Published for SISSA by Springer

Received: March 29, 2023
Accepted: April 20, 2023
Published: May 5, 2023

Quantization of Jackiw-Teitelboim gravity with a massless scalar

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Abstract: We study canonical quantization of Jackiw-Teibelboim (JT) gravity coupled to a massless scalar field. We provide concrete expressions of matter $\mathrm{SL}(2,\mathbf{R})$ charges and the boundary matter operators in terms of the creation and annihilation operators in the scalar field. The matter charges are represented in the form of an oscillator (Jordon-Schwinger) realization of the $\mathrm{SL}(2,\mathbf{R})$ algebra. We also show how the gauge constraints are implemented classically, by matching explicitly classical solutions of Schwarzian dynamics with bulk solutions. We introduce n-point transition functions defined by insertions of boundary matter operators along the two-sided Lorentzian evolution, which may fully spell out the quantum dynamics in the presence of matter. For the Euclidean case, we proceed with a two-sided picture of the disk geometry and consider the two-sided 2-point correlation function where initial and final states are arranged by inserting matter operators in a specific way. For some simple initial states, we evaluate the correlation function perturbatively. We also discuss some basic features of the two-sided correlation functions with additional insertions of boundary matter operators along the two-sided evolution.

Keywords: 2D Gravity, AdS-CFT Correspondence, Black Holes, Models of Quantum Gravity

ArXiv ePrint: 2303.05057

https://doi.org/10.1007/JHEP05(2023)045

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Contents

1 Introduction

Recently there are considerable interests in Jackiw-Teitelboim (JT) gravity in two dimensions with a negative cosmological constant [1, 2], which may serve as a simple model for quantum gravity (see [3–5] for reviews). In this model, there are no local dynamical degrees of freedom in the bulk while all the gravity dynamics are fully reflected in the boundary fluctuations of cutoff trajectories; these boundary (particle) dynamics are well-known to be described by the Schwarzian theories [6–8]. In addition to the Euclidean path integral approach [9], this boundary picture allows the canonical analysis in Lorentzian setup [10]. This analysis may be extended to the case of JT gravity including matter as far as the matter field does not couple directly to the dilaton field [11].

The relevant AdS_2 geometry in Lorentzian signature is intrinsically two-sided involving left and right cutoff trajectories near the AdS_2 boundary (see figure 1). In the context of AdS/CFT correspondence, it is rather natural to expect that the dual boundary theory has a description based on a tensor product structure of left and right theories. Hence it appears that one is able to construct a one-sided Hilbert space out of JT gravity in a natural manner. On the contrary, it has been argued that the quantized version of JT theory allows only a two-sided Hilbert space $\mathcal{H}$ whereas a one-sided Hilbert space cannot be defined [10, 12], which is coined as the *factorization problem* in JT gravity [10]. There

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are also related issues in higher dimensions on how to understand the behind-horizon interactions from the viewpoint of $\mathrm{CFT}_l \otimes \mathrm{CFT}_r$, which was emphasized in [13].

A recent observation says that the algebra of (bulk) operators $\mathcal{A}$ acting on a two-sided Hilbert space $\mathcal{H}$ can be constructed in such a way that two one-sided algebras $\mathcal{A}_l$ and $\mathcal{A}_r$ are well defined and commute with each other preserving the causality at the level of operator algebra [14–18] (see also [19], which appeared near completion of our work). It is noticeable that, even at the level of the algebra, the algebra $\mathcal{A}$ is not a tensor product of $\mathcal{A}_l$ and $\mathcal{A}_r$. In the JT gravity with matter, it is shown that the type of von Neumann algebra is type II_∞ and the corresponding algebra $\mathcal{A}_{l/r}$ is fully specified by the boundary Hamiltonian $H_{l/r}$ and the boundary matter operator $\hat{\varphi}_{l/r}$ derived from the bulk matter operator [11].

The Schwarzian theories involve higher derivative terms. However there is no inconsistency since these higher derivative terms are constrained by a gauge symmetry of $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ which leaves the full geometry including the cutoff boundaries invariant. By imposing the corresponding gauge constraints with an appropriate gauge-fixing, the quantization of JT theory with and without matter has been carried out in [11], which will be reviewed in the following. Upon quantization, the reduced Hilbert space exhibits a genuinely two-sided nature while the Hamiltonian $H_{l/r}$ generates the left and right time evolution respectively. From the viewpoint of this two-sided Hilbert space, the full dynamics of the system may be described by the total Hamiltonian $H_r + H_l$ with a single time parameter u evolving the left and right at the same time. In pure JT gravity, $H_r - H_l$ vanishes identically and merely induces a pure gauge transformation. In the presence of matter, $H_r - H_l$ becomes nontrivial and generates a relative (boostlike) time evolution.

For JT theory with a massless scalar field specifically, its full general solutions are presented explicitly in [20], which show some nontrivial aspects of two-sided black holes involving the matter field. For instance, the left and right temperatures of two-sided black holes become different from each other. Indeed, JT theory with matter seems to exhibit many more nontrivial features. In this note, we consider the explicit canonical quantization of JT gravity coupled to a massless scalar field. Based on explicit bulk expressions, we shall provide the concrete expressions of matter $\mathrm{SL}(2,\mathbf{R})$ charges and the boundary matter operators $\hat{\varphi}_{l/r}$. We also provide a comparison of the classical and quantum dynamics with some comments on the classical realization of the gauge constraints.

In [21], it was shown that the disk partition function of pure JT gravity [9] can be reproduced from an evaluation based on the two-sided picture (see the left diagram of figure 2); there one starts the two-sided evolution from an initial geodesic curve connecting two slightly separated boundary points in the bottom region of the disk and ends up with a final geodesic curve between two slightly separated boundary points in the top region. We generalize the computation to the case of JT theory with the massless scalar field, in which one may additionally arrange initial and final states including the matter part by inserting matter operators before and after the initial and final regularized curves (see the right diagram of figure 2). In this note, we specify prescribed states[1] at the initial and final

[1]The Hartle-Hawking construction on a half disk [19] may be a good alternative for preparing the initial or final state. However, it is not clear to us how to land there starting from our Lorentzian two-sided picture.

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regularized curves generalizing the proposal in [21]. One may additionally insert boundary matter operators along the two-sided evolution, which leads to higher two-sided correlation functions. We investigate basic properties of these two-sided correlation functions.

