New Brans-Dicke wormholes

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Two new classes of exact solutions in vacuum Brans-Dicke theory are obtained, each of which is a two-way traversable wormhole for the coupling parameter $\omega < -2$ or $-2 < \omega \leqslant 0$, respectively. Each of the two new classes of exact solutions satisfies not only the general constraints given by Morris and Thorne [Am. J. Phys. **56**, 395 (1988)], as concluded earlier, but also the constraints from a trip through a wormhole. It also follows that the scalar field ϕ plays the role of exotic matter violating the weak energy condition.

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Wormholes are topological handles in space-time linking widely separated regions of a single universe or "bridges" joining two different space-times. Interest in these configurations dates back at least as far as 1916 [1] with punctuated revivals of activity following both the work of Einstein and Rosen in 1935 [2] and the later series of works initiated by Wheeler in 1955 [3]. Recently wormhole physics has gradually drawn people's attention [4–9] as has the analysis of classical traversable wormholes for interstellar travel performed by Morris and Thorne [4]. If an advanced civilization could construct such wormholes, they could be used as a galactic or intergalactic transportation system and they might also be usable for backward time travel [4]. Morris and Thorne, and other people, have found traversable wormhole solutions within the framework of Einstein's general relativity theory (GRT). It is known that GRT can be recovered in the limiting case $\omega \to \infty$ of the Brans-Dicke theory (BDT). It seems natural that in the context of wormhole physics, one looks for wormhole solutions of BDT. The search for static wormhole geometry in BDT has been initiated by Agnese and La Camera [5]. They show that a static spherically symmetric Brans-Dicke (BD) solution, obtained in a certain gauge by Krori and Bhattacharjee [6], supports a two-way traversable wormhole for $\omega < -2$ and one way for $\omega > -3/2$. Nandi *et al.* [7] also show that three of the I–IV classes of Brans-Dicke (BD) solutions obtained by Brans [8] support a two-way traversable wormhole for $\omega < -2$ and $0 < \omega < \infty$. However, they only show that these three solutions satisfy the general constraints on the shape function $b(R)$ and the redshift function $\Phi(R)$, but not the constraints on them from a trip through the wormhole [4].

In this paper, we wish to obtain two new classes of exact solutions in vacuum BD theory and show that each of the two new classes of exact solutions satisfies not only the general constraints on the shape function $b(R)$ and the redshift function $\Phi(R)$, as concluded earlier, but also the constraints on them from a trip through a wormhole, to represent a two-way traversable wormhole. It will also be apparent that the presence of the BD scalar field ϕ cannot prevent weak energy condition violation, showing that it is not a consequence of the GRT alone.

The BD field equations are

$$(\phi^{;\rho})_{;\rho} = \frac{8\pi}{3+2\omega} T^{\mu}_{\mu}, \tag{1}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi}{\phi} T_{\mu\nu} - \frac{\omega}{\phi^2}\left[\phi_{;\mu}\phi_{;\nu} - \frac{1}{2} g_{\mu\nu}\phi_{;\rho}\phi^{;\rho}\right] - \frac{1}{\phi}[\phi_{;\mu;\nu} - g_{\mu\nu}(\phi^{;\rho})_{;\rho}], \tag{2}$$

where $T_{\mu\nu}$ is the matter energy-momentum tensor excluding the ϕ field, ω is a dimensionless coupling parameter. The general metric, in spherically symmetric coordinates, is given by $(G=c=1)$

$$d\tau^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\nu(r)}r^2[d\theta^2 + \sin^2\theta d\varphi^2]. \tag{3}$$

Our solutions are subject to the gauge $\beta - \nu = 0$. From Eqs. (1) and (2), we get the field equations in spherically symmetric coordinates:

$$\phi e^{-2\beta}\left[2\beta'' + \frac{4\beta'}{r} + (\beta')^2\right] = e^{-2\beta}\left[\phi'\alpha' - \frac{\omega(\phi')^2}{2\phi}\right] + 8\pi\left(T^0_0 - \frac{T}{2\omega+3}\right), \tag{4}$$

$$\phi e^{-2\beta}\left[2\alpha'\beta' + (\beta')^2 + \frac{2(\alpha'+\beta')}{r}\right]$$

