



# Bethe states for the two-site Bose–Hubbard model: A binomial approach



Gilberto Santos<sup>a,\*</sup>, Changrim Ahn<sup>b</sup>, Angela Foerster<sup>c</sup>, Itzhak Roditi<sup>a</sup>

<sup>a</sup> Centro Brasileiro de Pesquisas Físicas – CBPF, Rua Dr. Xavier Sigaud, 150, Urca, Rio de Janeiro, RJ, Brazil

<sup>b</sup> Department of Physics, Ewha Womans University, DaeHyun 11-1, Seoul 120-750, South Korea

<sup>c</sup> Instituto de Física, Universidade Federal do Rio Grande do Sul, Av. Bento Gonçalves, 9500, Agronomia, Porto Alegre, RS, Brazil

## ARTICLE INFO

### Article history:

Received 26 March 2015

Accepted 1 May 2015

Available online 5 May 2015

Editor: L. Alvarez-Gaumé

### Keywords:

Bethe ansatz

Bethe vectors

Form factors

Bose–Einstein condensation

## ABSTRACT

We calculate explicitly the Bethe vectors states by the algebraic Bethe ansatz method with the  $gl(2)$ -invariant  $R$ -matrix for the two-site Bose–Hubbard model. Using a binomial expansion of the  $n$ -th power of a sum of two operators we get and solve a recursion equation. We calculate the scalar product and the norm of the Bethe vectors states. The form factors of the imbalance current operator are also computed.

© 2015 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP<sup>3</sup>.

## 1. Introduction

The first experimental verification of the Bose–Einstein condensation (BEC) [1–3] occurred after a gap of more than seven decades following its theoretical prediction [4,5]. After its realization a great deal of progress has taken place both in the theoretical and experimental study of this physical phenomenon [6–13]. A particularly fruitful instrument in relation to ultracold physics are many-body atomic models related to BEC. In this direction the quantum inverse scattering method (QISM) [14–18] has been used to solve and study some prototypical many-body models that contribute to describe phenomena associated to BEC [19–21]. Some of these models, despite their simplicity, display a rich structure showing quantum phase transitions and interesting semi-classical behavior that have been studied in [22–27], and explored in different areas such as nuclear physics, condensed matter and atomic-molecular physics. To keep things as simple as possible we shall consider here the two-site Bose–Hubbard, also known in special cases as the canonical Josephson Hamiltonian [7]. This model may be viewed as a particular case of the bosonic multi-state two-well model studied in [21], and can be used to describe a quadrupolar nuclei system in nuclear magnetic resonance [28] by a  $N/2$  Schwinger pseudo-spin realization of the Hamiltonian. Conversely,

there is also a link with the Schwinger bosonic realization of the Lipkin–Meshkov–Glick model [29,30] used to study closed shells in a nuclei model. In spite, of course, from being a two-site specialization of the Bose–Hubbard model it is a very useful model in various realms as the understanding of tunneling phenomena using two BEC [31–37], as well as quantum phase transitions using tools of quantum computation and quantum information. The model is described by the Hamiltonian

$$\hat{H} = \frac{K}{8}(\hat{N}_1 - \hat{N}_2)^2 - \frac{\Delta\mu}{2}(\hat{N}_1 - \hat{N}_2) - \frac{\mathcal{E}_J}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), \quad (1)$$

where,  $\hat{a}_1^\dagger$ ,  $\hat{a}_2^\dagger$ , denote the single-particle creation operators in each site and,  $\hat{N}_1 = \hat{a}_1^\dagger \hat{a}_1$ ,  $\hat{N}_2 = \hat{a}_2^\dagger \hat{a}_2$ , are the corresponding boson number operators. The total boson particles number operator,  $\hat{N} = \hat{N}_1 + \hat{N}_2$ , is a conserved quantity,  $[\hat{H}, \hat{N}] = 0$ . The coupling  $K$  provides the interaction strength between the bosons and is proportional to the  $s$ -wave scattering length,  $\Delta\mu$  is the external potential and  $\mathcal{E}_J$  is the amplitude of tunneling.