This paper is organized as follows. In section 2, we give our basic setup of JT gravity. In section 3, we review the canonical quantization of JT gravity with matter following [11]. In section 4 , we specialize in JT gravity with a massless scalar field and provide full details of quantization. Especially we quantize the bulk scalar field leading to explicit expressions of the matter $\mathrm{SL}(2,\mathbf{R})$ charges. We find that the matter charges J_i^m form the oscillator (Jordan-Schwinger) realization of $\mathrm{SL}(2,\mathbf{R})$. In the case of the massless scalar especially, the mapping is given in terms of the matrices of $D^-_{j=1}$ representation. In section 5, we consider the classical solutions of Schwarzian dynamics and its relation to bulk solutions. We also check the gauge constraints in the classical setup. In section 6, we consider two-sided correlation functions in the presence of the matter field. We present some explicit evaluations of the two-sided correlation functions. In the final section, we summarize our results and give some comments on future directions.

2 Jackiw-Teitelboim gravity with matter

We shall consider JT gravity [1, 2, 22] coupled to a matter field which is described by action[2]

$$I = \frac{1}{2}\int_M d^2x\sqrt{-g}\,\phi\,(R+2) + I_{\text{surf}} + I_m(g,\varphi)\,, \tag{2.1}$$

where ϕ is for a dilaton field, φ for the matter field, and

$$\begin{aligned} I_{\text{surf}} &= \int_{\partial M} du\sqrt{-\gamma_{uu}}\,\phi\,(K-1)\,, \\ I_m &= -\frac{1}{2}\int_M d^2x\sqrt{-g}\,\left(g^{ab}\nabla_a\varphi\nabla_b\varphi + m^2\varphi^2\right)\,. \end{aligned} \tag{2.2}$$

Here, u is our boundary time coordinate and γ_{uu} and K respectively denote the induced metric and the extrinsic curvature on the boundary ∂M. The equation of motion following from the dilaton variation is given by

$$R+2=0\,, \tag{2.3}$$

which fixes the metric to be AdS_2. The remaining equations of motion read

$$\nabla_a\nabla_b\phi - g_{ab}\nabla^2\phi + g_{ab}\phi = -T_{ab}\,, \tag{2.4}$$

$$\nabla^2\varphi - m^2\varphi = 0\,, \tag{2.5}$$

where T_{ab} is the stress tensor of the matter field,

$$T_{ab} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}\left(g^{cd}\nabla_c\varphi\nabla_d\varphi + m^2\varphi^2\right)\,. \tag{2.6}$$

[2]Here, we have omitted a topological term which is irrelevant in our discussion below. We also set $8\pi G = 1$ and the AdS radius $\ell = 1$.

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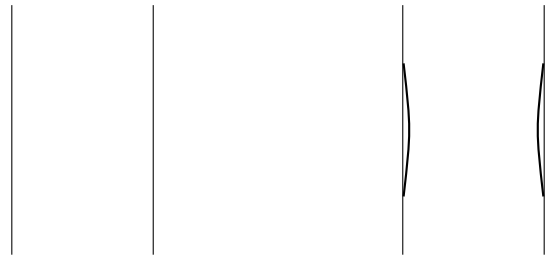

Figure 1. On the left we draw the global AdS_2 as a strip where the left and right lines denote the $\mu = -\pi/2, \pi/2$ boundaries of AdS_2, respectively. On the right we illustrate the left and right cutoff trajectories as curves near the AdS_2 boundaries.

In the global coordinates, the metric of the AdS_2 space is written as

$$ds^2 = \frac{-d\tau^2 + d\mu^2}{\cos^2\mu}\,, \tag{2.7}$$

where $\mu \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ which is strip-shaped as depicted on the left of figure 1. A vacuum solution of the dilaton field (in a gauge-fixed form) is given by

$$\phi = \bar{\phi}\, L\, \frac{\cos\tau}{\cos\mu}\,, \tag{2.8}$$

which is describing a two-sided black hole spacetime that is left-right symmetric.

As is well known, in this 2d gravity theory, there are no local dynamical gravity degrees of freedom in the bulk and all pure gravity dynamics are fully reflected in the boundary fluctuations of cutoff trajectories of AdS_2. For this, one introduces the cutoff trajectories $(\tau_{r/l}(u), \mu_{r/l}(u))$ parametrized by the boundary time u for the right and left cutoff boundaries. See the right diagram of figure 1. The prescription for metric and dilaton to get the cutoff boundary becomes

$$ds^2|_{\text{cutoff}} = -\frac{1}{\epsilon^2}du^2\,, \qquad \phi|_{\text{cutoff}} = \frac{\bar{\phi}}{\epsilon}\,, \tag{2.9}$$

and the corresponding boundary dynamics may be identified as a combination of Schwarzian theories [6–8],

$$S = \int du\, L_r + \int du\, L_l\,, \qquad L_{r/l} = \frac{\mathcal{C}}{2}\left[\left(\frac{\tau''_{r/l}}{\tau'_{r/l}}\right)^2 - \tau'^2_{r/l}\right], \tag{2.10}$$

where the total derivative terms are dropped and the coupling $\mathcal{C}$ may be identified with $\bar{\phi}$ that appears in the vacuum solution (2.8).

For each Schwarzian Lagrangian, one may follow a standard procedure in higher derivative theory by adding Lagrange multiplier terms $p_{\tau_{r/l}}(\tau'_{r/l} - e^{\chi_{r/l}}/\mathcal{C})$ to the Lagrangian where $p_{\tau_{r/l}}$ work as Lagrange multipliers at this stage. Using the multiplier equations of motion, the above may be rewritten as

$$L_{r/l} = \frac{\mathcal{C}}{2}\chi'^2_{r/l} - \frac{1}{2\mathcal{C}}e^{2\chi_{r/l}} + p_{\tau_{r/l}}\left(\tau'_{r/l} - \frac{1}{\mathcal{C}}e^{\chi_{r/l}}\right)\,. \tag{2.11}$$

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By a further Legendre transform with canonical momenta $p_{\chi_{r/l}}$ conjugated to $\chi_{r/l}$, one finds

$$L_{r/l} = p_{\tau_{r/l}} \tau'_{r/l} + p_{\chi_{r/l}} \chi'_{r/l} - H_{r/l} \,, \tag{2.12}$$

with the following left and right Hamiltonians [10, 12, 23]

$$H_{r/l} = \frac{1}{2\mathcal{C}} \left[p^2_{\chi_{r/l}} + 2 p_{\tau_{r/l}} \, e^{\chi_{r/l}} + e^{2\chi_{r/l}} \right] . \tag{2.13}$$

The linear dependence of $H_{r/l}$ in $p_{\tau_{r/l}}$ tells us that $H_{r/l}$ are not bounded from below, which may be viewed as an indication of instability of the system. This is, of course, the well-known aspect of higher derivative theory. However, in the present case, there would be a gauge symmetry described in detail below, which ensures the total Hamiltonian becomes positive on physical Hilbert space [11, 12].

