Index theory for the nonrelativistic Chern-Simons solitons

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We have demonstrated that the nonrelativistic Chern-Simons solitons have $4n$ parameters by explicit parameter counting, zero-mode calculation, and index theory, where $2n$ is the flux number of the soliton solutions. Despite the presence of the continuum mode, we have shown that the number of zero modes for the nonrelativistic Chern-Simons solitons is the same as the index.

I. INTRODUCTION

Solitons exist in U(1)-invariant gauge theories in $2+1$ space-time dimensions with a Chern-Simons term.[1] In particular it was recently shown that a self-coupled charged scalar field interacting with a pure Chern-Simons gauge field admits a Bogomol'nyi-type[2] lower bound on its energy, which makes it possible to obtain both topological and nontopological soliton solutions.[3,4] Especially in the nonrelativistic limit Jackiw and Pi[4] (JP) were able to obtain exact analytic soliton solutions by reducing the equations to the well-known Liouville equation.

In this work we shall use index theory[5] to count the number of parameters entering the JP general soliton solution with a given flux. The system we are studying is governed by the Hamiltonian

$$H=\int d^2r\left[\frac{1}{2m}(\mathbf{D}\psi)^*(\mathbf{D}\psi)-\frac{e^2}{2mK}(\psi^*\psi)^2\right],\quad K<0\ , \tag{1.1}$$

where ψ is the charged scalar field,

$$\mathbf{D}\psi=(\nabla-e\,\mathbf{A})\psi\ , \tag{1.2}$$

and $\mathbf{A}$ is the vector field which is determined by the Chern-Simons equation

$$B=\nabla\times\mathbf{A}=-\frac{e}{K}\psi^*\psi\equiv-\frac{e}{K}\rho\ . \tag{1.3}$$

The Hamiltonian achieves its minimum when ψ satisfies the static self-dual equation

$$(D_1+iD_2)\psi=0\ . \tag{1.4}$$

Decomposing ψ into phase and amplitude

$$\psi=e^{i\omega}\rho^{1/2}\ , \tag{1.5}$$

the self-dual equation yields

$$e\,\mathbf{A}=\nabla\omega+\tfrac{1}{2}\nabla\times\ln\rho\ . \tag{1.6}$$

This can be combined with Eq. (1.3) to arrive at the Liouville equation

$$\nabla^2\ln\rho=-\alpha\rho,\quad \alpha=\frac{e^2}{|K|}\ , \tag{1.7}$$

which holds away from the zeros of ρ. The soliton-type solutions of these equations have been studied in detail by JP.[4]

In Sec. II we shall explicitly count the degrees of freedom of the soliton solutions presented by JP and show that n-soliton solutions depend on $4n$ parameters, interpreted as $2n$ locations, n scales, and n phases, with one overall phase being irrelevant. In Sec. III we shall use an index theorem to count the parameters abstractly, confirming that $4n$ is indeed the correct number. In Sec. IV we determine the $4n$ zero modes that describe the infinitesimal deformations of the spherically symmetric n-soliton solution. These modes are obtained by solving for the deformations which preserve the self-duality condition. In the last section we discuss subtle points that arise when the index theorem is applied on an open infinite manifold.

II. EXPLICIT PARAMETER COUNTING FOR THE n-SOLITON SOLUTION

The matter density that solves the Liouville equation can be presented in the form[4]

$$\rho(\mathbf{r})=\frac{4}{\alpha}\frac{|f'(z)|^2}{[1+|f(z)|^2]^2}\ ,\qquad z=re^{i\theta}\ , \tag{2.1}$$

where $f(z)$ is an arbitrary analytic function. It is anticipated that an n-soliton solution depends on $4n$ parameters, since each soliton needs two parameters for position: one for scale and one for phase. We can thus expect that the most general solution for $f(z)$ which describes an n-soliton configuration is

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$$f(z)=\frac{a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\cdots+a_2z^2+a_1z+a_0}{(z-z_1)(z-z_2)\cdots(z-z_n)} . \tag{2.2a}$$

Equivalently, we may write

$$f(z)=\sum_{i=1}^{n}\frac{C_i}{z-z_i} . \tag{2.2b}$$

The locations of poles z_i describe the positions and C_i determines the scales and phases of solitons.