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$$=e^{-2\beta}\left[\phi''-\phi'\beta'+\frac{\omega(\phi')^2}{2\phi}\right]+8\pi\left(T_1^1-\frac{T}{2\omega+3}\right), \tag{5}$$

$$\phi e^{-2\beta}\left[\alpha''+(\alpha')^2+\frac{\alpha'+\beta'}{r}+\beta''\right]$$
$$=e^{-2\beta}\left[\phi'\beta'+\frac{\phi'}{r}-\frac{\omega(\phi')^2}{2\phi}\right]+8\pi\left(T_2^2-\frac{T}{2\omega+3}\right) \tag{6}$$

$$e^{-2\beta}\left[\phi''+\phi'\left(\alpha'+\beta'+\frac{2}{r}\right)\right]=\frac{8\pi T}{2\omega+3} \tag{7}$$

$$T^{\beta}_{\alpha;\beta}=0, \tag{8}$$

where α', α'', β' and β'', respectively, denote $d\alpha/dr$, $d^2\alpha/dr^2$, $d\beta/dr$ and $d^2\beta/dr^2$. For the vacuum case, $T_\alpha^\beta=0$. Equations (4)–(8) are resultant coupled nonlinear differential equations for which it is very difficult to obtain exact solutions. Fortunately, solving these equations, we obtain two new classes of exact solutions.

New class I solutions are given by

$$\alpha(r)=\alpha_0, \tag{9}$$

$$\beta(r)=\beta_0-2C\arctan\left(\frac{r}{B}\right)-\ln\left(\frac{r^2}{r^2+B^2}\right), \tag{10}$$

$$\phi(r)=\phi_0 e^{2C\arctan(r/B)}, \tag{11}$$

$$C^2\equiv-\left(\frac{\omega+2}{2}\right)^{-1}>0, \tag{12}$$

where α_0, β_0, B, C and ϕ_0 are integration constants. The constants α_0 and β_0 are determined by the asymptotic flatness condition as follows:

$$\alpha_0=0, \qquad \beta_0=\pi C. \tag{13}$$

In order to investigate whether a given solution represents a wormhole geometry, it is convenient to cast the metric (3) into Morris-Thorne canonical form:

$$d\tau^2=-e^{2\Phi(R)}dt^2+\left[1-\frac{b(R)}{R}\right]^{-1}dR^2+R^2[d\theta^2+\sin^2\theta d\varphi^2], \tag{14}$$

where $\Phi(R)$ and b(R) are called the redshift and shape functions, respectively. These functions are required to satisfy some constraints, given by Morris and Thorne [4], in order to represent a wormhole. However it is important to stress that the choice of coordinates by Morris-Thorne is purely a matter of convenience and not a physical necessity [9]. Redefining the radial coordinate as

$$R=r\left(1+\frac{B^2}{r^2}\right)\exp\left[1-\frac{2}{\pi}\arctan\left(\frac{r}{B}\right)\right]\beta_0, \tag{15}$$

comparing Eq. (3) and Eq. (14), we obtain the functions $\Phi(R)$ and $b(R)$ as

$$\Phi(R)=\alpha_0=0, \tag{16}$$

$$b(R)=R\left\{1-\left[1-\frac{2B[Cr(R)+B]}{r^2(R)+B^2}\right]^2\right\}. \tag{17}$$

The throat of the wormhole occurs at $R=R_0$ such that $b(R_0)=R_0$. This gives minimum allowed r-coordinate radii as

$$r_0^{\pm}=BC\left[1\pm\left(1+\frac{1}{C^2}\right)^{1/2}\right]. \tag{18}$$

The values $R_0^{\pm}$ can be obtained from Eq. (15) using this $r_0^{\pm}$. Noting that $R\to\infty$ as $r\to\infty$, we find that $b(R)/R\to 0$ as $R\to\infty$. The function $\Phi(R)$ is zero everywhere, so no horizon exists, and $\Phi(R)\to 0$ as $R\to\infty$. In order for a wormhole to be two-way traversable, we should have $r_0^{\pm}>0$. Therefore, we have $BC>0$ for $r_0^{+}>0$ or $BC<0$ for $r_0^{-}>0$. Equation (12) requires $\omega<-2$.

