

Construction and enlargement of dilatonic wormholes by impulsive radiationHiroko Koyama,^{*} Sean A. Hayward,[†] and Sung-Won Kim[‡]*Department of Science Education, Ewha Womans University, Seoul 120-750, Korea*

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The dynamical behavior of traversable wormholes and black holes under impulsive radiation is studied in an exactly soluble dilaton gravity model. Simple solutions are presented where a traversable wormhole is constructed from a black hole, or the throat of a wormhole is stably enlarged or reduced. These solutions illustrate the basic operating principles needed to construct similar analytic solutions in full Einstein gravity.

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I. INTRODUCTION

Wormholes are tunnels through space-time, linking otherwise separated regions of a single universe, or bridges joining two different universes. Since traversable wormholes were introduced by Morris and Thorne [1] as theoretically allowable space-times in Einstein gravity, wormholes have been pursued as an attractive research topic involving the possibility of rapid interstellar travel and even time machines, as reviewed by Visser [2]. The properties of static Morris-Thorne wormholes have been studied by many authors and it is known even in nonstatic situations that a negative-energy source is necessary for traversable wormholes to exist [3–6]. Assuming such a source, there are interesting and practical problems of clarifying the dynamical nature of wormholes, such as how to construct a traversable wormhole and how to enlarge the throat enough to enable human beings to pass through it.

Not long ago, a unified framework for black holes and traversable wormholes was proposed [5], indicating that they are dynamically interconvertible when the trapping horizons locally characterizing them [5,6–8] bifurcate or merge. This theory was first concretely confirmed in an exactly soluble model, Callan-Giddings-Harvey-Strominger (CGHS) two-dimensional dilaton gravity [9] with an additional negative-energy scalar field to support wormholes [10], and then in standard Einstein gravity numerically [11]. In seeking analytic solutions in full Einstein gravity, we have found nonstatic solutions to be difficult to obtain, except in the idealization of impulsive radiation, where the radiation is concentrated so as to deliver finite energy and momentum in an instant.

In this paper, we construct some of the simplest models of dynamic wormhole processes by employing impulsive radiation in a generalized dilaton gravity model; specifically, wormhole construction from a black hole, wormhole operation by energy balance and wormhole reduction or enlargement. The last point addresses a common belief that, while wormholes are possible or even expected at the Planck scale,

large-scale wormholes are unlikely or even impossible. Actually self-inflating wormholes were recently discovered numerically [11], but to date there have been no concrete examples of stable wormhole enlargement, where the wormhole size is controlled.

In Sec. II we review the static CGHS [9] black-hole and Hayward-Kim-Lee (HKL) [10] wormhole solutions and generalize the dilaton gravity model. We find solutions describing the construction of a wormhole from a black hole in Sec. III. In Secs. IV and V, we study processes to change the throat radius of the wormhole by controlling impulsive radiation, either from one universe or from both universes. The final section is devoted to summary.

II. BLACK HOLES AND WORMHOLES IN DILATON GRAVITY

The CGHS two-dimensional dilaton gravity [9] is generalized by the action [10]

$$\int_S \mu \left[e^{-2\phi} (R + 4(\nabla\phi)^2 + 4\lambda^2) - \frac{1}{2}(\nabla f)^2 + \frac{1}{2}(\nabla g)^2 \right] \quad (1)$$

where S is a 2-manifold, μ , R and ∇ are the area form, Ricci scalar and covariant derivative of a Lorentz 2-metric on S , λ represents a negative cosmological constant, ϕ is a scalar dilaton field, f is a Klein-Gordon field representing matter and g is a ghost Klein-Gordon field. The last term is added to the CGHS action in order that g provides the negative energy densities needed to support a traversable wormhole [1–6]. By choosing future-pointing null coordinates (x^+, x^-) , the line element may be written as

$$ds^2 = -2e^{2\rho} dx^+ dx^- \quad (2)$$

Taking the gauge choice $\rho = \phi$ and transforming the dilaton field ϕ to $r = 2e^{-2\phi}$, the field equations reduce to a simple form: the evolution equations

$$\partial_+ \partial_- f = 0, \quad (3)$$

$$\partial_+ \partial_- g = 0, \quad (4)$$

$$\partial_+ \partial_- r = -4\lambda^2 \quad (5)$$

and the constraints

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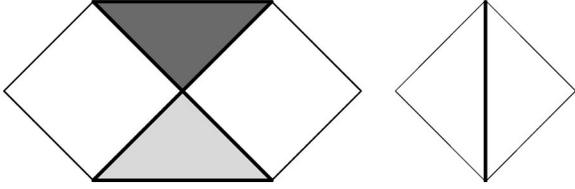


FIG. 1. Penrose diagrams of (left) a CGHS black hole and (right) a HKL traversable wormhole. The bold lines represent the trapping horizons, constituting the event horizons of the black hole and the throat of the wormhole.

$$\partial_{\pm}\partial_{\pm}r = (\partial_{\pm}g)^2 - (\partial_{\pm}f)^2. \quad (6)$$

The field r plays a similar role to the areal radius in spherical symmetry [8,12].