As reviewed in [24], the AdS_2 space has an $\mathrm{SL}(2,\mathbf{R})$ symmetry under the isometric coordinate transformations that are generated by Killing vectors

$$\begin{aligned} \xi_1 &= -\partial_\tau \,, \\ \xi_2 &= -\cos\tau \sin\mu \, \partial_\tau - \sin\tau \cos\mu \, \partial_\mu \,, \\ \xi_3 &= -\sin\tau \sin\mu \, \partial_\tau + \cos\tau \cos\mu \, \partial_\mu \,. \end{aligned} \tag{2.14}$$

Each of the above left-right boundary systems then possesses $\mathrm{SL}(2,\mathbf{R})$ symmetry under the transformations that are induced by the bulk $\mathrm{SL}(2,\mathbf{R})$ transformations along the left-right cutoff boundaries. By the standard Noether procedure, the corresponding (quantum) $\mathrm{SL}(2,\mathbf{R})$ generators may be constructed as [12]

$$\begin{aligned} J_1^{r/l} &= p_{\tau_{r/l}} \,, \\ J_2^{r/l} &= \pm e^{\chi_{r/l}} \cos\tau_{r/l} \mp \sin\tau_{r/l} \, p_{\chi_{r/l}} \pm \cos\tau_{r/l} \, p_{\tau_{r/l}} \pm \frac{i}{2} \sin\tau_{r/l} \,, \\ J_3^{r/l} &= \pm e^{\chi_{r/l}} \sin\tau_{r/l} \pm \cos\tau_{r/l} \, p_{\chi_{r/l}} \pm \sin\tau_{r/l} \, p_{\tau_{r/l}} \mp \frac{i}{2} \cos\tau_{r/l} \,, \end{aligned} \tag{2.15}$$

where the upper/lower signs are for the right/left quantities respectively. These generators satisfy the $\mathrm{SL}(2,\mathbf{R})$ algebra, $[J_i^{r/l}, J_j^{r/l}] = i\epsilon_{ijk}\eta^{kl} J_l^{r/l}$, where ϵ_{ijk} is a totally antisymmetric symbol with $\epsilon_{123} = 1$ and $\eta^{ij} = \mathrm{diag}(-1,1,1)$. It is then straightforward to check that

$$2\mathcal{C} H_{r/l} = \eta^{ij} J_i^{r/l} J_j^{r/l} - \frac{1}{4} \,, \tag{2.16}$$

which corresponds to the quadratic Casimir of $\mathrm{SL}(2,\mathbf{R})$ and so ensures the $\mathrm{SL}(2,\mathbf{R})$ invariance of the Hamiltonians.

Now, by turning on the bulk matter field, the corresponding boundary flux along the cutoff boundaries may be in general nonvanishing and the equations of motion along the boundaries are modified as [6]

$$\mathcal{C} \left\{ \tan \frac{\tau_{r/l}(u)}{2}, u \right\}' = \tau'^2_{r/l} \, T_{\tau\mu}\big|_{r/l} \,, \tag{2.17}$$

where the Schwarzian derivative is defined by $\{f(u), u\} \equiv -1/2(f''/f')^2 + (f''/f')'$ and the last r/l denote the evaluation of the stress tensor at the right/left cutoff boundary, respectively. Thus, with this nonvanishing boundary flux, the boundary Lagrangians have to be modified accordingly through explicit coupling to the bulk matter field along the cutoff trajectories. In the context of the AdS/CFT correspondence, however, one imposes the vanishing boundary condition

$$\varphi\big|_{r/l} = \mathcal{O}(\cos^{\Delta}\mu_{r/l}) = \mathcal{O}(\epsilon^{\Delta}) \tag{2.18}$$

where Δ denotes the dimension of the operator dual to the bulk matter field. In this paper, we shall limit our consideration to the matter field with the vanishing boundary condition (2.18) whose details will be further provided below. One is led to the vanishing boundary flux along the cutoff trajectories in the $\varepsilon \to 0$ limit. Then the forms of the boundary Hamiltonians in (2.13) remain intact while the effect of the bulk matter on the boundary systems are implicit through the constraints of the total conserved charges. With (2.18), the corresponding bulk matter charges may be evaluated as

$$J_i^m = -\int_{-\pi/2}^{\pi/2} d\mu\, \sqrt{-g}\, T^{\tau}{}_{a}\, \xi_i^a = \int_{-\pi/2}^{\pi/2} d\mu\, T_{\tau a}\, \xi_i^a\,, \tag{2.19}$$

which are conserved and satisfy the $\mathrm{SL}(2,\mathbf{R})$ algebra $[J_i^m, J_j^m] = i\epsilon_{ijk}\eta^{kl}J_l^m$. In fact, there is an $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ gauge symmetry generated by

$$\tilde{J}_i = J_i^r + J_i^l + J_i^m\,, \tag{2.20}$$

which leaves the full geometry, including the cutoff boundaries, invariant. Classically, one has the corresponding constraints, $\tilde{J}_i = 0$, and imposing these leads to consistent solutions of the left-right boundary dynamics once the bulk matter charges J_i^m are specified appropriately. These boundary descriptions agree with those of the bulk gravity description, as was explicitly verified in [20] for the case of $m^2 = 0$. Quantum mechanically, imposing the constraints properly on the wave function Ψ

$$\tilde{J}_i\, \Psi = 0 \tag{2.21}$$

will be the main part of our quantization of the system, whose details will be discussed in the next section.

For the resulting physical Hilbert space, we shall further impose $|\tau_r(u_1) - \tau_l(u_2)| < \pi$ at any real u_1, u_2 for any nonvanishing Ψ. This condition basically ensures the causality constraint as the left-right boundary systems are causally disconnected from each other through the bulk.[3] In fact one may show that the above condition automatically follows from the condition $|\tau_r(u_1) - \tau_l(u_2)| < \pi$ at some u_1 and u_2, say $u_1 = u_2 = 0$ [11].

[3] One can explicitly verify that this causality constraint is indeed respected for full general bulk solutions with $m^2 = 0$ [20].

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3 Canonical quantization

In this section, we consider the canonical quantization of JT theory without matter or with matter. The presentation in this section is mostly a review of the construction given in [11]. Our starting point is the unconstrained Hilbert space $\mathcal{H}_0 = L^2 \otimes \mathcal{H}_m$ where an L^2 function is further specified as a complex function of the variables $\tau_r, \tau_l, \chi_r, \chi_l$ that has a support only when $|\tau_r - \tau_l| < \pi$. Here in the presence of matter, the function is dependent upon the matter part but we shall not spell out this matter dependence explicitly in this section.