The Hamiltonian (1) is integrable in the sense that it can be solved by the quantum inverse scattering method (QISM) and it has been discussed in different ways using this method [37–45]. The algebraic formulation of the Bethe ansatz, associated to the QISM, was primarily developed in [14–18].

A very important problem in the algebraic Bethe ansatz method is the construction of the Bethe vectors states (BVS) [38,46,47] using the correspondent creation operator applied to the pseudo-vacuum. Employing this form of the BVS it is possible [47] to cal-

\* Corresponding author.

E-mail address: [gfilho@cbpf.br](mailto:gfilho@cbpf.br) (G. Santos).

culate their scalar product and then use it to calculate important physical quantities as the form factors. Form factors are defined as the matrix entries of operators in the base of the eigenvectors of the Hamiltonian. Another important application is in the calculation of the average values of the operators as for example correlation operators. Applying this method, some physical quantities for the Hamiltonian (1) were obtained in [38]. Recently, BVS have shown to be useful in fundamental issues of planar  $\mathcal{N} = 4$  super Yang–Mills (SYM) theory [48,49] in the context of the integrability in the AdS/CFT correspondence [50].

In the present work, we develop and use a new method to explicitly calculate the BVS and obtain the scalar product of two BVS for the two-site Bose–Hubbard model. Although we concentrate on this model the procedure is of general applicability. We use a Lax operator to construct a realization of the monodromy matrix and get an algebraic identity between the associated  $C$ -operator and the  $D$ -operator of the monodromy matrix (see next section) needed to calculate the BVS. We then use the binomial expansion for the  $n$ -th power of the sum of two operators: in a first step we will show that the binomial expansion can be written as a sum of permutations of the product of that operators or as a standard binomial expansion as in a commutative algebra plus a function of the commutator of these two operators; in a second step we write a recursion equation and give its solution. Next, we calculate the scalar product between one on-shell and one generic off-shell BVS, as well as the norm. As an application we obtain the form factors (non-normalized) for the imbalance current operator.

## 2. The algebraic Bethe ansatz method

The spectrum of the Hamiltonian (1) has appeared in different papers [37–40,42–45] using this method. To fix notation we will shortly describe the algebraic Bethe ansatz method, see [39,40,51] for more details. We begin with the  $gl(2)$ -invariant  $R$ -matrix, depending on the spectral parameter  $u$ ,

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2)$$

with  $b(u) = u/(u + \eta)$ ,  $c(u) = \eta/(u + \eta)$  and  $b(u) + c(u) = 1$ . Above,  $\eta$  is an arbitrary parameter, to be chosen later.

It is easy to check that  $R(u)$  satisfies the Yang–Baxter equation

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v), \quad (3)$$

where  $R_{jk}(u)$  denotes the matrix acting non-trivially on the  $j$ -th and the  $k$ -th spaces and as the identity on the remaining space.

Next we define the monodromy matrix  $\hat{T}(u)$ ,

$$\hat{T}(u) = \begin{pmatrix} \hat{A}(u) & \hat{B}(u) \\ \hat{C}(u) & \hat{D}(u) \end{pmatrix}, \quad (4)$$

such that the Yang–Baxter algebra is satisfied

$$R_{12}(u - v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{12}(u - v). \quad (5)$$

To obtain a solution for the two-site Bose–Hubbard model (1) we need to choose a realization for the monodromy matrix  $\pi(\hat{T}(u)) = \hat{L}(u)$ . In this construction, the Lax operator  $\hat{L}(u)$  has to satisfy the algebra

$$R_{12}(u - v)\hat{L}_1(u)\hat{L}_2(v) = \hat{L}_2(v)\hat{L}_1(u)R_{12}(u - v), \quad (6)$$

where we use the standard notation.