In vacuum, $f=g=0$, the general solution to the field equations is [9]

$$r = 2m - 4\lambda^2 x^+ x^- \quad (7)$$

where the origin has been fixed. The constant m may be interpreted as the mass of the space-time, whose global structure has been described previously [12]. For $m > 0$ this describes the CGHS static black hole, analogous to the Schwarzschild black hole. The Penrose diagram is shown in Fig. 1 (left panel).

Recently the solution [10]

$$r = a + 2\lambda^2(x^+ - x^-)^2, \quad g = 2\lambda(x^+ - x^-), \quad f = 0 \quad (8)$$

has been found, where the origin has again been fixed. If $a > 0$, this represents a traversable wormhole, hereafter called the HKL wormhole, with analogous global structure to a Morris-Thorne wormhole: a throat $r=a$ at $x^+ = x^-$, joining two regions with $r > a$, an $x^+ > x^-$ universe and a reflected $x^+ < x^-$ universe, as depicted in Fig. 1 (right panel).

Thus the model naturally contains both static black holes and static traversable wormholes. A characteristic feature of both cases is the trapping horizons, defined by $\nabla r \cdot \nabla r = 0$, or equivalently $\partial_+ r = 0$ or $\partial_- r = 0$ [5–8]. In the CGHS black hole, they coincide with the event horizons $r=2m$ at $x^- = 0$ and $x^+ = 0$ respectively. In the HKL wormhole, there is a double trapping horizon, $\partial_+ r = \partial_- r = 0$, at the throat $r=a$. This illustrates how trapping horizons of different type may be used to locally define both black holes and wormholes [5]. Also relevant are the locally trapped regions where $\nabla r \cdot \nabla r < 0$, consisting of future trapped regions if $\partial_{\pm} r < 0$ or past trapped regions if $\partial_{\pm} r > 0$, as occur in black holes or white holes respectively. Locating the trapping horizons and the locally trapped regions is a key feature of the analysis of dynamic situations.

In the following, we wish to take delta-function profiles for the radiation energy densities

$$\rho_{\pm} = (\partial_{\pm}f)^2 - (\partial_{\pm}g)^2 \quad (9)$$

where units have been fixed. To avoid ill-defined square roots of delta functions, the model may be formally generalized to

$$\partial_{\pm}\partial_{\pm}r = -\rho_{\pm}, \quad (10)$$

$$\partial_+\partial_-r = -4\lambda^2, \quad (11)$$

$$\partial_{\pm}\rho_{\mp} = 0, \quad (12)$$

where the energy densities ρ_{\pm} are now regarded as basic and need not be derived from Klein-Gordon fields. The evolution equations (11), (12) have the general solutions

$$r(x^+, x^-) = r_+(x^+) + r_-(x^-) - 4\lambda^2 x^+ x^-, \quad (13)$$

$$\rho_{\pm}(x^+, x^-) = \rho_{\pm}(x^{\pm}). \quad (14)$$

The constraints (10) are preserved by the evolution equations in the ∂_{\mp} directions, and so may be reduced to

$$\partial_{\pm}\partial_{\pm}r_{\pm} = -\rho_{\pm}. \quad (15)$$

The constraints are manifestly integrable for r_{\pm} given initial data:

$$\rho_{\pm} \quad \text{on} \quad x^{\mp} = x_0^{\mp}, \quad (16)$$

$$(r, \partial_+ r, \partial_- r) \quad \text{at} \quad x^+ = x_0^+, \quad x^- = x_0^- \quad (17)$$

for constants x_0^{\pm} . The data consist of the energy-density profiles of the left-moving and right-moving radiation, plus lower-dimensional data for the metric. Then the general procedure is to specify this initial data according to the desired physical situation, integrate the constraints (15) for r_{\pm} ; then the solution follows as Eqs. (13), (14). Consequently, the effect of the radiation is much easier to see than in Einstein gravity, although the model shares various physically important features including gravitational collapse to black holes satisfying cosmic censorship [12].