Let us impose the gauge constraint at quantum level. Since the $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ is noncompact, we may use a quantization scheme based on the equivalent classes defined by [25]

$$\Psi \cong g\Psi \tag{3.1}$$

where $g \in \widetilde{\mathrm{SL}}(2,\mathbf{R})$. These equivalent classes are called the coinvariant classes of the group $\widetilde{\mathrm{SL}}(2,\mathbf{R})$. We then introduce inner product by the integral

$$\langle\tilde{\Psi}|\Psi\rangle = \int dg\,(\tilde{\Psi},\, g\Psi)\,, \tag{3.2}$$

where dg is the left and right invariant measure of the group $\widetilde{\mathrm{SL}}(2,\mathbf{R})$. It is clear that this inner product depends only on the equivalent classes of Ψ and $\tilde{\Psi}$, so the formula defines the Hilbert space of coinvariants. It also ensures the constraints

$$\tilde{J}_i \int dg\, g\Psi \cong 0 \tag{3.3}$$

at the quantum level, or equivalently $\langle\tilde{\Psi}|\,\tilde{J}_i\,|\Psi\rangle = 0$ for any choice of $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$. Now we note that, for any $(\tau_r, \tau_l, \chi_r, \chi_l)$ with $|\tau_r - \tau_l| < \pi$, one may set $\tau_l = \tau_r = 0$ and $\chi_r = \chi_l$ by an appropriate gauge transformation. This implies that the physical Hilbert space of coinvariants is generated by a gauge-fixed wavefunction of the form [11]

$$\Psi = \delta(\tau_r)\delta(\tau_l)\delta(\chi_{rel})\psi(\chi) \tag{3.4}$$

where $\chi \equiv \frac{1}{2}(\chi_r + \chi_l)$ and $\chi_{rel} \equiv \chi_r - \chi_l$. With this gauge-fixing condition, the constraints (3.3) near $g = 1$ are realized as

$$\begin{aligned}
\tilde{J}_1 &\cong p_{\tau_r} + p_{\tau_l} + J_1^m \cong 0\,,\\
\tilde{J}_2 &\cong p_{\tau_r} - p_{\tau_l} + J_2^m \cong 0\,,\\
\tilde{J}_3 &\cong p_{\chi_r} - p_{\chi_l} + J_3^m \cong 0\,,
\end{aligned} \tag{3.5}$$

and the inner product is reduced to

$$\langle\tilde{\Psi}|\Psi\rangle = \int d\chi\, \tilde{\psi}^*(\chi)\psi(\chi)\,. \tag{3.6}$$

With help of the above relations, one may replace $p_{\tau_{r/l}}$ and $p_{\chi_{r/l}}$ by

$$p_{\tau_{r/l}} \cong -\frac{1}{2}\left(J_1^m \pm J_2^m\right)\,, \qquad p_{\chi_{r/l}} = \frac{1}{2}p_\chi \pm p_{\chi_{rel}} \cong \frac{1}{2}\left(p_\chi \mp J_3^m\right)\,, \tag{3.7}$$

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where the replacements are acting on the physical Hilbert space $\mathcal{H} = L^2(\chi) \otimes \mathcal{H}_m$. Then $H_{r/l}$ become [11]

$$2\mathcal{C}H_{r/l} = \frac{1}{4}\left(p_\chi \mp J_3^m\right)^2 - \left(J_1^m \pm J_2^m\right) e^\chi + e^{2\chi}\,, \tag{3.8}$$

which are acting on the physical Hilbert space $\mathcal{H}$ with $p_\chi = -i\partial_\chi$ satisfying $[\chi, p_\chi] = i$. It is straightforward to show that H_r commutes with H_l. In case of pure JT gravity, one finds that H_r is identical to H_l with the well known expression $2\mathcal{C}H^0_{r/l} = \frac{1}{4}p_\chi^2 + e^{2\chi}$ [10]. In this case, the boost generator $H_r - H_l$ becomes zero and merely induces a pure gauge transformation.

4 Explicit quantization with $m^2 = 0$

From now on, we shall consider JT gravity with a massless field to be specific. For this case, the bulk scalar field is dual to a dimension one ($\Delta = 1$) operator in the boundary side. The corresponding bulk field may be solved by[4]

$$\varphi = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n\pi}} \sin n(\mu + \frac{\pi}{2}) \left(a_n e^{-in\tau} + a_n^\dagger e^{-in\tau}\right)\,, \tag{4.1}$$

with the boundary condition (2.18). Upon quantization, the creation and annihilation operators satisfy

$$[a_m, a_n^\dagger] = \delta_{mn}\,, \tag{4.2}$$

while all the remaining commutators among them vanish identically. Starting with (4.1), the matter $\mathrm{SL}(2,\mathbf{R})$ charges can be identified as

$$\begin{aligned} J_1^m &= -\sum_{n=1}^{\infty} n\, a_n^\dagger a_n\,, \\ J_2^m &= \frac{1}{2}\sum_{n=1}^{\infty} \sqrt{n(n+1)}\left(a_n^\dagger a_{n+1} + a_n a_{n+1}^\dagger\right)\,, \\ J_3^m &= \frac{1}{2i}\sum_{n=1}^{\infty} \sqrt{n(n+1)}\left(a_n^\dagger a_{n+1} - a_n a_{n+1}^\dagger\right)\,, \end{aligned} \tag{4.3}$$

which follows from the definition of charges in (2.19). Here, J_1^m may involve an extra constant term but we fix this contribution to zero as it is required in order to allow a trivial representation on the vacuum sector of the bulk matter field. Of course, this vacuum sector should correspond to pure JT theory. Let us introduce a number operator defined by $N_m = \sum_{n=1}^{\infty} a_n^\dagger a_n$, which may be shown to be commuting with J_i^m. Then one has $[N_m, C_m] = 0$, where C_m denotes the Casimir operator given by $\eta^{ij} J_i^m J_j^m$. It then follows that[5]

$$[N_m, H_{r/l}] = [C_m, H_{r/l}] = 0\,. \tag{4.4}$$

[4]The most general solutions for arbitrary m are presented in [20, 26]. See also [27, 28].

[5]Since the number operator and the Casimir do not commute with the boundary matter operator $\hat{\varphi}_{l/r}$, they are not a center of the left or right algebra $\mathcal{A}_{l/r}$.

