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On a Stochastic Failure Model under Random Shocks

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Abstract. In most conventional settings, the events caused by an external shock are initiated at the moments of its occurrence. In this paper, we study a *new classes of shock model*, where each shock from a nonhomogeneous Poisson processes can trigger a failure of a system not immediately, as in classical extreme shock models, but with delay of some random time. We derive the corresponding survival and failure rate functions. Furthermore, we study the limiting behaviour of the failure rate function where it is applicable.

1. Introduction

Consider an orderly point process (without multiple occurrences) $N(t), t \geq 0$ of some 'initiating' events (IEs) with arrival times $T_1 < T_2 < T_3 < \ldots$. Let each event from this process triggers the 'effective event' (EE), which occurs after a random time (delay) $D_i, i = 1,2,\ldots$, since the occurrence of the corresponding IE at T_i. Obviously, in contrast to the initial ordered sequence $T_1 < T_2 < T_3 < \ldots$, the EEs $\{T_i + D_i\}, i = 1,2,\ldots$ are now not necessarily ordered. This setting can be encountered in many practical situations, when, e.g., initiating events start the process of developing the non-fatal faults in a system and we are interested in the number of these faults in $[0,t)$. Alternatively, effective events can result in fatal, terminating faults (failures) and then we are interested in the survival probability of our system. Therefore, the latter setting means that the *first EE ruins our system.* The corresponding stochastic survival model will be considered in this paper. When there are no delays, each shock (with the specified probability) results in the failure of the survived system and the described model obviously reduces to the classical extreme shock model (Gut and Hüsler, 2005; Finkelstein, 2008; Cha and Finkelstein, 2009).

The IEs can often be interpreted as some external shocks affecting a system, and for convenience, we will often use this wording (interchangeably with the "IE"). We will consider the case of the nonhomogeneous Poisson process (NHPP) of the IEs. The approach can, in principle, be applied to the case of renewal processes, but the corresponding formulas are too cumbersome. However, the obtained results for the NHPP case are in simple, closed forms that allow intuitive interpretations and proper analyses.

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The paper is organized as follows. In Section 2, the corresponding lifetime distribution and the failure rate are obtained for the model with delays. In Section 3, the limiting behaviour of the failure rate for $t \to \infty$ is analyzed for different limiting properties of the rate of the NHPP of shocks. Finally, in Section 4, concluding remarks are given.

2. Lifetime Distribution

Consider a system subject to the nonhomogeneous Poisson process of IEs $N(t), t \geq 0$ to be called shocks. Let the rate of this process be $\nu(t)$ and the corresponding arrival times be denoted as $T_1 < T_2 < T_3 \ldots$. Assume that the ith shock is 'harmless' to the system with probability $q(T_i)$, and with probability $p(T_i)$ it triggers the failure process of the system which results in its failure after a random time $D(T_i)$, $i = 1,2,\ldots$, where $D(t)$ is a non-negative, semi-continuous random variable with the point mass at "0" (at each fixed t). Note that this 'point mass' at 0 opens the possibility of 'immediate failure' of the system at the occurrence of a shock, which is practically very important, and, furthermore the case of the 'full point mass' of $D(t)$ at 0 reduces to the ordinary 'extreme shock model'. Obviously, without point mass at 0, we arrive at an absolutely continuous random variable. The distributions of $D(t)$ having point masses at other values could be considered similarly.

Let $G(t,x) \equiv P(D(t) \leq x)$, $\bar{G}(t,x) \equiv 1 - G(t,x)$, and $g(t,x)$ be the Cdf, the survival function and the pdf for the 'continuous part' of $D(t)$, respectively. Then, in accordance with the terminology in the Introduction, the failure in this case is the *Effective Event (EE)*.

First of all, we are interested in describing the lifetime of our system T_S. The corresponding conditional survival function is given by

$$P(T_S > t \mid N(s), 0 \leq s \leq t; D(T_1), D(T_2), \ldots, D(T_{N(t)}); J_1, J_2, \ldots, J_{N(t)})$$

$$= \prod_{i=1}^{N(t)} \big(J_i + (1 - J_i) I(D(T_i) > t - T_i)\big), \tag{1}$$

where the indicators are defined as

$$I(D(T_i) > t - T_i) = \begin{cases} 1, \ if \ D(T_i) > t - T_i \\ 0, \ otherwise \end{cases},$$

$$J_i = \begin{cases} 1, \ if \ the \ i\text{th} \ shock \ does \ not \ trigger \ the \ subsequent \ failure \ process, \\ 0, \ otherwise. \end{cases}$$

Assume the following conditions regarding 'conditional independence':

(i) Given the shock process, $D(T_i), i = 1,2,\ldots$, are mutually independent.

(ii) Given the shock process, J_i, $i = 1,2,\ldots$, are mutually independent. (It means that whether each shock triggers the failure process of the system or not is 'independently determined').