The axially symmetric embedded surface $z=z(R)$ shaping the wormhole's spatial geometry is obtained from

$$\frac{dz}{dR}=\pm\left[\frac{R}{b(R)}-1\right]^{-1/2}. \tag{19}$$

At the value $R=R_0$ (the wormhole throat) Eq. (19) is divergent, which means that the embedded surface is vertical there. For a coordinate-independent description of wormhole physics, one may use proper length l instead of R such that

$$l=\pm\int_{R_0^{\pm}}^{R}\frac{dR}{[1-b(R)/R]^{1/2}}. \tag{20}$$

In the present case,

$$l=\pm\int_{r_0^{\pm}}^{r}e^{\beta(r)}dr. \tag{21}$$

It can be seen that $l\to\pm\infty$ as $r\to\pm\infty$.

Consider a spaceship traveling radially through the wormhole with its propulsion power shut off [4], beginning at rest in a space station in the lower universe, at $l=-l_1$, and ending at rest in a space station in the upper universe, at $l=+l_2$. From Eq. (16), we get $\Phi=0$ everywhere, which corresponds to precisely zero tidal force as seen by stationary observers.

The constraints that gravity is weak at $-l_1$ and l_2 are [4]

$$b/R\ll 1, \Delta\lambda/\lambda=e^{-\Phi}-1\approx|\Phi|\ll 1,$$

$$g=-\left(1-\frac{b}{R}\right)^{1/2}\frac{d\Phi}{dR}\approx\left|\frac{d\Phi}{dR}\right|\leqslant g_{\oplus} \tag{22}$$

$$\text{at } l=-l_1 \text{ and } l=l_2,$$

where λ represents wavelength, $g_{\oplus}=980$ cm/s^2 (in cgs units) $\approx 1/0.97(1y)$ (in units $G=c=1$). Because $|\Phi|\ll 1$ at the stations, the proper time ticked by clocks there is equal to coordinate time t; cf. the space-time metric (14). Notice that $|\Phi|=0\ll 1$ and $g=-(1-b/R)^{1/2}d\Phi/dR=0\ll 1$. The constraints in Eq. (22) are all satisfied if we locate the two space stations at large enough radii that the factor $[1-b(R)/R]$ differs from unity by only 1%. That is corresponding to $l_1=l_2\approx 10^4R_0$, where we take the radial location of the two stations to be $R_1=R_2=10^4R_0$.

The acceleration that the traveler in the spaceship feels is [4]

$$a=\mp\left(1-\frac{b}{R}\right)^{1/2}e^{-\Phi}\frac{d}{dR}(\gamma e^{\Phi})\frac{d}{dl}(\gamma e^{\Phi}), \tag{23}$$

where $\gamma\equiv(1-\nu^2)^{-1/2}$ (in our units $G=c=1$). The traveler does not feel an acceleration larger than about 1 Earth gravity, which corresponds to

$$\left|e^{-\Phi}\frac{d}{dl}(\gamma e^{\Phi})\right|\leqslant g_{\oplus}\approx\frac{1}{0.97(1y)}. \tag{24}$$

Equation (23) tells us that, since travelers in the spaceship feel no acceleration (since a=0), the spaceship must travel with constant γe^{Φ}. For the zero-tidal-force solutions ($\Phi=0$), this corresponds to constant $\gamma=(1-\nu^2)^{-1/2}$ and hence to constant speed $\nu=dl/dt$ as measured by stationary observers,

$$\nu=\frac{dl}{dt}=\text{const, for an unpowered spaceship.} \tag{25}$$

The acceleration constraint (24) is trivially satisfied since $\Phi=0$ and γ keeps constant for the trip.

