GL(2,R) gauge theory of (1+1)-dimensional gravity

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We formulate a GL(2,R) gauge theory of (1+1)-dimensional gravity, which contains the Jackiw-Teitelboim model and the dilaton gravity model as special cases. A general classical solution of the theory is given, and it is shown that the solution reduces to those of the special cases when appropriate limits of parameters are taken.

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I. INTRODUCTION

Gravitational theories in (1+1)-dimensional space-time attracted much attention recently. Among various models the simplest one [1] is that proposed by Teitelboim and Jackiw, of which the action is

$$I_1=\int d^2x\sqrt{-g}\,\eta(R-\Lambda_1)\ , \tag{1.1}$$

where R is the Riemann curvature scalar, Λ_1 is a cosmological constant, and η is an invariant world-scalar field. One interesting feature of this model is the SO(2,1) [or SL(2,R)] gauge-invariant formulation proposed by several authors [2]. The gauge algebra of this theory is

$$[P_a,J]=\varepsilon_a{}^bP_b,\quad [P_a,P_b]=-\tfrac{1}{2}\Lambda_1\varepsilon_{ab}J\ , \tag{1.2}$$

where J is a Lorentz generator and P_a are translation generators.

Another model that has been recently studied for the purpose of modeling black-hole physics and Hawking radiation [3,4] is the "string-inspired" gravity, whose action is

$$I_2=\int d^2x\sqrt{-g}\,(\eta R-\Lambda_2)\ , \tag{1.3}$$

which differs from Eq. (1.1) in that the Lagrange multiplier η is absent from the cosmological contant. This action can be related to the usual formula

$$\bar{I}_2=\int d^2x\sqrt{-\bar{g}}\,e^{-2\phi}(\bar{R}-4\partial_\mu\phi\partial^\mu\phi-\Lambda_2) \tag{1.4}$$

by rescaling the metric with the "dilaton" field ϕ as $\bar{g}_{\mu\nu}=e^{2\phi}g_{\mu\nu}$ and defining $\eta=e^{-2\phi}$.

The gauge-theoretical formulation of I_2 was first given by Verlinde [3] based upon the nonsemisimple Poincaré group, with the algebra

$$[P_a,J]=\varepsilon_a{}^bP_b,\quad [P_a,P_b]=0\ , \tag{1.5}$$

which is the $\Lambda_1\to 0$ contraction of the algebra (1.2). After reviewing Verlinde's approach, Cangemi and Jackiw [5] and Jackiw [6] proposed a more natural gauge-invariant formulation of I_2 by changing the algebra to a centrally extended Poincaré algebra,

$$[P_a,J]=\varepsilon_a{}^bP_b,\quad [P_a,P_b]=\varepsilon_{ab}I\ , \tag{1.6}$$

which can be viewed as an unconventional contraction of (1.2).

It is the purpose of this article to combine the above two models into one such that the action becomes

$$I_3=\int d^2x\sqrt{-g}\,[\eta(R-\Lambda_1)-\Lambda_2]\ , \tag{1.7}$$

and to construct a gauge-invariant formalism based upon GL(2,R) [or SO(2,1)$\otimes$GL(1,R)] algebra,

$$[P_a,J]=\varepsilon_a{}^bP_b,\quad [P_a,P_b]=\varepsilon_{ab}\left[-\frac{\Lambda_1}{2}J+\xi I\right]\ ,$$
$$[I,P_a]=[I,J]=0\ , \tag{1.8}$$

where Λ_1 and ξ are nonvanishing constants. Notice that the ($\Lambda_1\neq 0$, $\xi=0$), ($\Lambda_1=0$, $\xi=0$), and ($\Lambda_1=0$, $\xi=1$) cases are the algebra of I_1, I_2 (Verlinde), and I_2 (Cangemi and Jackiw), respectively. Therefore the combined model offers a unified approach to the above two-dimensional gravity models.

In Sec. II the gauge-invariant formulation of I will be presented, and in Sec. III the most general solutions of the field equations will be obtained. Solving the field equations is simpler in the gauge formalism than in the usual geometric formalisms, and all possible solutions are easily codified by the algebra element. In the last section some discussion will be given.

The notation we use is the following: Time and space carry the metric tensor $g_{\mu\nu}$ with signature $(1,-1)$. Tangent space components are labeled by latin letters $a,b,\ldots$, and the flat metric tensor $h_{ab}=\mathrm{diag}(1,-1)$. The antisymmetric tensor ε^{ab} is such that $\varepsilon^{01}=-\varepsilon_{01}=1$.