Then, defining the transfer matrix, as usual, through

$$\hat{t}(u) = \text{Tr} \pi(\hat{T}(u)) = \pi(\hat{A}(u) + \hat{D}(u)), \quad (7)$$

it follows from (5) that the transfer matrix commutes for different values of the spectral parameter.

We are using the well-known [19] Lax operator, solution of Eq. (6),

$$\hat{L}_i(u) = \begin{pmatrix} u\hat{I} + \eta\hat{N}_i & \hat{a}_i \\ \hat{a}_i^\dagger & \eta^{-1}\hat{I} \end{pmatrix} \quad i = 1, 2, \quad (8)$$

for the boson operators  $\hat{a}_i^\dagger$ ,  $\hat{a}_i$ , and  $\hat{N}_i$ . These operators obey the standard canonical boson commutation rules.

Using the co-multiplication property of the Lax operators (8) we get the following realization for the monodromy matrix,

$$\pi(\hat{T}(u)) = \hat{L}_1(u + \omega)\hat{L}_2(u - \omega), \quad (9)$$

whose entries are,

$$\begin{aligned} \pi(\hat{A}(u)) &= (u^2 - \omega^2)\hat{I} + \eta u\hat{N} \\ &\quad + \eta^2\hat{N}_1\hat{N}_2 - \eta\omega(\hat{N}_1 - \hat{N}_2) + \hat{a}_2^\dagger\hat{a}_1, \end{aligned} \quad (10)$$

$$\pi(\hat{B}(u)) = (u + \omega + \eta\hat{N}_1)\hat{a}_2 + \eta^{-1}\hat{a}_1, \quad (11)$$

$$\pi(\hat{C}(u)) = (u - \omega + \eta\hat{N}_2)\hat{a}_1^\dagger + \eta^{-1}\hat{a}_2^\dagger, \quad (12)$$

$$\pi(\hat{D}(u)) = \hat{a}_1^\dagger\hat{a}_2 + \eta^{-2}\hat{I}. \quad (13)$$

Hereafter we will use the same symbol for the operators and its respective realization, so we define  $\pi(\hat{O}(u)) \equiv \hat{O}(u)$  for any operator in the entries of the monodromy matrix (4).

The parameters of the Hamiltonian (1) are all real numbers,  $K, \Delta\mu, \mathcal{E}_j \in \mathbb{R}$ . The parameters in the operators (10), (11), (12), (13) can be complex numbers,  $u, \eta, \omega \in \mathbb{C}$ , but in this case the transfer matrix is not Hermitian. We will only consider the Hermitian case.

We can apply the algebraic Bethe ansatz method, using the Fock vacuum as the pseudo-vacuum  $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$ , to find the BAE,

$$\eta^2(v_i^2 - \omega^2) = \prod_{j \neq i}^N \frac{v_i - v_j - \eta}{v_i - v_j + \eta}, \quad i, j = 1, \dots, N. \quad (14)$$

## 3. Bethe vectors states

In the algebraic Bethe ansatz method, the BVS are constructed by the application of the  $\hat{C}$ -operator to the pseudo-vacuum  $|0\rangle$ ,

$$|\Psi\rangle = \prod_{j=1}^N \hat{C}(v_j) |0\rangle, \quad (15)$$

where the  $\{v_j\}_1^N$  are solutions of the BAE (14).

Using the  $\hat{D}$ -operator (13) we can write the  $\hat{C}$ -operator (12) as

$$\hat{C}(v_j) = f_j \hat{a}_1^\dagger + \hat{\mathcal{D}}, \quad (16)$$

where

$$f_j \equiv v_j - \omega, \quad (17)$$

and

$$\hat{\mathcal{D}} \equiv \eta \hat{a}_2^\dagger \hat{D} = \eta \hat{a}_1^\dagger \hat{N}_2 + \hat{a}_2^\dagger \eta^{-1}. \quad (18)$$