III. CONSTRUCTION OF A WORMHOLE FROM A BLACK HOLE

We study how to construct a traversable wormhole from a black hole by irradiating it with negative energy. Although a similar idea has been studied previously [10], now we present a simpler solution involving impulsive radiation, which is a preliminary to constructing an analytic solution in four-dimensional Einstein gravity. We consider a CGHS black hole subjected to impulsive negative-energy radiation at some positive value x_0 of the Kruskal-like coordinates x^{\pm} , with energy density chosen in order to close up the future trapped region by merging its trapping horizons, followed by the constant irradiation needed to maintain the static wormhole:

$$\rho_{\pm} = -4\lambda^2 x_0 \delta(x^{\pm} - x_0) - 4\lambda^2 \Theta(x^{\pm} - x_0) \quad (18)$$

where Θ is the unit Heaviside step function and δ the Dirac (delta-function) distribution. When differentiating to check solutions, it may be useful to remember that the derivative δ' of the delta function, as a distribution acting on test functions f , satisfies $\delta'f = -\delta f$ or $(\delta f)' = 0$. Of course $\Theta' = \delta$.

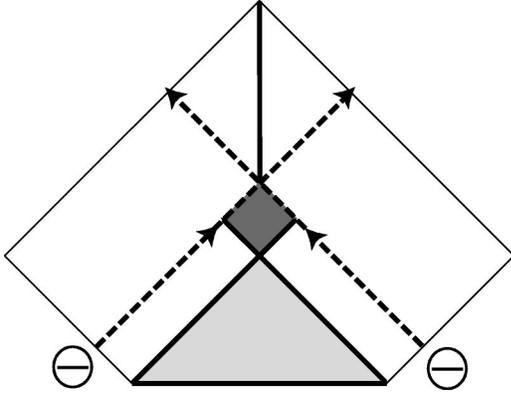


FIG. 2. A HKL wormhole is constructed from a CGHS black hole by irradiating with impulsive negative-energy radiation, represented by dashed lines. The bold lines represent the trapping horizons, light shading indicates past trapped regions and darker shading indicates future trapped regions. The impulses shift the black-hole trapping horizons suddenly, making them coincide, then constant non-impulsive radiation supports the resulting wormhole.

Assuming a black hole of mass m in the initial region, we obtain the solution

$$r = 2m - 4\lambda^2 x^+ x^- + 2\lambda^2 (x^{+2} - x_0^2) \Theta(x^+ - x_0) + 2\lambda^2 (x^{-2} - x_0^2) \Theta(x^- - x_0). \quad (19)$$

The solution in the final region $x^\pm > x_0$ can be recognized as a HKL wormhole (8). In more detail, the trapping horizons

$$0 = \partial_\pm r = 4\lambda^2 [x^\pm \Theta(x^\pm - x_0) - x^\mp] \quad (20)$$

are located at

$$\begin{aligned} x^\mp &= 0, & x^\pm < x_0, \\ x^+ &= x^-, & x^\pm > x_0 \end{aligned} \quad (21)$$

and their radii are

$$r_0 = \begin{cases} 2m, & x^\pm < x_0, \\ 2m - 4\lambda^2 x_0^2, & x^\pm > x_0, \end{cases} \quad (22)$$

as depicted in Fig. 2. In this solution, the throat radius $a = 2m - 4\lambda^2 x_0^2$ of the wormhole is smaller than the horizon radius $2m$ of the initial black hole. Thus we require $2\lambda^2 x_0^2 < m$ in order that a wormhole is constructed.

The results are similar to the previous solution [10], except that here the trapping horizons move discontinuously rather than continuously, a well-known property under infinitesimally thin mass shells [13]. This is a general feature of the solutions presented in this article, stemming from the fact that the field equations or Einstein equations relate $\partial_\pm \partial_\pm r$ to the energy densities ρ_\pm , so that delta-function ρ_\pm causes discontinuous $\partial_\pm r$. This is, of course, an idealization of a situation where a concentrated packet of radiation causes swift movement of the trapping horizon.

A recently discovered four-dimensional wormhole solution [14] can be similarly constructed from a Schwarzschild

black hole in full Einstein gravity [15]. By the time reverse, we can also obtain a picture where a wormhole collapses into a black hole by beaming in impulsive radiation at the moment of switching off the supporting ghost radiation. In this case, the horizon radius of the black hole is larger than the throat radius of the initial wormhole. This reduces to the sudden collapse case [10] without the impulsive radiation and $x_0 = 0$.