One may wonder whether the $f(z)$ given in (2.2) is the most general solution. In fact, a solution that depends on $(4n+2)$ parameters can also describe the n-soliton configuration

$$f(z)=f_0+\sum_{i=1}^{n}\frac{C_i}{z-z_i} . \tag{2.3}$$

However, we can show that the additional two parameters are not independent of the others.

For convenience, we first write $f(z)$ of (2.3) in a different form as

$$f(z)=f_0+\frac{\sum_{k=0}^{n-1}a_kz^k}{\prod_{i=1}^{n}(z-z_i)} . \tag{2.4}$$

If we define new parameters $\tilde{a}_k$ and $\tilde{z}_i$

$$\tilde{a}_k=\frac{a_k}{1+|f_0|^2} ,$$

$$\prod_{i=1}^{n}(z-\tilde{z}_i)=\prod_{i=1}^{n}(z-z_i)+f_0^*\sum_{k=1}^{n-1}\tilde{a}_kz^k ,$$

the $\rho(\mathbf{r})$ can be presented as

$$\rho(\mathbf{r})=\frac{\left[\frac{Y}{x}\right]'\left[\frac{Y}{x}\right]'^{*}}{\left[1+\left[\frac{Y}{x}\right]\left[\frac{Y}{x}\right]^{*}\right]^2} \tag{2.5}$$

where

$$Y=\sum_{k=0}^{n-1}\tilde{a}_kz^k ,$$

$$X=\prod_{i=1}^{n}(z-\tilde{z}_i) .$$

The $\rho(\mathbf{r})$ in (2.5) is the same as that obtained from

$$f(z)=\frac{\sum_{k=1}^{n-1}\tilde{a}_kz^k}{\prod_{i=1}^{n}(z-\tilde{z}_i)} .$$

This is the same form as (2.2a); therefore, f_0 in (2.3) does not introduce new parameters beyond those contained in (2.2). Thus the known n-soliton solutions depend at most on $4n$ parameters. By use of the index theory, we now show that this is indeed the maximum number.

III. PARAMETER COUNTING FROM INDEX THEORY

In this section using index theory we shall count parameters by methods similar to those which have been used in other self-dual systems.[6] If an arbitrary self-dual solution for $\mathbf{A}$ and ψ is given, variation of a parameter yields a set of differential equations for the infinitesimal fluctuation fields. If we require that the fluctuation fields preserve the self-duality equation, the modes must lie in the kernel of a matrix linear differential operator D. One can then determine the number of independent parameters from the dimension of D.

The index is defined as

$$\begin{aligned}I(D)&=\dim(\mathrm{Ker}D)-\dim(\mathrm{Ker}D^*)\\&=\dim(\mathrm{Ker}D^*D)-\dim(\mathrm{Ker}DD^*) .\end{aligned} \tag{3.1}$$

If the adjoint operator D^* has a vanishing kernel, the index is equal to the dimension of Ker$D$.

Infinitesimal fluctuations preserving self-duality satisfy the equations

$$(Di+iD_2)\delta\psi-ie\psi(\delta A^1+i\delta A^2)=0 , \tag{3.2}$$

$$e\nabla\times\delta\mathbf{A}=\alpha\delta(\psi^*\psi)=0 . \tag{3.3}$$

In order to remove gauge degrees of freedom from Eqs. (3.2) and (3.3) we impose a gauge condition. There are two convenient gauges: the Coulomb gauge and the background gauge. In our case it is particularly convenient to work with the Coulomb gauge in order to obtain exact solutions for normalizable zero eigenmodes of D when the background is spherically symmetric. The background gauge condition requires that the fluctuations that we consider be orthogonal to those that are merely gauge transformations whose gauge parameter vanishes at spatial infinity. Our gauge condition is to take

$$e\nabla\cdot\delta\mathbf{A}+\frac{i\epsilon}{2}(\psi^*\delta\psi-\psi\delta\psi^*)=0 . \tag{3.4}$$

The Coulomb gauge and the background gauge correspond to $\epsilon=0$ and $\epsilon=1$ in Eq. (3.4), respectively.