II. GL(2,R) GRAVITY

From the combined action

$$I_3=\int d^2x\sqrt{-g}\,[\eta(R-\Lambda_1)-\Lambda_2]\ , \tag{2.1}$$

we obtain the equations of motion by varying η and $g_{\mu\nu}$:

$$R=\Lambda_1\ , \tag{2.2}$$

$$(D_\mu D_\nu-g_{\mu\nu}D^2)\eta+\tfrac{1}{2}g_{\mu\nu}(\Lambda_1\eta+\Lambda_2)=0\ . \tag{2.3}$$

These geometric dynamics may be presented in a gauge-theoretical fashion. To this end we use the extended de Sitter group algebra (1.8), which can be reexpressed in the form of the $\widetilde{\mathrm{SO}}(2,1)\otimes \mathrm{GL}(1,R)$ algebra, whose generators P_a, J, and I satisfy the relations

$$[P_a,\tilde{J}]=\varepsilon_a{}^bP_b,\quad [P_a,P_b]=\frac{-\Lambda_1}{2}\varepsilon_{ab}\tilde{J}\ , \tag{2.4}$$

$$[I,P]=[I,\tilde{J}]=0\ , \tag{2.5}$$

where

$$\tilde{J}=J-\frac{2\xi}{\Lambda_1}I\ . \tag{2.6}$$

We use a tilde over SO(2,1) to remind us that the Lorentz generator is not J but $\tilde{J}$. This replacement of J by $\tilde{J}$ is due to the well-known ambiguity of two-dimensional angular momentum.

Based upon the algebra (1.8) or (2.4) and (2.5) the gauge connection one-form is expanded in terms of the generators:

$$\begin{aligned}A&=e^aP_a+\omega J+aI\\&=e^aP_a+\omega\tilde{J}+\tilde{a}I\equiv\tilde{A}+\tilde{a}I\ ,\end{aligned} \tag{2.7}$$

where

$$\tilde{a}=a+\omega\frac{2\xi}{\Lambda_1}\ . \tag{2.8}$$

Here e^a_μ is the zweibein, ω_μ is the spin connection, and a is an additional connection which has no direct geometrical meaning. The curvature two-forms

$$F=dA+A^2\ , \tag{2.9a}$$

$$\tilde{F}=d\tilde{A}+\tilde{A}^2 \tag{2.9b}$$

become

$$\begin{aligned}F&=f^aP_a+fJ+gI\\&=f^aP_a+f\tilde{J}+\tilde{g}I=\tilde{F}+\tilde{g}I\ ,\end{aligned} \tag{2.10}$$

where

$$\tilde{g}=g+f\frac{2\xi}{\Lambda_1} \tag{2.11}$$

and

$$f^a=(De)^a=de^a+\varepsilon^a{}_b\omega e^b\ , \tag{2.12}$$

$$f=d\omega-\frac{\Lambda_1}{4}e^a\varepsilon_{ab}e^b\ , \tag{2.13}$$

$$g=da+\frac{\xi}{2}e^a\varepsilon_{ab}e^b\ , \tag{2.14a}$$

$$\tilde{g}=d\tilde{a}\ . \tag{2.14b}$$

It is seen that $d\omega$ is the scalar curvature density, f^a is the torsion density, each expressed in terms of e^a and ω, but g is the additional two-form without a geometric counterpart.

A finite gauge transformation generated by the gauge function Θ,

$$\Theta=\theta^aP_a+\alpha J+\beta I=\theta^aP_a+\alpha\tilde{J}+\tilde{\beta}I\ , \tag{2.15}$$

where

$$\tilde{\beta}=\beta+\alpha\frac{2\xi}{\Lambda_1}\ , \tag{2.16}$$

is denoted by $U=e^{\Theta}$, which can be written as

$$U=e^{\tilde{\beta}I}\tilde{U}\ , \tag{2.17}$$

where $\tilde{U}$ is a $\widetilde{\mathrm{SO}}(2,1)$ element.

In the usual way the gauge transformations of A and F are given by

$$A'=U^{-1}AU+U^{-1}dU=\tilde{U}^{-1}\tilde{A}\tilde{U}+\tilde{U}^{-1}d\tilde{U}+(\tilde{a}+d\tilde{\beta})\ , \tag{2.18}$$

$$F'=U^{-1}FU=\tilde{U}^{-1}\tilde{F}\tilde{U}+\tilde{g}\ . \tag{2.19}$$

It is apparent that $\tilde{A}$ and $\tilde{F}$ are the $\widetilde{\mathrm{SO}}(2,1)$ parts of A and F, and $\tilde{a}$ and $\tilde{g}$ are the GL(1,R) parts.