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Hence, the time evolution of the system occurs within a sector with a given total oscillator number. The matter Hilbert space $\mathcal{H}_m$ has a basis

$$|\vec{k}\rangle = |k_1 k_2 k_3 \cdots\rangle\,, \tag{4.5}$$

with $a_n^\dagger a_n|\vec{k}\rangle = k_n|\vec{k}\rangle$ where k_n is a nonnegative integer. With this basis, one has

$$N_m|\vec{k}\rangle = \sum_{n=1}^{\infty} k_n|\vec{k}\rangle\,, \quad J_1^m|\vec{k}\rangle = -\sum_{n=1}^{\infty} n k_n|\vec{k}\rangle\,, \tag{4.6}$$

which in particular shows that the generator J_1^m is nonpositive definite. The $\mathrm{SL}(2,\mathbf{R})$-invariant matter vacuum state belongs to the trivial representation of $\mathrm{SL}(2,\mathbf{R})$. It is also clear that the matter part of Hilbert space above the vacuum is given by a direct sum of the discrete series representation $D^-_{j=q}$ of $\mathrm{SL}(2,\mathbf{R})$ specified with $C_m = q(1-q)$ and a given N_m. See appendix A for some details of representations of matter charges.

Let us introduce the dual boundary operators which may be inserted along the left and right cutoff trajectories. In the present case of $\Delta = 1$, the boundary operators may be identified as

$$\hat{\varphi}_{r/l} = \lim_{\varepsilon\to 0}\left(\frac{\bar{\phi}}{\epsilon}\right)^{\Delta} \varphi\Big|_{r/l}\,. \tag{4.7}$$

In fact, this is a slight generalization of the standard AdS/CFT dictionaries reviewed in [29]. In two dimensions, the cutoff trajectories become dynamical and one needs to take into account of these dynamical fluctuations of boundary geometries. Using the relation $\cos\mu\big|_{r/l} = \epsilon\tau'_{r/l} = \frac{\epsilon}{\mathcal{C}} e^{\chi_{r/l}}$, one finds

$$\hat{\varphi}_{r/l} = \frac{e^{\chi_{r/l}}}{\sqrt{\pi}} \sum_{n=1}^{\infty} (\mp)^{n+1}\sqrt{n}\left(a_n e^{-in\tau_{r/l}} + a_n^\dagger e^{in\tau_{r/l}}\right)\,. \tag{4.8}$$

We note that

$$[\hat{\varphi}_r, \hat{\varphi}_l] = \frac{2i}{\pi} e^{\chi_r+\chi_l} \sum_{n=1}^{\infty} (-)^{n+1} n \sin(\tau_r - \tau_l) = 2ie^{\chi_r+\chi_l}\left(\delta'(\tau_r-\tau_l+\pi) + \delta'(\tau_r-\tau_l-\pi)\right) \tag{4.9}$$

where the second equality is defined over the interval $\tau_r - \tau_l \in [-\pi, \pi]$. Since $|\tau_r - \tau_l| < \pi$, one has $[\hat{\varphi}_r, \hat{\varphi}_l] = 0$. It is also straightforward to check that $[\tilde{J}_i, \hat{\varphi}_{r/l}] = 0$, so the left and right boundary operators are gauge invariant. Expressed in the physical Hilbert space variables after the gauge-fixing, these boundary operators become

$$\hat{\varphi}_{r/l} = \frac{e^{\chi}}{\sqrt{\pi}} \sum_{n=1}^{\infty} (\mp)^{n+1}\sqrt{n}\left(a_n + a_n^\dagger\right)\,. \tag{4.10}$$

Finally, one may also check that $[H_{r/l}, \hat{\varphi}_{l/r}] = 0$. Any left-side operators that are constructed out of H_l and $\hat{\varphi}_l$ are commuting with the right-side operators that are generated by combinations of H_r and $\hat{\varphi}_r$.

There are two types of time evolutions in our theory. One is our system time defined for the action (2.10), where the time parameter u evolves the left and right system equally with the Hamiltonian

$$H_{\mathrm{tot}} = H_r + H_l = \frac{1}{\mathcal{C}}\left(\frac{1}{4}\left(p_\chi^2 + J_3^{m2}\right) - J_1^m\, e^{\chi} + e^{2\chi}\right)\,. \tag{4.11}$$

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In this note, we use this time evolution primarily. One may introduce a time-evolved operator by

$$\hat{\varphi}_{r/l}(u) = e^{iuH_{\rm tot}}\hat{\varphi}_{r/l}\, e^{-iuH_{\rm tot}} \tag{4.12}$$

Alternatively, one may add an independent evolution by $H_{rel} = \frac{1}{2}(H_r - H_l)$ with a boost evolution parameter u_{rel}, Unlike the case of pure JT, this boost operator becomes physical. Then one may evolve the left and right operators separately by

$$\hat{\varphi}_{r/l}(u_{r/l}) = e^{iu_{r/l}H_{r/l}}\hat{\varphi}_{r/l}\, e^{-iu_{r/l}H_{r/l}}\,. \tag{4.13}$$

Since the left operators are fully commuting with the right operators, the two definitions agree with each other. Thus, the latter evolution works equally well and is equivalent to the former in our case. In either ways, the time-ordered $n(= n_r + n_l)$-point transition function

$$G(u_1^r, \cdots, u_{n_r}^r; u_1^l, \cdots, u_{n_l}^l) = \langle\widetilde{\Psi}|\mathcal{T}_r \prod_{k=1}^{n_r} \hat{\varphi}_r(u_k^r)\, \mathcal{T}_l \prod_{k'=1}^{n_l} \hat{\varphi}_l(u_{k'}^l)|\Psi\rangle\,, \tag{4.14}$$

may be defined irrespective of the orderings between the left and the right boundary operators. This reflects the fact that the left and right cutoff boundaries are causally disconnected with each other. Note also that $[N_m, \hat{\varphi}_{r/l}] \neq 0$ and $[C_m, \hat{\varphi}_{r/l}] \neq 0$. Hence with insertion of operators, the total number of oscillators and the value for the matter Casimir are not preserved in general. Of course, without insertion of extra operators, these two are preserved under the left and right Hamiltonian evolutions.