Now we can write the BVS in the product form as

$$|\Psi\rangle = \prod_{j=1}^N [f_j \hat{a}_1^\dagger + \hat{\mathcal{D}}] |0\rangle, \quad (19)$$

or in the summation form as

$$|\Psi\rangle = \sum_{n=0}^N \mathcal{F}_n (\hat{a}_1^\dagger)^{N-n} \hat{D}^n |0\rangle, \quad (20)$$

with the identification

$$\mathcal{F}_0 = \prod_{j=1}^N f_j, \quad \mathcal{F}_N = \sum_{\substack{j_1, j_2, \dots, j_N=1 \\ (j_1 > j_2 > \dots > j_N)}}^N \frac{\mathcal{F}_0}{f_{j_1} f_{j_2} \dots f_{j_N}}. \quad (21)$$

To explicitly write the BVS (19) or (20) we need to expand the powers of the  $\hat{D}$ -operator (18). We thus consider the following  $n$ -th power binomial expansion for any two operators  $\hat{X}$  and  $\hat{Y}$ , with  $n \geq 2$ , proved by induction,

$$\begin{aligned} (\hat{X} + \hat{Y})^n &= \sum_{j=0}^n \sum_{\alpha=1}^{\frac{n!}{j!(n-j)!}} \hat{P}_\alpha (\hat{X}^j \hat{Y}^{n-j}) \\ &= \sum_{j=0}^n \binom{n}{j} \hat{X}^{n-j} \hat{Y}^j + f([\hat{X}, \hat{Y}]), \end{aligned} \quad (22)$$

with

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}, \quad (23)$$

where the  $\hat{P}_\alpha$ -operator stand for the set of all permutations of the  $\hat{X}$  and  $\hat{Y}$  operators, not respecting the commutation rule between them, and  $f([\hat{X}, \hat{Y}])$  is a function of that commutation rule. Below we show  $f([\hat{X}, \hat{Y}])$  for two values of  $n$ :

I - For  $n = 2$ :

$$f([\hat{X}, \hat{Y}]) = [\hat{Y}, \hat{X}]. \quad (24)$$

II - For  $n = 3$ :

$$f([\hat{X}, \hat{Y}]) = [\hat{Y}, \hat{X}^2] + [\hat{Y}^2, \hat{X}] + [\hat{X}\hat{Y}, \hat{X}] + [\hat{Y}, \hat{X}\hat{Y}]. \quad (25)$$

If the operators commute,  $f([\hat{X}, \hat{Y}]) = 0$ , we clearly get the standard commutative binomial formula. In this case the  $n$ -th power of the  $\hat{D}$ -operator (13) is given by

$$\hat{D}^n = \sum_{j=0}^n \binom{n}{j} (\hat{a}_1^\dagger \hat{a}_2)^{n-j} \eta^{-2j}. \quad (26)$$

The  $n$ -th power of the  $\hat{D}$ -operator (26) is acting in the pseudo-vacuum as

$$\hat{D}^n |0\rangle = \eta^{-2n} |0\rangle. \quad (27)$$

Redefining the  $\hat{D}$ -operator (18) as

$$\hat{D} = \hat{X} + \hat{Y}, \quad (28)$$

with

$$\hat{X} = \eta \hat{a}_1^\dagger \hat{N}_2, \quad \hat{Y} = \eta^{-1} \hat{a}_2^\dagger, \quad (29)$$

we get the binomial expansion of the  $\hat{D}$ -operator (18) for the  $n$ -th power,

$$\hat{D}^n = \sum_{j=0}^n \sum_{\alpha=1}^{\frac{n!}{j!(n-j)!}} \hat{P}_\alpha (\hat{X}^j \hat{Y}^{n-j}). \quad (30)$$

Applying the  $n$ -th power of the  $\hat{D}$ -operator (30) to the pseudo-vacuum we get

$$\hat{D}^n = \sum_{j=0}^n C_{n,j} |j\rangle_1 \otimes |n-j\rangle_2, \quad (31)$$

where the coefficients  $C_{n,j}$  satisfies the recursion equation

$$C_{n+1,j} = \eta \sqrt{j} (n+1-j) C_{n,j-1} + \eta^{-1} \sqrt{n+1-j} C_{n,j}, \quad (32)$$

with the condition  $C_{j,j} = 0$ .