For either value of ϵ, there will be one surviving gauge mode which is not eliminated by condition (3.4), as long as $\psi(\mathbf{r})\xrightarrow[r\to\infty]{}0$. This is the global U(1) mode of the multisoliton solution, and corresponds to the irrelevant, overall phase factor mentioned previously.

If we write $\psi=\psi_1+i\psi_2$, the equations (3.2) through (3.4) take the matrix form

$$D\eta=0 , \tag{3.5}$$

where

$$D=\begin{pmatrix} \nabla_1+eA^2 & -\nabla_2+eA^1 & \psi_2 & \psi_1 \\ \nabla_2-eA^1 & \nabla_1+eA^2 & -\psi_1 & \psi_2 \\ 2\alpha\psi_1 & 2\alpha\psi_2 & \nabla_2 & -\nabla_1 \\ \epsilon\psi_2 & -\epsilon\psi_1 & \nabla_1 & \nabla_2 \end{pmatrix}, \tag{3.6}$$

$$\eta=\begin{pmatrix} \delta\psi_1 \\ \delta\psi_2 \\ e\delta A^1 \\ e\delta A^2 \end{pmatrix}. \tag{3.7}$$

The index of D can be defined in terms of a spatial integral[6] as

$$\hat{I}(D)=\left[\mathrm{Tr}\left[\frac{M^2}{D^*D+M^2}\right]-\mathrm{Tr}\left[\frac{M^2}{DD^*+M^2}\right]\right], \tag{3.8}$$

where "Tr" denotes a functional trace, and M^2 is an arbitrary parameter on which $\hat{I}(D)$ does not depend. It is convenient to calculate $\hat{I}(D)$ in the limit $M^2\to\infty$.

Though Eq. (3.1) requires the index to be integer, the formula (3.8) does not necessarily yield an integer when the problem is considered on an unbounded space, as in our application. In such a space the differential operator D can have a continuous spectrum in addition to normalizable zero eigenmodes.[7,8] If the continuum spectrum is separated by a finite gap from the zero eigenvalues, the continuum does not cause any problem. When $\psi\to 0$ at infinity, the continuum extends to zero and can affect the calculation of the index. Analysis of this, in the case that $\hat{I}(D)$ is nonintegral, indicates that the true index is the highest integer less than $\hat{I}(D)$. However, $\hat{I}(D)$ can come out an integer even on an open space. In that case the index may still be one integer less than $\hat{I}(D)$, depending on the normalizability properties of the zero modes. In our case $\hat{I}(D)$ is an integer, but all relevant modes are normalizable so that $\hat{I}(D)$ coincides with the index. We shall elaborate on this in the last section.

For the calculation of the index, we shall first show that $\mathrm{Ker}D^*=0$. The eigenvalue equation for zero modes of D^* is

$$D^*\phi=\begin{pmatrix} -\nabla_1+eA^2 & -\nabla_1-eA^1 & 2\alpha\psi_1 & \epsilon\psi_2 \\ \nabla_2+eA^1 & -\nabla_1+eA^2 & 2\alpha\psi_2 & -\epsilon\psi_1 \\ \psi_2 & -\psi_1 & -\nabla_2 & -\nabla_1 \\ \psi_1 & \psi_2 & \nabla_1 & -\nabla_2 \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}=0. \tag{3.9}$$

It is straightforward to reduce these four coupled first-order linear differential equations to the following second-order equations:

$$(\nabla^2-\epsilon|\psi|^2)\phi_4=0\ , \tag{3.10}$$

$$(\nabla^2+2\alpha|\psi|^2)\phi_3=0\ , \tag{3.11}$$

$$\phi_1=\frac{1}{|\psi|^2}[(\psi_2\nabla_2-\psi_1\nabla_1)\phi_3+(\psi_1\nabla_2+\psi_2\nabla_1)\phi_4]\ , \tag{3.12}$$

$$\phi_2=\frac{1}{|\psi|^2}[-(\psi_1\nabla_2+\psi_2\nabla_1)\phi_3+(\psi_2\nabla_2-\psi_1\nabla_1)\phi_4]\ . \tag{3.13}$$

Equation (3.10) shows that square-integrable bound-state solutions for ϕ_4 do not exist for $\epsilon=0,1$, and thus $\phi_4=0$. Then ϕ_1 and ϕ_2 are determined by (3.12) and (3.13) from ϕ_3 in (3.11).

