setminus (\sigma_p(T) \cup \sigma_p(T^*)^*)$ does not intersect $\sigma^*(T)$.*

Proof. Suppose that $\lambda_0 \in \partial(W(T)) \setminus (\sigma_p(T) \cup \sigma_p(T^*)^*)$. By a harmless rotation and translation, it suffices to consider the case $\lambda_0 = 0$ and

$$W(T) \subset H = \{\zeta \in \mathbb{C} \mid \operatorname{Re} \zeta \leq 0\}. \tag{3.5}$$

Since $0 \in \mathbb{C} \setminus (\sigma_p(T) \cup \sigma_p(T^*)^*)$, by Theorem 3.4 it suffices to show that there exists a sequence $\{\lambda_n\} \subset \text{unbd}(\mathbb{C} \setminus \sigma(T))$ such that $\lambda_n \to 0$ and

$$\|(T - \lambda_n)^{-1}\| \leq 1/|\lambda_n|, \quad n \in \mathbb{N}.$$

But this is obvious with $\lambda_n := 1/n$, since H is a spectral set for T by (3.5) and Lemma 3.7 (a). □

Corollary 3.10. *Suppose T is a quasinilpotent quasiaffinity in $\mathcal{L}(\mathcal{H})$ and some closed half-plane H determined by a line through the origin contains the numerical range $W(T)$ of T. Then $C_w(\mathcal{A}_T) = \sigma^*(T) = \emptyset$. (In particular, this applies to the Volterra operator $V \in \mathcal{L}(L^2([0,1]))$ defined by $(Vf)(x) = \int_0^x f(t)\,dt$.)*

Proposition 3.11. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is an absolutely continuous contraction, $\varphi \in C_w(\mathcal{A}_T)$, and $\lambda_\varphi = \varphi(T)$. Then $\lambda_\varphi \in \mathbb{D}$ and for every $f \in H^\infty(\mathbb{D})$, $\varphi(f(T)) = f(\lambda_\varphi)$. (Here $f(T)$ is given by the Sz.-Nagy-Foias functional calculus.)*

Proof. One knows that the sequence $\{T^n\}_{n\in\mathbb{N}}$ converges weak* to zero, and thus $\lambda_\varphi^n \to 0$, so $\lambda_\varphi \in \mathbb{D}$. Moreover, if $f \in H^\infty(\mathbb{D})$, then (since $\mathbb{D}$ is a Carathéodory domain) there is a sequence $\{p_n\}_{n\in\mathbb{N}}$ of polynomials in $H^\infty(\mathbb{D})$ such that $\{p_n\}_{n\in\mathbb{N}}$ converges weak* to f. Thus $\{p_n(T)\}_{n\in\mathbb{N}}$ converges weak* to $f(T)$, and since $\varphi(p_n(T)) = p_n(\lambda_\varphi)$, we get $p_n(\lambda_\varphi) \to \varphi(f(T))$. But point evaluation at λ_φ belongs to $C_w(H^\infty(\mathbb{D}))$ (defined in the obvious way), so $p_n(\lambda_\varphi) \to f(\lambda_\varphi)$ and $\varphi(f(T)) = f(\lambda_\varphi)$. □

Corollary 3.12. *Let T be any C_0-contraction in $\mathcal{L}(\mathcal{H})$ such that the minimal function m of T does not vanish on $\mathbb{D}$ (for definitions and examples, see* [1], [2], *or* [17]*). Then $C_w(\mathcal{A}_T) = \sigma^*(T) = \emptyset$.*

Proof. One knows that T is an absolutely continuous contraction, so if $\varphi \in C_w(\mathcal{A}_T)$, then, by Proposition 3.11, for every $f \in H^\infty(\mathbb{D})$, $\varphi(f(T)) = f(\lambda_\varphi)$ where $\lambda_\varphi \in \mathbb{D}$. But then,

$$m(\lambda_\varphi) = \varphi(m(T)) = \varphi(0) = 0,$$

contradicting the hypothesis. □

For completeness, we include here the following known result.