The solution of the recursion equation (32) is

$$C_{n,j} = \eta^{2j-n} \frac{j!}{(n-j)!} \sum_{l=0}^{n-j} (-1)^{n+l-j} l^n \binom{n-j}{l}. \quad (33)$$

Finally, using the binomial expansion of the  $\hat{D}$ -operator (31) we can write the BVS (19) as

$$\begin{aligned} |\Psi\rangle &= \mathcal{F}_0 \sqrt{N!} |N\rangle_1 \otimes |0\rangle_2 \\ &+ \sum_{n=1}^N \sum_{j=0}^n \mathcal{F}_n C_{n,j} \sqrt{\frac{(N-n+j)!}{j!}} \\ &\times |N-n+j\rangle_1 \otimes |n-j\rangle_2. \end{aligned} \quad (34)$$

The scalar product between one on-shell and one generic off-shell BVS (34),  $|\Psi\rangle$  and  $|\tilde{\Psi}\rangle$ , with the set of solutions of the BAE (14)  $\{v_j\}_1^N$  and the generic set  $\{\tilde{v}_j\}_1^N$  [52], is

$$\begin{aligned} \langle \tilde{\Psi} | \Psi \rangle &= N! \tilde{\mathcal{F}}_0^* \mathcal{F}_0 \\ &+ \sum_{r,n=1}^N \sum_{j=0}^n \tilde{\mathcal{F}}_r^* \mathcal{F}_n C_{r,r-n+j} C_{n,j} \\ &\times \frac{(N-n+j)!}{\sqrt{j!(r-n+j)!}}. \end{aligned} \quad (35)$$

From the scalar product (35) we can write the norm of the BVS (34),

$$\begin{aligned} \langle \Psi | \Psi \rangle &= N! |\mathcal{F}_0|^2 + \sum_{r,n=1}^N \sum_{j=0}^n \mathcal{F}_r^* \mathcal{F}_n C_{r,r-n+j} C_{n,j} \\ &\times \frac{(N-n+j)!}{\sqrt{j!(r-n+j)!}}. \end{aligned} \quad (36)$$

Because the total number of atoms is a conserved quantity all the BVS (34) are eigenfunctions of the  $\hat{N}$ -operator with the same eigenvalue  $N$ , and so they are degenerate states for this operator. As  $\hat{N}_1$  and  $\hat{N}_2$  are not conserved quantities the BVS are not eigenfunction of these operators but we can still write the form factors of these operators using the BVS (34). For instance, the non-normalized form factors of the imbalance current between the two BEC is written as

$$\begin{aligned} \langle \tilde{\Psi} | \frac{\hat{N}_1 - \hat{N}_2}{N} | \Psi \rangle &= N! \tilde{\mathcal{F}}_0^* \mathcal{F}_0 + \sum_{r,n=1}^N \sum_{j=0}^n \tilde{\mathcal{F}}_r^* \mathcal{F}_n \\ &\times C_{r,r-n+j} C_{n,j} \left[ 1 - \frac{2(n-j)}{N} \right] \\ &\times \frac{(N-n+j)!}{\sqrt{j!(r-n+j)!}}. \end{aligned} \quad (37)$$

#### 4. Summary

We have explicitly written the Bethe vectors states (BVS) by the algebraic Bethe ansatz method using the  $gl(2)$ -invariant  $R$ -matrix and an algebraic relation between the  $\hat{C}$  and the  $\hat{D}$  operators. We use a binomial expansion of the  $n$ -th power of the sum of two operators obtained from that algebraic relation to write a recursion equation and solve it. The binomial expansion of the  $n$ -th power of the sum of two operators can be written as a commutative binomial expansion plus a function of the commutator of the operators. We calculate the scalar product and the norm of those BVS. The BVS are degenerate eigenfunctions of the total number of particles  $\hat{N}$ -operator with eigenvalue  $N$ . As an example of application of the BVS we calculate the form factors for the imbalance current operator.