IV. WORMHOLE ENLARGEMENT AND REDUCTION

We are interested in how to create wormholes with throat large enough for human beings to pass from one universe to another. It is practically useful if it can be achieved by processes from our universe only. In this section, we study wormhole operation by energy balance from one universe only. We irradiate the wormhole from our universe with impulsive radiation of equal positive and negative energy at different times:

$$\rho_+ = -4\lambda^2 - \beta^2 \delta(x^+ - x_0) + \beta^2 \delta(x^+ - x_1), \quad (23)$$

$$\rho_- = -4\lambda^2,$$

where $x_1 > x_0$, so that the positive-energy impulse follows the negative-energy impulse. Assuming a wormhole of throat radius a initially, we obtain the solution

$$r = a + 2\lambda^2 (x^+ - x^-)^2 + \beta^2 (x^+ - x_0) \Theta(x^+ - x_0) - \beta^2 (x^+ - x_1) \Theta(x^+ - x_1). \quad (24)$$

The locations of the trapping horizons $\partial_\pm r = 0$ are given by

$$x^- = \begin{cases} x^+, & x^+ < x_0, \\ x^+ + \beta^2/4\lambda^2, & x_0 < x^+ < x_1, \\ x^+, & x^+ > x_1, \end{cases} \quad (25)$$

$$x^- = x^+.$$

One sees that the wormhole mouth $\partial_+ r = 0$ is shifted out by the negative-energy radiation and back again by the positive-energy radiation, merging with the unmoved $\partial_- r = 0$ mouth to leave a static wormhole again, Fig. 3 (left panel). As in the previous section, the sudden shift of the wormhole mouth is due to the impulsive nature of the radiation. The throat radii of the initial and final wormholes are

$$r_0 = \begin{cases} a, & x^\pm < x_0, \\ a + \beta^2(x_1 - x_0), & x^\pm > x_1, \end{cases} \quad (26)$$

so that the throat of the final state becomes larger than that of the initial state.

On the other hand, if $x_1 < x_0$, where the negative-energy impulse follows the positive-energy impulse, the throat of the final state becomes smaller than that of the initial state, Fig. 3 (right panel). Then we need $\beta^2(x_0 - x_1) < a$ to obtain a wormhole rather than a naked singularity. If one thinks of the positive-energy impulse as an idealized model of a fast-moving traveller traversing the wormhole, this suggests how

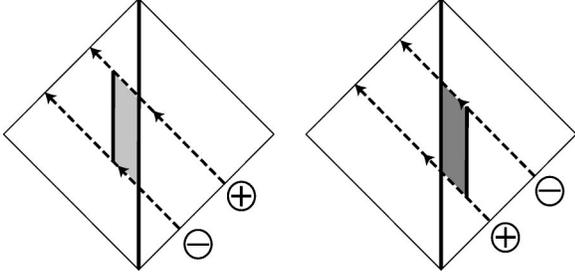


FIG. 3. A HKL wormhole is subjected to a double burst of impulsive radiation with equal positive and negative energy. The operation shifts one wormhole mouth away from, then back to, its original position. The shaded regions are (left) past or (right) future trapped, respectively expanding or contracting, so the wormhole becomes respectively larger or smaller.

to operate and maintain the wormhole for transport, including the back reaction of the traveller [5]. This also demonstrates the stability of the wormhole to such dynamic perturbations.

In summary, the throat radius of the wormhole can be adjusted at will by controlling the energy and timing of impulses. The example demonstrates the property, following from the second law of wormhole dynamics [5], that the wormhole becomes smaller (larger) after an operation in which the wormhole mouths bifurcate to open up a contracting (expanding) region of future (past) trapped surfaces and subsequently merge again. Note that the energies $\pm\beta^2$ of the impulsive radiation are equal and opposite, to ensure that the wormhole mouths merge again, as expected from the first law of wormhole dynamics [5].