The constraints of radial tidal acceleration and lateral tidal acceleration can be written, respectively, as

$$\left|\left(1-\frac{b}{R}\right)\left[-\frac{d^2\Phi}{dR^2}+\frac{Rdb/dR-b}{2R(R-b)}\frac{d\Phi}{dR}-\left(\frac{d\Phi}{dR}\right)^2\right]\right|$$
$$\leqslant\frac{g_{\oplus}}{2m}\approx\frac{1}{(10^{10}\ \text{cm})^2}, \tag{26}$$

$$\left|\frac{\gamma^2}{2R^2}\left[\nu^2\left(\frac{db}{dR}-\frac{b}{R}\right)+2(R-b)\frac{d\Phi}{dR}\right]\right|$$
$$\leqslant\frac{g_{\oplus}}{2m}\approx\frac{1}{(10^{10}\ \text{cm})^2}, \tag{27}$$

where $2m$ is the size of the traveler's body. The radial tidal constraint (26) is satisfied as $\Phi=0$ everywhere. We are thus left with constraint (27) limiting the tidal forces associated with motion through the tunnel:

$$\frac{\gamma^2\nu^2}{2R^2}\left|\frac{db}{dR}-\frac{b}{R}\right|\leqslant\frac{1}{(10^{10}\ \text{cm})^2}. \tag{28}$$

Substituting Eqs. (15) and (17) into the above equation yields

$$\frac{2\gamma^2\nu^2}{r^3}\left|BC+\frac{2B^2}{r}-\frac{B^3C}{r^2}\right|e^{-2\pi C[1-(2/\pi)\arctan(r/B)]}$$
$$\times\left(1+\frac{B^2}{r^2}\right)^{-4}\leqslant\frac{1}{(10^{10}\ \text{cm})^2}. \tag{29}$$

This constraint is most severe for the smallest radius $r=r_0^{\pm}$ (at the throat). We choose $r=r_0^{+}$ to estimate the magnitude of travel velocity ν. Supposing $|C|>1$, Eq. (18) gives $r_0^{+}\approx 2BC$. Computing Eq. (29) approximately gives

$$\gamma\nu\leqslant 2\times 10^{-10}A,$$
$$A^2\equiv B^2C^2\left(1+\frac{1}{4C^2}\right)^4e^{2\pi C[1-(2/\pi)\arctan(2C)]}$$
$$\times\left(1+\frac{3}{4C^2}\right)^{-1}, \tag{30}$$

where A is measured in units of cm. In the limit that the motion is nonrelativistic ($\gamma\approx 1$) we obtain (in cgs units)

$$\nu\leqslant 6A\ \ \text{cm/s}. \tag{31}$$

Correspondingly, the total time lapse (in cgs units) for travel from station 1 to station 2 [4] is the same (since $\gamma\approx 1$, $\Phi=0$) for clocks ticking in the stations and on board the spaceship:

$$\Delta\tau_T\approx\Delta t\approx\int_{-l_1}^{l_2}\frac{dl}{\nu}\approx 2\times 10^4\frac{R_0}{\nu}$$
$$\approx\frac{2\times 10^4}{3}\left(1+\frac{1}{4C^2}\right)^{-1}\left(1+\frac{3}{4C^2}\right)^{1/2}\quad\text{s}. \tag{32}$$

The energy density ρ_ϕ of the scalar field ϕ is obtained by computing the Einstein tensor G_{00} such that

$$G_{00}=-\frac{1}{8\phi}(T_\phi)_{00}=\frac{\rho_\phi}{8\phi}=\frac{1}{R^2}\frac{db}{dR}. \tag{33}$$

From Eqs. (15) and (17), we obtain

$$\frac{db}{dR}=-\frac{4B^2r^2(C^2+1)}{(r^2+B^2)^2}. \tag{34}$$

From the above equation, we obtain $db/dR\leqslant 0$. This implies $\rho_\phi\leqslant 0$, with ϕ everywhere non-negative. This shows that the scalar field ϕ plays the role of exotic matter at the wormhole throat and there is consequently a violation of the weak energy condition.