#### Acknowledgements

The authors acknowledge CAPES/FAPERJ (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior/Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro), CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) and the Ewha Womans University by the grant WCU No. R32-2008-000-101300 for the financial support.

#### References

- [1] J.R. Anglin, W. Ketterle, *Nature* 416 (2002) 211.
- [2] M.H. Anderson, J.R. Ensher, M.R. Mathews, C.E. Wieman, E.A. Cornell, *Science* 269 (1995) 198.
- [3] J. Williams, R. Walser, J. Cooper, E.A. Cornell, M. Holland, *Phys. Rev. A* 61 (2000) 0336123.
- [4] S.N. Bose, *Z. Phys.* 26 (1924) 178.
- [5] A. Einstein, *Phys. Math.* K1 22 (1924) 261.
- [6] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, S. Stringari, *Rev. Mod. Phys.* 71 (1999) 463.
- [7] A.J. Leggett, *Rev. Mod. Phys.* 73 (2001) 307.
- [8] P.W. Courteille, V.S. Bagnato, V.I. Yukalov, *Laser Phys.* 11 (2001) 659.
- [9] E.A. Cornell, C.E. Wieman, *Rev. Mod. Phys.* 74 (2002) 875.
- [10] E.A. Donley, N.R. Claussen, S.T. Thompson, C.E. Wieman, *Nature* 417 (2002) 529.
- [11] A.F.R.T. Piza, *Braz. J. Phys.* 34 (3B) (2004) 1102.
- [12] I. Bloch, J. Dalibard, W. Zwerger, *Rev. Mod. Phys.* 80 (2008) 875.
- [13] I. Carusotto, C. Ciuti, *Rev. Mod. Phys.* 85 (2013) 299.
- [14] L.D. Faddeev, E.K. Sklyanin, L.A. Takhtajan, *Theor. Math. Phys.* 40 (1979) 194.
- [15] P.P. Kulish, E.K. Sklyanin, *Integrable Quantum Field Theories*, in: J. Hietarinta, C. Montonen (Eds.), *Proceedings of the Symposium Held at Tvärminne, March 23–27, 1981, Finland*, in: *Lecture Notes in Physics*, vol. 151, Springer-Verlag Berlin Heidelberg, Berlin, 1982, p. 61.
- [16] L.A. Takhtajan, *Quantum groups*, in: H.-D. Doebner, J.-D. Hennig (Eds.), *Proceedings of the 8th International Workshop on Mathematical Physics Held at the Arnold Sommerfeld Institute, Clausthal, FRG, on 19–26 July 1989*, in: *Lecture Notes in Physics*, vol. 370, Springer-Verlag Berlin Heidelberg, Berlin, 1990, p. 3.
- [17] V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, Cambridge, 1993.
- [18] L.D. Faddeev, *Int. J. Mod. Phys. A* 10 (1995) 1845.
- [19] G. Santos, A. Foerster, I. Roditi, Z.V.T. Santos, A.P. Tonel, *J. Phys. A, Math. Theor.* 41 (2008) 295003.
- [20] G. Santos, *J. Phys. A, Math. Theor.* 44 (2011) 345003.
- [21] G. Santos, A. Foerster, I. Roditi, *J. Phys. A, Math. Theor.* 46 (2013) 265206.
- [22] G. Santos, A. Tonel, A. Foerster, *J. Links, Phys. Rev. A* 73 (2006) 023609.
- [23] A.P. Tonel, C.C.N. Kuhn, G. Santos, A. Foerster, I. Roditi, Z.V.T. Santos, *Phys. Rev. A* 79 (2009) 013624.