V. SYMMETRIC WORMHOLE ENLARGEMENT

Now we construct solutions to enlarge the throat by beaming in impulsive radiation symmetrically from both universes. We give two solutions, a simple one which is difficult to generalize to full Einstein gravity, and a more delicate one which can be so generalized. The first example is a symmetrized version of that of the previous section:

$$\rho_{\pm} = -4\lambda^2 - \beta^2 \delta(x^{\pm} - x_0) + \beta^2 \delta(x^{\pm} - x_1). \quad (27)$$

Assuming a wormhole with initial throat radius a , we obtain the solution

$$\begin{aligned} r = & a + 2\lambda^2(x^+ - x^-)^2 + \beta^2(x^+ - x_0)\Theta(x^+ - x_0) \\ & + \beta^2(x^- - x_0)\Theta(x^- - x_0) - \beta^2(x^+ - x_1)\Theta(x^+ - x_1) \\ & - \beta^2(x^- - x_1)\Theta(x^- - x_1). \end{aligned} \quad (28)$$

This also describes a HKL wormhole (8) in the final region $x^{\pm} > x_1$. The initial and final wormhole regions have throats $\partial_+ r = \partial_- r = 0$ at $x^+ = x^-$, with radii

$$r_0 = \begin{cases} a, & x^{\pm} < x_0, \\ a + 2\beta^2(x_1 - x_0), & x^{\pm} > x_1, \end{cases} \quad (29)$$

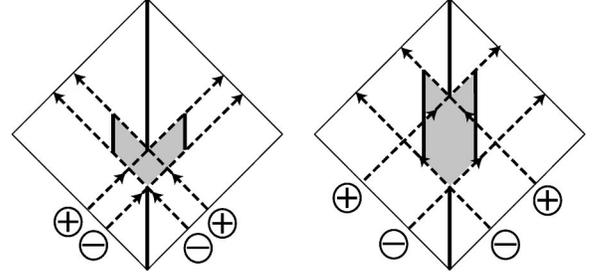


FIG. 4. Wormhole enlargement by symmetric bursts of impulsive radiation, keeping the non-impulsive radiation constant. Both wormhole mouths are shifted out, then back again, opening and closing an expanding (past trapped) region, shaded for a rapid burst (left) and a slow burst (right).

respectively. Then the wormhole is enlarged if $x_1 > x_0$, so that the negative-energy impulse precedes the positive-energy impulse, as before. The final formation of a static wormhole is again dependent on the additional radiation having equal and opposite energies $\pm\beta^2$. For a slow burst of duration $x_1 - x_0 > \beta^2/4\lambda^2$, the wormhole mouths $\partial_{\pm} r = 0$ are shifted so that the oppositely moving positive-energy impulses intersect them, while for a rapid burst of duration $x_1 - x_0 < \beta^2/4\lambda^2$, the mouths enclose the entire middle region $x_0 < x^{\pm} < x_1$ between the impulses, Fig. 4.

In full Einstein gravity, we wish to patch together Schwarzschild, static-wormhole [14] and Vaidya regions. However, the four-dimensional analogue of the middle region in the above example is not known analytically. So for the second example, we switch off the non-impulsive radiation between the impulses and switch it back on afterward:

$$\begin{aligned} \rho_{\pm} = & -4\lambda^2 \Theta(x_0 - x^{\pm}) - \beta^2 \delta(x^{\pm} - x_0) + \alpha^2 \delta(x^{\pm} - x_1) \\ & - 4\lambda^2 \Theta(x^{\pm} - x_1). \end{aligned} \quad (30)$$

Then the middle region is vacuum and therefore part of a CGHS white hole. Assuming the initial region $x^{\pm} < x_0$ to be a HKL wormhole of throat radius a , we find the solution

$$\begin{aligned} r = & a - 4\lambda^2(x^+ x^- + x_0^2) + 4\lambda^2 x_0(x^+ + x^-) + 2\lambda^2(x^+ - x_0)^2 \\ & \times \Theta(x_0 - x^+) + 2\lambda^2(x^- - x_0)^2 \Theta(x_0 - x^-) + \beta^2(x^+ - x_0) \\ & \times \Theta(x^+ - x_0) + \beta^2(x^- - x_0)\Theta(x^- - x_0) - \alpha^2(x^+ - x_1) \\ & \times \Theta(x^+ - x_1) - \alpha^2(x^- - x_1)\Theta(x^- - x_1) + 2\lambda^2(x^+ - x_1)^2 \\ & \times \Theta(x^+ - x_1) + 2\lambda^2(x^- - x_1)^2 \Theta(x^- - x_1). \end{aligned} \quad (31)$$