5 Comparison with bulk solutions

In this section, we consider the gauge constraints and the gauge-fixing of the Schwarzian theories in the classical limit. First, one may recall the equations of motion given by the Hamiltonians (2.13)

$$p'_{\tau_{r/l}} = 0\,, \qquad p_{\chi_{r/l}} = \mathcal{C}\chi'_{r/l}\,, \qquad e^{2\chi_{r/l}} + p_{\tau_{r/l}}e^{\chi_{r/l}} = -\mathcal{C}p'_{\chi_{r/l}}\,, \qquad e^{\chi_{r/l}} = \mathcal{C}\tau'_{r/l}\,, \tag{5.1}$$

which retain their forms even in the presence of the matter. The solutions to these equations are given by

$$\frac{1}{\mathcal{C}}p_{\tau_{r/l}} = \pm C_{r/l}\,, \tag{5.2}$$

$$\frac{1}{\mathcal{C}}e^{\chi_{r/l}} = A_{r/l}\cos\tau_{r/l} + B_{r/l}\sin\tau_{r/l} \mp C_{r/l}\,, \tag{5.3}$$

$$\frac{1}{\mathcal{C}}p_{\chi_{r/l}} = -A_{r/l}\sin\tau_{r/l} + B_{r/l}\cos\tau_{r/l}\,, \tag{5.4}$$

where $A_{r/l}, B_{r/l}$ and $C_{r/l}$ are integration constants. The last equation in (5.1), $e^{\chi_{r/l}} = \mathcal{C}\tau'_{r/l}$, together with the solution (5.3) leads to boundary cutoff trajectories [20] parametrized as

$$\tanh\frac{1}{2}L_{r/l}(u - u_0^{r/l}) = \sqrt{\frac{1+q_{r/l}}{1-q_{r/l}}}\ \tan\frac{1}{2}\Big(\tau_{r/l}(u) - \tau_B^{r/l}\Big)\,, \tag{5.5}$$

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where $u_0^{r/l}$ are another integration constants and[6]

$$\tan\tau_B^{r/l} \equiv \frac{B_{r/l}}{A_{r/l}}\,, \qquad q_{r/l} \equiv \frac{\pm C_{r/l}}{\sqrt{A_{r/l}^2+B_{r/l}^2}}\,, \qquad L_{r/l} \equiv \sqrt{A_{r/l}^2+B_{r/l}^2-C_{r/l}^2}\,. \tag{5.6}$$

For these solutions, the on-shell values of $\mathrm{SL}(2,\mathbf{R})$ generators $J_i^{r/l}$ in (2.15) are given by

$$J_1^{r/l}\big|_{sol} = \pm\mathcal{C}\,C_{r/l}\,, \qquad J_2^{r/l}\big|_{sol} = \pm\mathcal{C}\,A_{r/l}\,, \qquad J_3^{r/l}\big|_{sol} = \pm\mathcal{C}\,B_{r/l}\,, \tag{5.7}$$

and so the on-shell values of the Hamiltonians are given by

$$H_{r/l}\big|_{sol} = \frac{\mathcal{C}}{2}(A_{r/l}^2+B_{r/l}^2-C_{r/l}^2)\,. \tag{5.8}$$

As is well-known as the Darboux's theorem, the solution space of the equations of motion is symplectomorphic to the phase space in classical mechanics [30, 31]. In this regard, the eight constants $A_{r/l}, B_{r/l}, C_{r/l}$ and $u_0^{r/l}$ describing classical solutions correspond to the eight-dimensional (unconstrained) phase space described by four variables $(\tau_{r/l}, \chi_{r/l})$ and their canonical conjugate momenta $(p_{\tau_{r/l}}, p_{\chi_{r/l}})$. To obtain physical phase space with the causality constraint $|\tau_r - \tau_l| < \pi$, we need to take a symplectic quotient by the constraint group $\widetilde{\mathrm{SL}}(2,\mathbf{R})$. This quotient or reduction of variables can be understood as fixing the integration constants. The constants $u_0^{r/l}$ may be chosen, which corresponds to a certain gauge choice, such as

$$\tanh\frac{1}{2}L_{r/l}\,u_0^{r/l} = \sqrt{\frac{1+q_{r/l}}{1-q_{r/l}}}\,\tan\frac{1}{2}\tau_B^{r/l}\,, \tag{5.9}$$

and then, $\tau_{r/l}(u=0)=0$. One may note that the gauge invariant combination of remaining constants $A_{r/l}, B_{r/l}$ and $C_{r/l}$ appears in the cut-off trajectory expression in (5.5), which is given by $L_{r/l}$ or equivalently the right/left energies $H_{r/l}|_{sol}$. So, the solution space is described by the variables $L_{r/l}$ (i.e. $H_{r/l}|_{sol}$). In pure JT gravity, $H_r + H_l = 2H_r = 2H_l$ and its conjugate variable $\frac{1}{2}(\tau_r+\tau_l)$ form two-dimensional phase space [10, 11], while in JT gravity with matter $H_r + H_l$ and $H_r - H_l$ give us different time evolutions and energies. Of course, this reduction can also be understood from the canonical variables. Concretely, by using $\tilde{J}_1$ and $\tilde{J}_2$, one can set $\tau_r = \tau_l = 0$. Using the remaining constraint generator $\tilde{J}_3$, one can set $e^{\chi_r} = e^{\chi_l}$. See appendix B for the details of the gauge-fixing.

Now, let us consider the bulk scalar solution and its on-shell matter J_i^m charges to check the gauge constraints on the classical solutions. First of all, one may note that the on-shell matter charges, computed by the bulk integral (2.19) for the classical scalar field solution φ given in (4.1), takes the same form with (4.3). This can be rewritten as

$$J_i^m\big|_{sol} = -(Q_i^r - Q_i^l)\,, \tag{5.10}$$

[6] In pure JT gravity, $L_{r/l}$ reduces to the horizon radius L since $A_r = A_l = L$ for the vacuum black hole solution in (2.8). This will become clearer after we discuss the match between the bulk solution and the boundary Schwarzian solution provided below.

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where $Q_i^{r/l}$ read

$$
\begin{aligned}
Q_1^{r/l} &= \pm \sum_{n=1}^{\infty} \frac{1}{2} n a_n^\dagger a_n \,, \\
Q_2^{r/l} &= \bar{\phi} L \mp \sum_{n=1}^{\infty} \frac{1}{4} \sqrt{n(n+1)}\, (a_n^\dagger a_{n+1} + a_{n+1} a_n^\dagger) \,, \\
Q_3^{r/l} &= \mp \sum_{n=1}^{\infty} \frac{1}{4i} \sqrt{n(n+1)}\, (a_n^\dagger a_{n+1} - a_{n+1}^\dagger a_n) \,,
\end{aligned} \tag{5.11}
$$

where $a_n^\dagger$ should be interpreted as the complex conjugate of a_n in the classical solutions.