Proposition 3.13 (Cassier)***.*** *Suppose N is a normal operator in $\mathcal{L}(\mathcal{H})$ without point spectrum such that $N^* \in \mathcal{A}_N$ (which happens, of course, if $\sigma(N)^\circ = \emptyset$ and $\sigma(N)$ doesn't separate the plane). Then $C_w(\mathcal{A}_N) = \sigma^*(N) = \emptyset$.*

Proof. Clearly $\mathcal{A}_N$ is a von Neumann algebra, and since elements of $C_w(\mathcal{A}_N)$ are in one-to-one correspondence with the maximal weak* closed ideals $\mathcal{J}$ in $\mathcal{A}_N$, it suffices to show that there are no such ideals. But such a $\mathcal{J}$ is a von Neumann algebra, and, as such, has an identity $E_\mathcal{J} \neq 1_\mathcal{H}$ which is a maximal projection in $\mathcal{A}_T$. Thus $1_{\mathcal{H}} - E_{\mathcal{J}}$ is a minimal projection in $\mathcal{A}_T$. But, by the spectral theorem, normal operators without point spectrum correspond to measure spaces containing no atoms, so there are no nonzero minimal projections. ◻

4. SOME REMARKS AND OPEN QUESTIONS

In this section we make some remarks and set forth some open problems. The one most pertinent to the invariant subspace problem is the following.

Problem 4.1. If $T \in \mathcal{L}(\mathcal{H})$ and $\sigma(T)$ contains a nonempty open set, must $C_w(\mathcal{A}_T)$ be nonvoid?

Problem 4.2. If T is a completely nonunitary contraction in $\mathcal{L}(\mathcal{H})$ and $\sigma(T)$ contains a nonempty open set, must every $A \in \mathcal{A}_T$ such that $A \neq \lambda 1_{\mathcal{H}}$ satisfy $\sigma(A)^\circ \neq \emptyset$? (In this connection, using a transfinite induction argument, one sees that it is enough to show that every $A \in \mathcal{A}_T$ which is a weak* limit of a sequence $\{p_n(T)\}_{n\in\mathbb{N}}$ of polynomials has this property.)

Problem 4.3. For an operator T in $\mathcal{L}(\mathcal{H})$ with connected spectrum and with $\sigma_p(T) \cup \sigma_p(T^*)^* = \emptyset$, is $\sigma^*(T)$ always either an open set or a closed set? Can $\sigma^*(T)$ of such a T be a circle? (With respect to the first question, we note that without the hypothesis that $\sigma(T)$ is connected, the answer is obviously no by Corollary 3.2, Corollary 3.3, and Remark 4.5.)

Problem 4.4 (D. Larson)**.** Does there exist some T in $\mathcal{L}(\mathcal{H})$ and some $\varphi \in C_w(\mathcal{A}_T)$ such that φ is not continuous in the weak operator topology?

Remark 4.5. It is easy to construct an example of a quasinilpotent bilateral weighted shift B in $\mathcal{L}(\mathcal{H})$ that is a quasiaffinity but $\sigma^*(B) = \{0\}$. (Simply construct B to have a semi-invariant subspace $\mathcal{M} \neq (0)$ such that $0 \in \sigma_p(B_{\mathcal{M}})$ and apply Proposition 3.1.) Thus for quasinilpotent quasiaffinities Q all (both) possible sets $\emptyset$ and $\{0\}$ can occur as $\sigma^*(Q)$.

Remark 4.6. Corollary 3.3 is equivalent to the statement that $C_w(H^\infty(\mathbb{D}))$, suitably defined, consists exactly of the point evaluations at points of $\mathbb{D}$. But it is known that there exists a bounded domain of holomorphy $G \subset \mathbb{C}^n$ such that $C_w(H^\infty(G))$ contains a character φ that is not point evaluation at some point of G ([15]). It is also worth pointing out that if $G \subset \mathbb{C}$ and $\mathcal{R}(G)$, the algebra of rational functions with poles off G is weak* dense in $H^\infty(G)$, then all elements of $C_w(H^\infty(G))$ are point evaluations at points of G.