Now we want the final region $x^{\pm} > x_1$ to be a HKL wormhole in the usual coordinates, Fig. 5. Then we find the relations

$$x_0 = -\frac{\beta^2}{4\lambda^2}, \quad x_1 = -\frac{\alpha^2}{4\lambda^2} \quad (32)$$

between the energy and timing of the impulses. This simplifies the solution in the initial, middle and final regions:

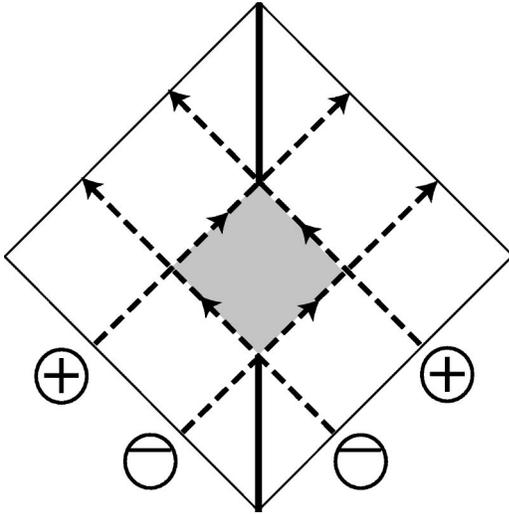


FIG. 5. Wormhole enlargement process by symmetric bursts of impulsive radiation, with negative energy followed by positive energy, timed as described in the text. The non-impulsive radiation is switched off between the impulses. Then the middle shaded region is vacuum and expanding.

$$r = \begin{cases} a + 2\lambda^2(x^+ - x^-)^2, & x^\pm < x_0, \\ a + 4\lambda^2x_0^2 - 4\lambda^2x^+x^-, & x_0 < x^\pm < x_1, \\ a + 4\lambda^2(x_0^2 - x_1^2) + 2\lambda^2(x^+ - x^-)^2, & x_1 < x^\pm, \end{cases} \quad (33)$$

which are recognizable as CGHS white-hole (7) or HKL wormhole (8) regions. The wormhole throats are at $x^+ = x^-$ with radii

$$r_0 = \begin{cases} a, & x^\pm < x_0, \\ a + 4\lambda^2(x_0^2 - x_1^2), & x_1 < x^\pm. \end{cases} \quad (34)$$

Thus the throat radius of the final wormhole is larger than that of the initial one, again due to the negative-positive energy ordering of the burst.

Note that $x_0 < x_1 < 0$, so that $\alpha^2 < \beta^2$, meaning that the positive-energy impulsive radiation does not completely balance the negative-energy impulsive radiation, since some supporting negative energy has already been removed. The difference $\alpha^2 - \beta^2 = -4\lambda^2(x_1 - x_0)$ equals the energy $\int_{x_0}^{x_1} \rho_\pm dx^\pm$ missing as compared with the static wormhole.

Combined with the results from the previous section, we have confirmed that the radius of the wormhole throat is enlarged when an expanding region of past trapped surfaces is opened and closed between the initial and final static wormholes, by bifurcating and merging the wormhole mouths, defined as trapping horizons. In addition, the solution just presented is one of the simplest where the wormhole is enlarged, in the sense that each relevant region is part of either a static white hole, a static wormhole or a pure-radiation region, joined at null boundaries, which is also possible in full Einstein gravity [15].

VI. SUMMARY

In this paper, we have used an exactly soluble dilaton gravity model to study wormhole dynamics under impulsive radiation, finding solutions where a traversable wormhole is created from a black hole or the throat radius of a wormhole is enlarged or reduced, the size being controlled by the energy and timing of the impulses. Where the solutions consist of black-hole, static-wormhole and pure-radiation regions matched along null boundaries, we can succeed in constructing similar analytic solutions in full Einstein gravity, although the analytical details are much more complex [15].

The recipe for enlarging the wormhole is to cause the wormhole mouths to bifurcate, opening up an expanding region of past trapped surfaces, and then merge again, by adding additional negative energy followed by compensating positive energy. The general proof involves the first and second laws of wormhole dynamics and is a future important work in the unified framework for black-hole and wormhole dynamics [5]. The second law determines whether the area increases or decreases, and the first law quantifies it in terms of the energy supplied and work done.

The results in this paper show how to create a traversable wormhole of human size in principle, if negative-energy matter can be controlled. Self-inflating wormholes were discovered recently [11], but the present solutions are the first to describe stable wormhole enlargement. Clarifying the dynamical behavior of wormholes is a quite attractive subject, since the cosmic shortcuts and time travel usually considered as science fiction are thereby closer to being realized.

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