We have judiciously rewritten the on-shell matter charges $J_i^m|_{sol}$ in terms of $Q_i^{r/l}$'s, since those are related to the asymptotic form of the bulk dilaton field ϕ. For the bulk scalar field solution, φ_{sol} under a vanishing boundary condition, the bulk dilaton solution can be obtained by solving (2.4). From the explicit asymptotic expressions of the bulk dilaton solution, one can see that the dilaton solution ϕ at the cutoff boundaries takes the form of the vacuum solution as (see appendix C for a summary of these solutions in [20])

$$
\phi \underset{\mu \to \mu_c^{r/l}}{\longrightarrow} \eta^{ij} Q_i^{r/l} Y_j \Big|_{\mu_c^{r/l}} \,, \qquad Y_i = \left(\tan\mu, \frac{\cos\tau}{\cos\mu}, \frac{\sin\tau}{\cos\mu} \right) . \tag{5.12}
$$

Equivalently, using the equations of motion (2.4), one immediately sees that the relevant bulk integral reduces to a surface term, resulting in the above expression of $J_i^m|_{sol}$. For instance, $J_1^m|_{sol}$ can be computed as

$$
J_1^m|_{sol} = -\int_{\mu_c^l}^{\mu_c^r} d\mu \; T_{\tau\tau}|_{sol} = \frac{1}{\cos\mu} \frac{\partial}{\partial\mu} (\phi \cos\mu) \Big|_{\mu_c^l}^{\mu_c^r} = -(Q_1^r - Q_1^l) \,, \tag{5.13}
$$

which shows why the matter charges $J_i^m|_{sol}$ are related to the asymptotic forms of the dilaton field ϕ. Note that the left and right constants $Q_i^{r/l}$ are not independent but related by

$$
Q_1^r = -Q_1^l \,, \qquad \bar{\phi} L - Q_2^r = -(\bar{\phi} L - Q_2^l) \,, \qquad Q_3^r = -Q_3^l \,. \tag{5.14}
$$

Now, let us check the gauge constraints by relating the bulk solutions to the boundary solutions in Schwarzian variables $(e^{\chi_{r/l}}, \tau_{r/l})$ through the cutoff conditions given in (2.9). By using the relation of the dilaton at the cutoff trajectories with the Schwarzian variables, one obtains

$$
e^{\chi_{r/l}} = \mathcal{C} \tau'_{r/l} = \frac{\mathcal{C}}{\epsilon} \cos\mu_c^{r/l} = \phi \; \cos\mu_c^{r/l} \big|_{\mu \to \mu_c^{r/l}} = \mp Q_1^{r/l} + Q_2^{r/l} \cos\tau + Q_3^{r/l} \sin\tau \,, \tag{5.15}
$$

where the metric cutoff condition (2.9) is used in the second equality, the dilaton cutoff condition (2.9) with $\mathcal{C} = \bar{\phi}$ is used in the third equality, and the asymptotic form of ϕ in (5.12) is used in the last equality. By matching this expression to the boundary solution in (5.3), one can deduce that it is consistent with the gauge constraint $\tau_r = \tau_l$ by taking $\tau = \tau_{r/l}$, and that

$$
\mathcal{C}\, C_{r/l} = Q_1^{r/l} \,, \qquad \mathcal{C}\, A_{r/l} = Q_2^{r/l} \,, \qquad \mathcal{C}\, B_{r/l} = Q_3^{r/l} \,. \tag{5.16}
$$

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Using this matching of constants, (5.7) and (5.10), it is straightforward to check

$$\tilde{J}_i\big|_{sol} = (J_i^r + J_i^l + J_i^m)|_{sol} = 0\,, \tag{5.17}$$

which tells us that the gauge constraints are automatically satisfied in classical solutions when the Schwarzian variables and the dilaton at the cutoff trajectories are related by the cutoff conditions (2.9).

As a side remark, we would like to note that matter charges are related to the left or right coefficients $Q_i^{r/l}$ as

$$Q_1^{r/l} = \mp\frac{1}{2}J_1^m\big|_{sol}\,, \qquad Q_2^{r/l} = \bar{\phi}L \mp \frac{1}{2}J_2^m\big|_{sol}\,, \qquad Q_3^{r/l} = \mp\frac{1}{2}J_3^m\big|_{sol}\,. \tag{5.18}$$

6 Two-sided correlation functions

In this section, we shall consider the partition function and two-sided correlation functions of JT theory with matter from the viewpoint of two-sided picture. Let us begin with the case of pure JT theory. Without matter contribution, the total Hamiltonian becomes

$$H_{\rm tot} = H_r + H_l = 2H_r = \left(\frac{1}{4}p_\chi^2 + e^{2\chi}\right)/\mathcal{C}\,. \tag{6.1}$$

This is a Liouville quantum-mechanical system that involves an exponential potential. Note that the renormalized geodesic length between two boundary points $\tau_l(u)$ and $\tau_r(u)$ is given by

$$\ell_{\rm ren} \equiv \ell_{\rm bare} - \ln 2\phi|_r - \ln 2\phi|_l = \ln\left(\frac{\cos^2\frac{\tau_r-\tau_l}{2}}{\mathcal{C}^2\tau_l'\tau_r'}\right) = -2\chi \tag{6.2}$$

where, for the first equality, (2.9) is used and the last equality follows from the gauge-fixing condition in the above. The corresponding eigenvalue problem,

$$H_{\rm tot}\,\psi_s(\chi) = \frac{s^2}{\mathcal{C}}\,\psi_s(\chi)\,, \tag{6.3}$$

can be solved by [10]

$$\psi_s(\chi) = N_s\,K_{2is}(2e^\chi)\,, \qquad N_s = \frac{2}{\pi}(2s\sinh 2\pi s)^{\frac{1}{2}}\,, \tag{6.4}$$

which satisfies the scattering normalization

$$\int_{-\infty}^{\infty} d\chi\,\psi_s^*(\chi)\psi_{s'}(\chi) = \delta(s-s')\,. \tag{6.5}$$

In the scattering regime of $\chi \to -\infty$, the wavefunction behaves as

$$\psi_s \to \frac{\Gamma(-2is)}{\sqrt{\pi}\,|\Gamma(-2is)|}\left(e^{2is\chi} + R(s)e^{-2is\chi}\right), \tag{6.6}$$

where the reflection amplitude may be identified as $R(s) = \frac{\Gamma(2is)}{\Gamma(-2is)}$. In the forbidden region of $\chi \to \infty$, on the other hand, the wavefunction decays doubly-exponentially as

$$\psi_s(\chi) \to N_s\sqrt{\frac{\pi}{4e^\chi}}\,e^{-2e^\chi}\,. \tag{6.7}$$

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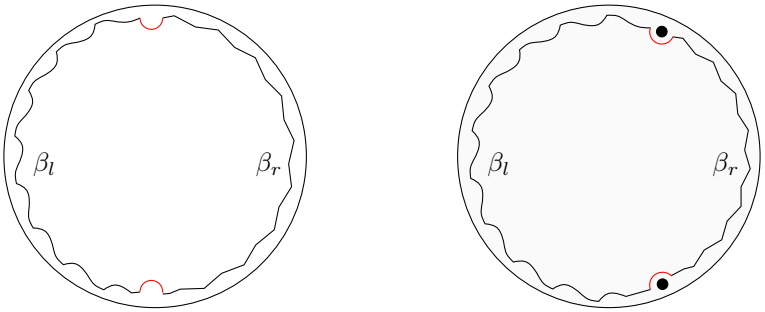

Figure 2. On the left we draw the two-side evolution for pure JT theory. On the right, we depict the evolution for JT theory with matter. The big dots represent insertions of the boundary operators at the initial and the final points of the evolution. The red curves in each diagram represent the initial and final cutoff geodesics.