(iii) Given the shock process, $\{D(T_i), i = 1,2,\ldots\}$ and $\{J_i, i = 1,2,\ldots\}$, are mutually independent.

Integrating out all conditional random quantities in (1) under the basic assumptions described above results in the following theorem.

Theorem 1. *Suppose that* $\nu(+0) > 0$. *Then*

$$P(T_S \geq t) = \exp\left\{-\int_0^t G(x;t-x)p(x)\nu(x)dx\right\}, t \geq 0,$$

and the failure rate function of the system is

$$\lambda_S(t) = \int_0^t g(x;t-x)p(x)\nu(x)dx + G(t,0)p(t)\nu(t), t \geq 0.$$

Proof.
The proof is omitted due to the page limit.

Remark 1. It has been recognized that many queuing systems are most appropriately described by nonstationary queuing models, in which "both the arrival and service rates" are functions of time. Therefore, there is a growing literature on methods for calculating time-dependent performance measures in these models. It can be seen that in a queuing context, when $G(t,x) = G(x)$, our system's failure time can be interpreted as the first departure time in the $M_t / G / \infty$ system that starts empty (Ross, 1996). The model considered in this section is a time variant of it and, accordingly, the failure time can be interpreted in terms of the first departure time from the corresponding $M_t / G_t / \infty$ system that starts empty.

Remark 2. Speaking formally, the split of effects to effective and ineffective shocks does not add any mathematical complexity due to the NHPP nature of the arrival process. This means that the result would be the same if we had only one type of effects and the NHPP with the rate function $p(t)\nu(t)$. However, from the practical point of view and keeping in mind that we are generalizing here the classical extreme shock model with two types of effects, this splitting seems to be reasonable. Furthermore, we can consider the case of multi-type delayed consequences of shocks ($n > 1$), where the shock that occurs at time t causes the delayed (with distribution $G_i(t,x)$) effect of type i with probability $p_i(t)$, whereas the probability of 'no effect' is $1 - \sum_{i=1}^{n} p_i(t)$. Obviously, this model is the same as the single-type model with $G(t,x) = \sum_{i=1}^{n} p_i^*(t)G_i(t,x)$ and $p(t) = \sum_{i=1}^{n} p_i(t)$, where $p_i^*(t) = p_i(t) / \sum_{i=1}^{n} p_i(t)$. Therefore, similar to Theorem 1,

$$P(T_S \geq t) = \exp\left\{-\int_0^t \left(\sum_{i=1}^{n} p_i(x)G_i(x,t-x)\right)\nu(x)dx\right\}, t \geq 0$$

and

$$\lambda_S(t) = \int_0^t \left(\sum_{i=1}^{n} p_i(x)g_i(x,t-x)\right)\nu(x)dx + \left(\sum_{i=1}^{n} p_i(t)G_i(t,0)\right)\nu(t).$$

3. Limiting behaviour

In this subsection, we study the limiting behavior of the failure rate $\lambda_S(t)$. Without loss of generality, assume that $p(t)$ and $\nu(t)$ are continuous functions with $p(t) > 0$, for all $t \geq 0$. We further assume that

$$D(t) \to D(\infty) \equiv D \text{ in distribution as } t \to \infty,$$

where $D(t)$, $t > 0$, and D are semi-continuous random variables with supports $[0, \infty)$ and point masses at "0", and

$$\frac{g(t,x)}{g(0,x)} \text{ is bounded for all } t, x \geq 0. \tag{2}$$

Obviously, (2) is a rather weak condition. The simplest example of a continuous random variable is

$$g(t,x) = (1/\mu(t))\exp\{-(1/\mu(t))x\},\ x \geq 0, \tag{3}$$

where $\mu(t)$ is continuous and decreasing with

$$\mu(t) \to \mu, \text{ as } t \to \infty.$$

Observe that the distribution which corresponds to the pdf in (3) is an exponential distribution for each fixed t.

Theorem 2. *In addition to the assumptions of Theorem 1, let* $\lim_{t\to\infty} p(t) \equiv p(\infty)$ *and* $\lim_{t\to\infty} \nu(t) \equiv \nu(\infty) \leq \infty$ *exist. Then*

$$\lim_{t\to\infty} \lambda_S(t) \equiv p(\infty)\nu(\infty).$$

Proof.
The proof is omitted due to the page limit.

4. Concluding remarks

One can find a lot of different shock models in the vast literature on this topic (see, e.g., Nakagawa, 2007 and references therein). These models usually deal with systems that are subject to shocks of random magnitudes at random times.

In this paper, we study a new shock model. The model generalizes the extreme shock model to the case when each shock from a nonhomogeneous Poisson processes can be fatal only after some time delay. We derive the corresponding survival probabilities and failure rates in a rather simple, meaningful form that allows probabilistic analysis and prompts for further generalizations.

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