New class II solutions are given by

$$\alpha(r)=\alpha_0, \tag{35}$$

$$e^{\beta(r)}=e^{\beta_0}(1+B/r)^2\left(\frac{1-B/r}{1+B/r}\right)^{1-C}, \tag{36}$$

$$\phi(r)=\phi_0\left(\frac{1-B/r}{1+B/r}\right)^{C}, \tag{37}$$

$$C^2\equiv\left(\frac{\omega+2}{2}\right)^{-1}>0, \tag{38}$$

where α_0,β_0,B,C and ϕ_0 are integration constants. The constants α_0 and β_0 are determined by the asymptotic flatness condition as follows:

$$\alpha_0=0, \qquad \beta_0=0. \tag{39}$$

With the same method as in the new class I solutions, in order to investigate whether the new class II solutions represent a wormhole geometry, it is convenient to cast the metric (3) into Morris-Thorne canonical form [see Eq. (14)]. Redefining the radial coordinate as

$$R=re^{\beta_0}(1+B/r)^2\left(\frac{1-B/r}{1+B/r}\right)^{1-C}, \tag{40}$$

comparing Eq. (3) and Eq. (14), we obtain the functions $\Phi(R)$ and $b(R)$ as

$$\Phi(R)=\alpha_0=0, \tag{41}$$

$$b(R)=R\left\{1-\left[\frac{r^2(R)-2BCr(R)+B^2}{r^2(R)-B^2}\right]^2\right\}. \tag{42}$$

The throat of the wormhole occurs at $R=R_0$ such that $b(R_0)=R_0$. This gives minimum allowed r-coordinate radii as

$$r_0^{\pm}=BC\left[1\pm\left(1-\frac{1}{C^2}\right)^{1/2}\right]. \tag{43}$$

The values $R_0^{\pm}$ can be obtained from Eq. (40) using this $r_0^{\pm}$. Noting that $R\to\infty$ as $r\to\infty$, we find that $b(R)/R\to 0$ as $R\to\infty$. The function $\Phi(R)$ is zero everywhere, so no horizon exists, and $\Phi(R)\to 0$ as $R\to\infty$. In order for a wormhole to be two-way traversable, we should have $r_0^{\pm}>0$. If $r_0^{\pm}>0$, then $R_0^{\pm}>0$. From $r_0^{\pm}>0$ and using Eq. (43), we obtain that $C^2\geqslant 1$ and $BC>0$. Substituting $C^2\geqslant 1$ into Eq. (38) yields $-2<\omega\leqslant 0$.

The energy density ρ_ϕ of the scalar field ϕ is obtained by computing the Einstein tensor G_{00} such that

$$G_{00}=-\frac{1}{8\phi}(T_\phi)_{00}=\frac{\rho_\phi}{8\phi}=\frac{1}{R^2}\frac{db}{dR}. \tag{44}$$

From Eqs. (42) and (40), we obtain

$$\frac{db}{dR}=-\frac{4r^2B^2}{(r^2-B^2)^2}(C^2-1). \tag{45}$$

From the above equation, we obtain $db/dR\leqslant 0$. This implies $\rho_\phi\leqslant 0$, with ϕ everywhere non-negative. This also shows that the scalar field ϕ plays the role of exotic matter at the wormhole throat, which violates the weak energy condition.

Using the same method as in the new class I solutions, we can easily see that the new class II solutions satisfy all the other constraints proposed by Morris and Thorne [4].

It was shown in the foregoing that two new classes of exact solutions to vacuum BD field equations are obtained and each of them gives rise to a two-way traversable wormhole provided the constants are chosen appropriately. Each of the two new classes of exact solutions satisfies not only the general constraints on the shape function $b(R)$ and the redshift function $\Phi(R)$, but also the constraints on them from a trip through a wormhole, and represents a two-way traversable wormhole. It was also shown that the presence of the BD scalar field ϕ cannot prevent the weak energy condition violation.

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