For the radially symmetric n-soliton solution which corresponds to $f(z)=C_n z^{-n}$ in (2.1):

$$|\psi|^2=\rho(r)=\frac{4n^2}{\alpha r^2}\frac{1}{\left[\left[\frac{r}{r_0}\right]^n+\left[\frac{r_0}{r}\right]^n\right]^2}\ , \tag{3.14}$$

where $|C_n|^2=r_0^{2n}$. In this case Eq. (3.11) possesses one square-integrable mode:

$$\phi_3=\frac{\left[\frac{r}{r_0}\right]^n}{\left[\frac{r}{r_0}\right]^{2n}+1}\ . \tag{3.15}$$

However, ϕ_1 and ϕ_2 are not square integrable, as can be seen from the expression for the norm density

$$\phi_1^2+\phi_2^2=\frac{1}{\rho}(\nabla\phi_3)^2\ . \tag{3.16}$$

Simple power counting shows that $\phi_1^2+\phi_2^2$ is not integrable with ϕ_3 given in (3.15).

Since $\mathrm{Ker}D^*$ vanishes, the index $\hat{I}(D)$ indeed counts the zero modes if we treat carefully the continuum. We now evaluate the index $\hat{I}(D)$.

It is straightforward to show that

$$D^*D=-\nabla^2 I-L_1\ ,$$

$$DD^*=-\nabla^2 I-L_2\ ,$$

and

$$\mathrm{tr}(L_1-L_2)=4eB\ , \tag{3.17}$$

where "tr" denotes the matrix trace. We now expand the two terms in (3.8) about $M^2/(-\nabla^2+M^2)$ and take the limit $M^2\to\infty$. The result is

$$\hat{I}(D)=\lim_{M\to\infty}\int d^2r\,(4eB)\left\langle x\left|\frac{M^2}{(-\nabla^2+M^2)^2}\right|x\right\rangle\ . \tag{3.18}$$

The factor $\langle x|M^2/(-\nabla^2+M^2)^2|x\rangle$ is easily evaluated in momentum space,

$$\left\langle x\left|\frac{M^2}{(-\nabla^2+M^2)^2}\right|x\right\rangle=\int\frac{d^2p}{(2\pi)^2}\frac{M^2}{(p^2+M^2)^2}=\frac{1}{4\pi}\ . \tag{3.19}$$

The $\hat{I}(D)$ is then

$$\hat{I}(D)=\frac{e}{\pi}\int d^2r\,B=4n\ , \tag{3.20}$$

which is an integer.

IV. EXPLICIT ZERO MODES

In this section, we solve Eq. (3.6) exactly with the radially symmetric n-soliton background. Equation (3.5) implies

$$\delta\psi_1 = -\psi_2\delta\omega + \psi_1\frac{\delta\rho}{2\rho}\ , \tag{4.1a}$$

$$\delta\psi_2 = \psi_1\delta\omega + \psi_2\frac{\delta\rho}{2\rho}\ , \tag{4.1b}$$

$$e\delta A^i = \nabla_i\delta\omega + \tfrac{1}{2}\epsilon^{ij}\nabla_j\frac{\delta\rho}{\rho}\ , \tag{4.1c}$$

where $\delta\omega$ and $\delta\rho/\rho$ are determined by the equations

$$(\nabla^2 + 2\alpha|\psi|^2)\left[\frac{\delta\rho}{\rho}\right] = 0\ , \tag{4.2}$$

$$(\nabla^2 - \epsilon|\psi|^2)\delta\omega = 0\ . \tag{4.3}$$

With the radially symmetric solution for $|\psi|^2$ presented in (3.14) we have to solve the equations

$$\left[\nabla^2 + \frac{v}{r^2}\frac{4n^2}{\left[\left[\frac{r}{r_0}\right]^n + \left[\frac{r_0}{r}\right]^n\right]^2}\right]F = 0\ , \tag{4.4}$$

where

$$v = 2,\quad \text{for } F = \frac{\delta\rho}{\rho};\quad v = -\frac{\epsilon}{\alpha},\quad \text{for } F = \delta\omega\ .$$