The gauge-invariant Lagrange density is then

$$\mathcal{L}=\tilde{\eta}_af^a+\tilde{\eta}_2f+\tilde{\eta}_3\tilde{g}=\sum_{A=0}^{2}\tilde{\eta}_A\tilde{F}^A+\tilde{\eta}_3\tilde{g}\ , \tag{2.20}$$

where the Lagrange multiplier triplet $\tilde{\eta}_A$ is taken to transform by the coadjoint representation of $\widetilde{\mathrm{SO}}(2,1)$, and $\tilde{\eta}_3$ is a gauge scalar.

By introducing η as

$$\eta_a\equiv\tilde{\eta}_a,\quad \eta_2=\tilde{\eta}_2+\frac{2\xi}{\Lambda_1}\tilde{\eta}_3,\quad \eta_3=\tilde{\eta}_3\ , \tag{2.21}$$

one can rewrite the Lagrangian density as

$$\begin{aligned}\mathcal{L}&=\eta_af^a+\eta_2f+\eta_3g\\&=\eta_a(De)^a+\eta_2\left[d\omega-\frac{\Lambda_1}{4}e^a\varepsilon_{ab}e^b\right]\\&\quad+\eta_3\left[da+\frac{\xi}{2}e^a\varepsilon_{ab}e^b\right]\ .\end{aligned} \tag{2.22}$$

In order to see the gauge-transformation properties it is better to use $\tilde{\eta}$ variables, but for the purpose of taking limits ($\Lambda\to0$ or $\xi\to0$) and comparing with previous models it is more convenient to use η variables. The field equations may be derived from either (2.20) or (2.22).

The equations obtained by varying η give the condition of vanishing torsion,

$$f^a=(De)^a=de^a+\varepsilon_b{}^a\omega e^b=0 , \qquad (2.23)$$

which allows evaluation of the spin connection in terms of the zweibein

$$\omega=e^a(h_{ab}\varepsilon^{\mu\nu}\partial_\mu e_\nu{}^b)/\det(e) , \qquad (2.24)$$

if $\det(e)\neq 0$. The equation which follows upon variation of η_2 gives

$$f=d\omega=\frac{\Lambda_1}{4}e^a\varepsilon_{ab}e^b=0 , \qquad (2.25)$$

which regains (2.2) ($R=\Lambda_1$) once (2.24) is used.

Variations of e^2, ω, and a produce equations for the Lagrange multipliers η_a, η_2, and η_3, respectively:

$$d\eta_a+\varepsilon_a{}^b\omega\eta_b-\frac{\Lambda_1}{2}\varepsilon_{ab}e^b\eta_2+\xi\varepsilon_{ab}e^b\eta_3=0 , \qquad (2.26a)$$

$$d\eta_2+\eta_a\varepsilon^a{}_b e^b=0 , \qquad (2.26b)$$

$$d\eta_3=0 . \qquad (2.26c)$$

The last equation is solved as

$$\eta_3=-\frac{\Lambda_2}{2_\xi} , \qquad (2.27)$$

where the second cosmological constant Λ_2 is an integration constant, and not inserted *a priori* into the theory as Cangemi and Jackiw [5] and Jackiw [6] pointed out. After elimination of η_a by using (2.26b), and after some manipulations one can recover (2.3) by letting $\eta_2=\eta$.

The remaining equation obtained by varying η_3 is

$$d\left[a+\frac{2\xi}{\Lambda_1}\omega\right]=0 , \qquad (2.28)$$

which becomes

$$da=-\frac{\xi}{2}e^a\varepsilon_{ab}e^b , \qquad (2.29)$$

where $d\omega=-(\Lambda_1/4)e^a\varepsilon_{ab}e^b$ is used. Equation (2.29) can be always solved at least locally, because the $e^a\varepsilon_{ab}e^b$ is a closed two-form.

Upon eliminating ω in $\mathcal{L}$ of (2.22) with the zero-torsion condition $(De)^a=0$ and evaluating η_3 at $-\Lambda_2/2\xi$, we regain the geometrical form of $\mathcal{L}$ in (2.1), apart from the total derivative $-(\Lambda_2/2\xi)da$, which does not contribute to the equations of motion because it is a total divergence.