- [24] G. Santos, A. Foerster, J. Links, E. Mattei, S.R. Dahmen, *Phys. Rev. A* 81 (2010) 063621.
- [25] M. Duncan, A. Foerster, J. Links, E. Mattei, N. Oelkers, A. Tonel, *Nucl. Phys. B* 767 (2007) 227.
- [26] L. Zhou, W. Zhang, H.Y. Ling, L. Jiang, H. Pu, *Phys. Rev. A* 75 (2007) 043603.
- [27] L. Zhou, J. Qian, H. Pu, W. Zhang, H.Y. Ling, *Phys. Rev. A* 78 (2008) 053612.
- [28] A.G. Araujo-Ferreira, R. Auccaise, R.S. Sarthour, I.S. Oliveira, T.J. Bonagamba, I. Roditi, *Phys. Rev. A* 87 (2013) 053605.
- [29] H.J. Lipkin, N. Meshkov, A.J. Glick, *Nucl. Phys.* 62 (1965) 188.
- [30] H. S. Lerma, J. Dukelsky, *Nucl. Phys. B* 870 (2013) 421.
- [31] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, M.K. Oberthaler, *Phys. Rev. Lett.* 95 (2005) 010402.
- [32] G.J. Milburn, J. Corney, E.M. Wright, D.F. Walls, *Phys. Rev. A* 55 (1997) 4318.
- [33] A.P. Hines, R.H. McKenzie, G.J. Milburn, *Phys. Rev. A* 67 (2003) 013609.
- [34] A.P. Tonel, J. Links, A. Foerster, *J. Phys. A, Math. Gen.* 38 (2005) 6879.
- [35] A.P. Tonel, J. Links, A. Foerster, *J. Phys. A, Math. Gen.* 38 (2005) 1235.
- [36] A.P. Hines, R.H. McKenzie, G.J. Milburn, *Phys. Rev. A* 71 (2005) 042303.
- [37] J. Links, A. Foerster, A.P. Tonel, G. Santos, *Ann. Henri Poincaré* 7 (2006) 1591.
- [38] J. Links, H.-Q. Zhou, *Lett. Math. Phys.* 60 (2002) 275.
- [39] J. Links, H.-Q. Zhou, R.H. McKenzie, M.D. Gould, *J. Phys. A, Math. Gen.* 36 (2003) R63.
- [40] A. Foerster, J. Links, H.-Q. Zhou, in: A. Kundu (Ed.), *Classical and Quantum Non-linear Integrable Systems: Theory and Applications*, Institute of Physics Publishing, Bristol and Philadelphia, 2003, pp. 208–233.
- [41] A. Foerster, E. Ragoucy, *Nucl. Phys. B* 777 (2007) 373.
- [42] J. Links, K.E. Hibberd, *SIGMA* 2 (2006) 095.
- [43] J. Links, I. Marquette, *J. Phys. A, Math. Theor.* 48 (2015) 045204.
- [44] D. Rubeni, A. Foerster, E. Mattei, I. Roditi, *Nucl. Phys. B* 856 (2012) 698.
- [45] A.P. Tonel, L.H. Ymai, *J. Phys. A, Math. Theor.* 46 (2013) 125202.
- [46] P.P. Kulish, N. Manojlovic, *Lett. Math. Phys.* 55 (2001) 77.
- [47] N.A. Slavnov, *Theor. Math. Phys.* 79 (1989) 502.
- [48] J. Escobedo, N. Gromov, A. Sever, P. Vieira, *J. High Energy Phys.* 09 (2011) 028.
- [49] N. Gromov, P. Vieira, *Phys. Rev. Lett.* 111 (2013) 211601.
- [50] N. Beisert, C. Ahn, et al., *Lett. Math. Phys.* 99 (2012) 3.
- [51] I. Roditi, *Braz. J. Phys.* 30 (2000) 357.
- [52] S. Belliard, S. Pakuliak, E. Ragoucy, N.A. Slavnov, *J. Stat. Mech.* (2012) P10017.