Setting

$$F = F_m(A_m\cos m\theta + B_m\sin m\theta) \tag{4.5}$$

and inserting into Eq. (4.4) yields

$$\left[r\frac{d}{dr}r\frac{d}{dr} - m^2 + \frac{4vn^2}{r^n + r^{-n}}\right]F_m = 0\ , \tag{4.6}$$

where r_0 has been scaled to unity. Equation (4.6) can be cast into the hypergeometric equation. Changing variable

$$u = \frac{1}{r^{2n}+1}$$

and setting

$$F = \frac{r^m}{(1+r^{2n})^{m/n}}y_m$$

yields

$$u(u-1)\frac{d^2y_m}{du^2} + \left[2\left[\frac{m}{n}+1\right]u - \left[\frac{m}{n}+1\right]\right]\frac{dy_m}{du} + \left[\frac{m}{n}\left[\frac{m}{n}+1\right] - v\right]y_m = 0\ , \tag{4.7}$$

whose solutions are recognized as hypergeometric functions.

For $v=2$, i.e., for $(\delta\rho/\rho)_m$, solutions become polynomials. We can obtain the solutions for $(\delta\rho/\rho)_m$ and $\delta\omega_m$ in the form

$$(\delta\rho/\rho)_m = A_m f_m(r) + A_{-m}f_{-m}(r)\ , \tag{4.8a}$$

$$\delta\omega_m = B_m F_m(r) + B_{-m}F_{-m}(r)\ , \tag{4.8b}$$

where

$$f_m(r) = \frac{r^m}{r^{2n}+1}[(m+n)+(m-n)r^{2n}]\ , \tag{4.9a}$$

$$F_m(r) = r^m\,{}_2F_1\left[a,b;1-\frac{m}{n};\frac{1}{r^{2n}+1}\right]\ , \tag{4.9b}$$

$$a = \frac{1}{2} - \left[\frac{1}{4} - \frac{\epsilon}{\alpha}\right]^{1/2},\quad b = \frac{1}{2} + \left[\frac{1}{4} - \frac{\epsilon}{\alpha}\right]^{1/2},$$

$$\epsilon = 0 \text{ or } 1\ .$$

[Note that the $F_m(r)$ is real for arbitrary value of α since the properties of hypergeometric functions depend only on the value of ab and $(a+b)$ which are real in our case, $ab=\epsilon/\alpha$ and $a+b=1$.] Taking $\epsilon=0$ in (4.9b) yields the Coulomb gauge solutions

$$\delta\omega_m = A_m r^m + B_m r^{-m}\ . \tag{4.10}$$

$(\delta\rho/\rho)$ can also be obtained directly by deforming the solution of the Liouville equation as given in (2.1):

$$\delta\rho = \partial_z\left[\frac{\rho\delta f}{f'}\right] + \text{c.c.} \tag{4.11}$$

For the spherically symmetric solutions, inserting

$$f = cz^{-n},\quad \delta f = az^N \tag{4.12}$$

into (4.11) immediately yields (4.8a) and (4.9a) for $(\delta\rho/\rho)$.

For $m=0$, $\pm n$, the two sets of independent solutions (f_m, F_m) and (f_{-m}, F_{-m}) in (4.7) coincide, respectively, apart for overall constant factors. In these cases we then need second solutions in order to obtain the most general solutions. We have found them, but they never make the fluctuation fields normalizable, so we rule them out. We may then write the most general solution for $(\delta\rho/\rho)$ in the form

$$\frac{\delta\rho}{\rho} = \delta\rho_0 + \delta\rho_n + \sum_{\substack{m=1\\ m\neq n}}^{\infty}[f_m(r)P_m^{(1)}(\theta) + f_{-m}(r)P_{-m}^{(2)}(\theta)]\ , \tag{4.13}$$