III. EXPLICIT SOLUTIONS

Finding explicit solutions of the geometric field equations (2.2) and (2.3) or equivalently the gauge-theoretic equations (2.23), (2.25), and (2.26) becomes straightforward once the latter equations are recast in a group-theoretical fashion. The field equations following from (2.20), upon respective variation of η and A, are

$$\tilde{F}=0 , \qquad (3.1)$$

$$d\tilde{H}+[\tilde{A},\tilde{H}]=0 , \qquad (3.2)$$

$$\tilde{g}=0 , \qquad (3.3)$$

$$d\tilde{\eta}_3=0 , \qquad (3.4)$$

where $\tilde{A}$, $\tilde{F}$, and $\tilde{H}$ belong to the $\widetilde{\mathrm{SO}}(2,1)$ algebra, and $\tilde{g}$ and $\tilde{\eta}_3$ are GL(1,R) elements. Here $\tilde{H}$ is defined as

$$\tilde{H}=\tilde{\eta}_a h^{ab}P_b+\frac{\Lambda_1}{2}\tilde{\eta}_2\tilde{J} , \qquad (3.5)$$

where the factor $\Lambda_1/2$ is a consequence of the group metric.

Equation (3.1) implies that $\tilde{A}$ is a pure gauge given by an arbitrary element $\tilde{U}$ of the $\widetilde{\mathrm{SO}}(2,1)$ group,

$$\tilde{A}=\tilde{U}^{-1}d\tilde{U} , \qquad (3.6)$$

while $\tilde{H}$ can be immediately determined as

$$\tilde{H}=\tilde{U}^{-1}\tilde{\Phi}\tilde{U} , \qquad (3.7)$$

where $\tilde{\Phi}$ is a constant element in the algebra. The solutions of (3.3) and (3.4) are straightforward. Thus the field equations are completely solved by the choice of an element of the algebra $\tilde{\Phi}$ and a group element $\tilde{U}$.

For calculational purposes it is convenient to use a matrix realization of the algebra. Specifically, we will use the representation

$$p_0=\left[\frac{\Lambda_1}{8}\right]^{1/2}\sigma_3,\quad p_1=i\left[\frac{\Lambda_1}{8}\right]^{1/2}\sigma_2,\quad \tilde{J}=\tfrac{1}{2}\sigma_1 . \qquad (3.8)$$

These are just the generators of SL(2,R), which is due to the isomorphism SL(2,R)$\simeq$SO(2,1). An arbitrary element of the algebra is realized as a 2×2 real matrix

$$\tilde{\Phi}=\begin{bmatrix}\tilde{\Phi}_{11} & \tilde{\Phi}_{12}\\ \tilde{\Phi}_{21} & \tilde{\Phi}_{22}\end{bmatrix},\quad \mathrm{tr}\tilde{\Phi}=0 , \qquad (3.9)$$

while a group element is

$$\tilde{U}=\begin{bmatrix}\tilde{U}_{11} & \tilde{U}_{12}\\ \tilde{U}_{21} & \tilde{U}_{22}\end{bmatrix},\quad \det\tilde{U}=1 . \qquad (3.10)$$

The realization of the Abelian GL(1,R) algebra may also be done by a 2×2 matrix simply as

$$I=\tfrac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} . \qquad (3.11)$$

Then combining both the above algebras one obtains a GL(2,R) realization, and the solutions of the field equations can be succinctly expressed as

$$A=U^{-1}dU , \qquad (3.12)$$

$$H=U^{-1}\Phi U , \qquad (3.13)$$

where $U\in\mathrm{GL}(2,R)$ and Φ is an arbitrary 2×2 real matrix.

Extraction of geometrical information from the above solution is straightforward:

$$e^0=\left[\frac{2}{\Lambda_1}\right]^{1/2}\mathrm{tr}(U^{-1}dU\,\sigma_3)\ ,$$
$$e^1=\left[\frac{2}{\Lambda_1}\right]^{1/2}\mathrm{tr}[U^{-1}dU(-i\sigma_2)]\ , \qquad (3.14)$$
$$\omega=\mathrm{tr}(U^{-1}dU\,\sigma_1)\ ,\quad \tilde{a}=\mathrm{tr}(U^{-1}dU)$$

and

$$\tilde{\eta}_2=\frac{2}{\Lambda_1}\mathrm{tr}(U^{-1}\Phi U\sigma_1),\quad \tilde{\eta}_3=\mathrm{tr}(U^{-1}\Phi U)\ . \qquad (3.15)$$