Acknowledgements. The second and the third authors were supported by a grant from the Korean Research Foundation (KRF 2002–070–C00006). The fourth author appreciates the support of the National Science Foundation. The authors gratefully acknowledge several conversations with Ciprian Foias about the subject matter of this note.

References

[1] Hari Bercovici, *Factorization theorems and the structure of operators on Hilbert space*, Ann. of Math. (2) **128** (1988), 399–413. MR 89i:47032

[2] ______, Operator Theory and Arithmetic in H^∞, Mathematical Surveys and Monographs, vol. 26, American Mathematical Society, Providence, RI, 1988, ISBN 0-8218-1528-8. MR 90e:47001

[3] Hari Bercovici, Ciprian Foias, and Carl Pearcy, Dual Algebras with Applications to Invariant Subspaces and Dilation Theory, CBMS Regional Conference Series in Mathematics, vol. 56, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1985, ISBN 0-8218-0706-4. MR 87g:47091

[4] Arlen Brown and Carl Pearcy, Introduction to Operator Theory. I, Springer-Verlag, New York, 1977, ISBN 0-398-90257-0. MR 58 #23463

[5] Scott W. Brown, *Some invariant subspaces for subnormal operators*, Integral Equations Operator Theory **1** (1978), 310–333. MR 80c:47007

[6] ______, *Hyponormal operators with thick spectra have invariant subspaces*, Ann. of Math. (2) **125** (1987), 93–103. MR 88c:47010

[7] Scott W. Brown, Bernard Chevreau, and Carl Pearcy, *Contractions with rich spectrum have invariant subspaces*, J. Operator Theory **1** (1979), 123–136. MR 80m:47002

[8] ______, *On the structure of contraction operators. II*, J. Funct. Anal. **76** (1988), 30–55. MR 90b:47030b

[9] Gilles Cassier, *Algèbres duales uniformes d'opérateurs sur l'espace de Hilbert*, Studia Math. **95** (1989), 17–32. MR 91a:47060 (French, with English summary)

[10] ______, *Sur la structure d'algèbres duales uniformes d'opérateurs sur l'espace de Hilbert*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), 479–482. MR 91f:47059 (French, with English summary)

[11] ______, *Champs d'algèbres duales et algèbres duales uniformes d'opérateurs sur l'espace de Hilbert*, Studia Math. **106** (1993), 101–119. MR 95b:47054

[12] Bernard Chevreau, *Sur les contractions à calcul fonctionnel isométrique. II*, J. Operator Theory **20** (1988), 269–293. MR 90f:47021 (French)

[13] Jacques Dixmier, Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann), Cahiers scientifiques, Fascicule XXV, Gauthier-Villars, Paris, 1957. MR 20 #1234 (French)

[14] Heydar Radjavi and Peter Rosenthal, Invariant Subspaces, Springer-Verlag, New York, 1973. MR 51 #3924

[15] Nessim Sibony, *Prolongement analytique des fonctions holomorphes bornées*, C. R. Acad. Sci. Paris Sér. A-B **275** (1972), A973–A976. MR 47 #7062 (French)

[16] Joseph G. Stampfli and James P. Williams, *Growth conditions and the numerical range in a Banach algebra*, Tôhoku Math. J. (2) **20** (1968), 417–424. MR 39 #4674

[17] Béla Sz.-Nagy and Ciprian Foias, Harmonic Analysis of Operators on Hilbert Space, Translated from the French and revised, North-Holland Publishing Co., Amsterdam, 1970. MR 43 #947

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KEY WORDS AND PHRASES: dual algebra, weak* continuous character, weak spectrum
2000 MATHEMATICS SUBJECT CLASSIFICATION: 47D27, 47A10 (47A15)
Received: *September 10th, 2003; revised: March 1st, 2004.*