where

$$P_m^{(i)}(\theta)=A_m^{(i)}\cos m\theta+B_m^{(i)}\sin m\theta\ , \tag{4.14}$$

$$\delta\rho_0=A_0 f_0(r)\ , \tag{4.15}$$

$$\delta\rho_n=f_n(r)(A_n\cos n\theta+B_n\cos n\theta)\ . \tag{4.16}$$

For later use, we record the asymptotic behaviors of $f_m(r)$:

$$f_{m\neq n}(r)\underset{r\to 0}{\sim}(m+n)r^m\left[1-\frac{2n}{m+n}r^{2n}+\cdots\right]\to r^m$$
$$\underset{r\to\infty}{\sim}(m-n)r^m\left[1+\frac{2n}{m-n}r^{-2n}+\cdots\right]\to r^m\ , \tag{4.17}$$

$$f_{-m\neq -n}(r)\underset{r\to 0}{\sim}(n-m)r^{-m}\left[1-\frac{2n}{n-m}r^{2n}+\cdots\right]\to r^{-m}$$
$$\underset{r\to\infty}{\sim}(-m-n)r^{-m}\left[1-\frac{2n}{m+n}r^{-2n}+\cdots\right]\to r^{-m}\ , \tag{4.18}$$

$$f_{m=n}(r)\underset{r\to 0}{\sim}2nr^n(1-r^{2n}+\cdots)\to r^n$$
$$\underset{r\to\infty}{\sim}2nr^{-n}(1-r^{-2n}+\cdots)\to r^{-n}\ . \tag{4.19}$$

Note that the asymptotic behavior of $f_{m=n}(r)$ at large r switches from its behavior at the origin, while the $f_{m\neq n}(r)$'s behave in the same manner at the origin and at infinity. This switching in $f_{m=n}$ is the reason why the continuum modes do not contribute to our evaluation of the index. This point will be discussed in detail in the next section.

Before writing down the most general solution for $\delta\omega$, we first define a set of new function $\bar{F}_m(r)$'s:

$$\bar{F}_m(r)\equiv C_2^{-m}F_{-m}(r)-C_1^m F_m\ , \tag{4.20}$$

where

$$C_1^m=\frac{\Gamma\left[\frac{m}{n}+1\right]\Gamma\left[\frac{m}{n}\right]}{\Gamma\left[\frac{m}{n}+a\right]\Gamma\left[\frac{m}{n}+b\right]}\ ,$$
$$C_2^m=\frac{\Gamma\left[\frac{m}{n}+1\right]\Gamma\left[-\frac{m}{n}\right]}{\Gamma(a)\Gamma(b)}\ . \tag{4.21}$$

The most general solution for $\delta\omega$ can be written in the form

$$\delta\omega=\delta\omega_0+\delta\omega_n+\sum_{\substack{m=1\\ m\neq n}}^{\infty}[\bar{F}_m(r)Q_m^{(1)}(\theta)+F_{-m}(r)Q_m^{(2)}(\theta)]\ , \tag{4.22}$$

where

$$Q_m^{(i)}(\theta)=C_m^{(i)}\cos m\theta+D_m^{(i)}\sin m\theta\ , \tag{4.23}$$

$$\delta\omega_0=C_0F_0(r)\ , \tag{4.24a}$$

$$\delta\omega_n=F_n(r)(C_n\cos n\theta+D_n\cos n\theta)\ . \tag{4.24b}$$

The types of asymptotic behavior of $\bar{F}_m(r)$ and $F_{-m}(r)$ are

$$\bar{F}_m(r)\underset{r\to 0}{\sim}(C_2^mC_2^{-m}-C_1^mC_1^{-m})r^m[1+O(r^{2n-2m})]\to r^m$$
$$\underset{r\to\infty}{\sim}-C_1^m r^m[1+O(r^{-2n})]\to r^m\ , \tag{4.25}$$

$$F_{-m}\underset{r\to 0}{\sim}r^{-m}[C_1^m+C_2^m r^{2m}+O(r^{2n+2m})]\to r^{-m}$$
$$\underset{r\to\infty}{\sim}r^{-m}\left[1+\frac{ab}{1+\frac{m}{n}}r^{-2n}+O(r^{-4n})\right]\to r^{-m}\ . \tag{4.26}$$