As an illustration of an explicit solution we construct a simple example [6]. The group and algebra elements we choose are

$$U=\exp\left\{\frac{-i\pi}{2\sqrt{K}}\left[\left[\frac{\Lambda_1}{8}\right]^{1/2}(x^0 i\sigma_2+x^1\sigma_3)-\sigma_1\right]\right\} \qquad (3.16)$$

and

$$\Phi=\left[\frac{2}{\Lambda_1}\right]^{1/2}\left[\alpha_0\frac{i\sigma_2}{2}-\alpha_1\frac{\sigma_3}{2}\right]-\frac{\alpha_2}{2}\sigma_1+\frac{\alpha_3}{2}I\ , \qquad (3.17)$$

where

$$K=1-\frac{\Lambda_1}{8}x^2,\quad x^2=x^0x^0-x^1x^1\ , \qquad (3.18)$$

and α_a, α_2, α_3 are constants. Using (3.14) and (3.15) we obtain

$$e^0=\frac{dx^0}{K},\quad e^1=\frac{dx^1}{K},\quad \omega=\frac{\Lambda_1}{4}(-x^0dx^1+x^1dx^0)\ , \qquad (3.19)$$

$$\tilde{a}=a+\frac{2\xi}{\Lambda_1}\omega=0 \qquad (3.20)$$

and

$$\tilde{\eta}_2=\frac{2}{\Lambda_1}\frac{\alpha_a x^a+\alpha_2\left[1+\frac{\Lambda_1}{8}x^2\right]}{K}\ , \qquad (3.21)$$

$$\tilde{\eta}_3=\frac{-\Lambda_2}{2\xi}=\alpha_3\ . \qquad (3.22)$$

From the relation $\tilde{\eta}_2=\eta_2+(2\xi/\Lambda_1)\tilde{\eta}_3$ we have

$$\eta_2=\frac{2}{\Lambda_1}\left[\frac{\alpha_a x^a+\alpha_2\left[1+\frac{\Lambda_1}{8}x^2\right]}{K}+\xi\alpha_3\right]\ . \qquad (3.23)$$

These solutions reproduce those of the Jackiw-Teiteltoim model by letting $\xi=0$, and also those of a dilaton gravity model by letting $\Lambda_1\to 0$, $\xi=1$ [5,6]. For the latter one must be careful because $\alpha_a,\alpha_2,\alpha_3$ must also banish as $\Lambda_1\to 0$. Taking

$$\alpha_a=\Lambda_1\hat{\alpha}_a,\quad \alpha_2=\Lambda_1\left[\hat{\alpha}_2-\frac{\alpha_3}{\Lambda_1}\right]\ , \qquad (3.24)$$

one can regain the well-known black-hole-type solution

$$-2\eta_2=M-\frac{\Lambda_2}{2}(x-x_0)^2\ . \qquad (3.25)$$

The solutions (3.19) and (3.23) are general solutions to the geometric field equations (2.2) and (2.3) modulo an arbitrary coordinate transformation of this configuration. There are four free parameters $(\alpha_a,\alpha_2,\alpha_3)$ which characterize the geometry: the black-hole mass M, location x_0^a, and the cosmological contant Λ_2.

IV. DISCUSSION

The gauge theoretical framework of (1+1)-dimensional gravity has solutions which have no geometrical sense. For example, the simplest solution is $A=0$, $H=\Phi$, in which case both the zweibein and the connection e^a and ω vanish. But by performing suitable gauge transformations with the group element U, geometrically meaningful solutions can be regained. As Jackiw [6] has pointed out, the relevant geometric information is succinctly encoded in the algebra: Φ carries all the information that characterizes the intrinsic geometry.

The gauge formulations of I_1 and I_2 can be obtained by dimensional reduction from (2+1)-dimensional Chern-Simons models [7]. Both of them arise as different dimensional reductions of a single, extended (2+1)-dimensional gravity [8]. We expect that the combined model I_3 could also be obtainable from a (2+1)-dimensional theory.

Quantization in the gauge formulation was discussed in various works [2]. Since the models have no degrees of freedom it is possible to quantize them, but they lack interesting physics. Therefore it is necessary to couple matter fields in a gauge-invariant form [9], which is not a straightforward problem, and may provide new insight into the black-hole radiation problems when quantized.

Note added in proof. We note that the action I_3 of (1.7) can be obtained from I_1 of (1.1) by shifting $\eta\to\eta+\Lambda_2/\Lambda_1$. In the gauge theory formulation there exist two different ways to realize the shifting of the η field. One is to put Λ_2/Λ_1 in the same representation as η, namely, to transform $(\eta+\Lambda_2/\Lambda_1)$ as J of the SO(2,1) algebra, in which case the result becomes that of Verlinde [3] in the limit $\Lambda_1\to 0$. The other way is to put Λ_2/Λ_1 in a different representation of the extended algebra, which is the case discussed in this paper. We thank Professor R. Jackiw and Professor C. K. Lee for pointing out the relation between I_1 and I_3.

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