From the square integrability of the fluctuation fields

$$\int d^2r[(e\delta A^i)^2+(\delta\psi_1)^2+(\delta\psi_2)^2]$$
$$=\int d^2r\left\{(1+2\alpha)\rho\left[\frac{\delta\rho}{2\rho}\right]^2+\nabla_i\left[\left[\delta\omega\delta^{ij}-\frac{\delta\rho}{2\rho}\epsilon^{ij}\right]\left[\nabla_j\delta\omega+\frac{1}{2}\epsilon^{jk}\nabla_k\frac{\delta\rho}{\rho}\right]\right]\right\}<\infty\ , \tag{4.27}$$

and from the asymptotic behavior of the relevant functions, we obtain the conditions, for $1\le m\le n-1$, $Q_m^{(i)}$ and $P_m^{(i)}$ must be matched as

$$P_m^{(i)}=(A_m^{(i)}\cos m\theta+B_m^{(i)}\sin m\theta)\ ,$$
$$Q_m^{(i)}=(-B_m^{(1)}\cos m\theta+A_m^{(1)}\sin m\theta)\ , \tag{4.28}$$
$$Q_m^{(2)}=(B_m^{(2)}\cos m\theta-A_m^{(2)}\sin m\theta)\ ,$$

for $m=0$ the constants A_0 and C_0 defined in Eqs. (4.15) and (4.24) may remain arbitrary, for $m=n$ two constants A_n and B_n can be arbitrary while C_n and D_n must be set to zero. We thus have $4n$ independent parameters: $4n-4$ from $1\le m\le n-1$, 2 from $m=0$, and 2 from $m=n$. Note that our parameter counting includes the global U(1) mode, which is seen from Eq. (4.24a) where the $\delta\omega_0$ does not vanish at infinity.

One interesting point in our zero-mode calculation is that the maximum angular momentum state ($m=n$) is included as a normalizable eigenmode. This is in contrast with related studies of zero modes on the plane[7,8] where the highest angular momentum state is unnormalizable and produces a mismatch between the index and number of zero modes.

V. DISCUSSIONS AND CONCLUSIONS

The Atiyah-Singer index theory[5] requires that the manifold on which the differential operators act is compact. In our case the manifold is a two-dimensional Euclidean space, and thus the operator D has a continuous spectrum in addition to the discrete set of eigenvalues. This continuum extends to zero when $\phi \underset{r\to\infty}{\longrightarrow} 0$ and affects the calculation of the index, which now requires a subtle treatment.

Since the continuum correction arises from the zero eigenvalue end of the continuum spectrum we can evaluate it by studying the asymptotic behavior of Eq. (3.5). Neglecting all terms which fall faster than $1/r^2$ yields the simple equation

$$(D_1+iD_2)\delta\psi=0 \;, \tag{5.1}$$

where we set $\delta A_1=\delta A_2=0$. Its normalizable solutions for $\delta\psi$ agree with the asymptotic part of the full solutions given in (4.13) except the highest angular momentum mode, the $m=n$ case. The exact solutions include this state as a normalizable mode, while the asymptotic Eq. (5.1) leaves it as an unnormalizable one.

The asymptotic equation (5.1) and its adjoint equation

$$(-D_1+iD_2)\delta\psi^*=0 \tag{5.2}$$

are essentially equivalent to the equations studied by Ansourian[7] and Kiskis.[8] The solutions of our asymptotic equations agree completely with theirs including the unnormalizable mode ($m=n$ state) in the asymptotic behavior. They found that the index is larger than the number of normalizable solutions by one unit when the index is an integer. In our case the mode in question is normalizable in the full equation (3.5). Therefore the number of normalizable zero modes agrees with the index.

By comparing the Coulomb gauge solution for $\delta\omega$ given in (4.10) with the background gauge solution given in (4.8b) we see that the computations and results are much simpler in the Coulomb gauge, while the physically relevant information such as the asymptotic behavior and the number of normalizable zero modes are the same in both gauge conditions.

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