Now let us turn to the evaluation of the disk partition function in the two-sided picture. The relevant density of states is basically one-sided quantity whereas our physical Hilbert space is inherently two-sided. In this respect, currently there is no well-defined procedure computing the disk partition function based on the two-sided description. Here we follow the proposal in [21]

$$Z(\beta) \propto \lim_{\chi^c \to \infty} \langle \chi^c | \, e^{-\frac{\beta}{2} H_{\rm tot}} | \chi^c \rangle \,, \tag{6.8}$$

which is based on the picture in the left side of figure 2. Since $H_l = H_r$ in the present case, $\beta H_{\rm tot}/2$ may be replaced by $\beta_l H_l + \beta_r H_r$ with $\beta_l + \beta_r = \beta$.

In this two-sided picture, one starts the evolution from an initial geodesic connecting two slightly separated boundary points somewhere on the bottom side as illustrated in the left side of figure 2. Its renormalized length $\ell^c_{\rm ren}$ $(= -2\chi^c)$ goes to negative infinity as the above two points approach each other. Then we evolve through the bulk leading to the final geodesic between two regularized points on the top side. Basically the evolution is based on the propagator with an appropriate Boltzmann weight, which defines the path integral computation in the two-sided picture. With this prescribed regularization, one finds

$$Z(\beta) \propto \lim_{\chi^c \to \infty} W(\chi^c) \int_0^\infty ds s \sinh 2\pi s \, e^{-\beta \frac{s^2}{2\mathcal{C}}} \tag{6.9}$$

where

$$W(\chi) = \frac{2\, e^{-4e^{\chi}}}{\pi \, e^{\chi}} \,. \tag{6.10}$$

Note here that the factor $W(\chi^c)$ is independent of the variable s and may be absorbed into the overall coefficient of the partition function or the constant part of the entropy S_0. Hence up to this overall coefficient, the disk partition function may be identified as

$$Z(\beta) = \int_0^\infty ds s \sinh 2\pi s \, e^{-\beta \frac{s^2}{2\mathcal{C}}} = \frac{\sqrt{2}\, \mathcal{C}^{\frac{3}{2}}}{\beta^{\frac{3}{2}}} e^{\frac{2\pi^2 \mathcal{C}}{\beta}} \tag{6.11}$$

which agrees with the previous results based on the one-sided picture [9, 32] (See also [33–36] for related works). The question yet remaining is how to specify the initial (or final) state and an alternative based on the Hartle-Hawking state on a half disk is given in [19].

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We now turn to the case of JT gravity with a massless matter field. The two-sided function now depends on β_r and β_l since H_r and H_l differ from each other.[7] In the semiclassical regime, the left and right black holes involving a nontrivial matter field indeed become different from each other as was constructed explicitly in [20]. The corresponding Euclidean disk geometry becomes two-sided[8] with insertion of operators in the bottom and the top region. This insertion of matter state (as a linear combination of $|\vec{k}\rangle$) induces a state at the initial curve as $|\Phi_I\rangle = \sum_{\vec{k}} \delta(\chi - \chi^c_{\vec{k}})\, c_{\vec{k}}\, |\vec{k}\rangle$, by which the bulk will be affected in general. We also assume the final cutoff state $|\Phi_F\rangle = |\Phi_I\rangle$ for simplicity. Of course, this assumption can be relaxed and the definition may be generalized to the case where $|\Phi_F\rangle \neq |\Phi_I\rangle$. With this preparation, we consider

$$Z_I(\beta_r, \beta_l) \propto \lim_{\chi^c_{\vec{k}} \to \infty} \langle \Phi_I|\, e^{-(\beta_r H_r + \beta_l H_l)} |\Phi_I\rangle \tag{6.12}$$

where the way to send $\chi^c_{\vec{k}}$ to infinity will be specified further below. The corresponding two-sided evolution is depicted in figure 2. Again one begins with an initial geodesic curve connecting two slightly separated points in the bottom region. The precise locations of these two points may be adjusted by an infinitesimal amount depending on each matter basis state $|\vec{k}\rangle$ (see below). From this, we evolve the two-sided system with a Boltzmann weight $e^{-(\beta_r H_r + \beta_l H_l)}$ which ends up with the final geodesic curve prescribed by the same way as the initial one. The corresponding left/right evolution times are given by β_l/β_r with the Hamiltonians H_l/H_r, respectively. Due to the initial and final insertions of operators, the bulk state and the left and right Hamiltonians are affected in general. In the semiclassical limit, this corresponds to the so-called vev deformation whose details are studied in [26].

To be specific, let us consider the case with $\beta_r = \beta_l = \frac{\beta}{2}$ and $|\Phi_I\rangle = \delta(\chi - \chi^c_{\vec{k}})\, |\vec{k}\rangle$. In this case, one has $\beta_r H_r + \beta_l H_l = \frac{\beta}{2} H_{\rm tot}$. The relevant eigenvalue problem $H_{\rm tot}|\Phi\rangle = E|\Phi\rangle$ may be solved perturbatively by decomposing

$$H_{\rm tot} = H_{(0)} + H_{(1)} \tag{6.13}$$

where

$$H_{(0)} = \frac{1}{\mathcal{C}}\left(\frac{1}{4}p_\chi^2 - J_1^m\, e^\chi + e^{2\chi}\right), \qquad H_{(1)} = \frac{1}{4\mathcal{C}}(J_3^m)^2\,. \tag{6.14}$$

We solve the zeroth-order eigenvalue problem $\mathcal{C}H_{(0)}|\Phi\rangle_{(0)} = s^2|\Phi\rangle_{(0)}$ with a state of the form $|\Phi\rangle_{(0)} = \psi_{q,s}(\chi)\, |\vec{k}\rangle$. This leads to an eigenvalue equation

$$\left(\frac{1}{4}p_\chi^2 + q\, e^\chi + e^{2\chi}\right)\psi_{q,s}(\chi) = s^2 \psi_{q,s}(\chi)\,, \tag{6.15}$$

where $q = \sum_{n=1}^\infty n k_n \geq 0$. This q-dependent potential is everywhere nonnegative definite and becomes zero as $\chi \to -\infty$. This problem is solved by

$$\psi_{q,s} = N_{q,s}\, Y_{q,s}(4e^\chi)\,, \qquad N_{q,s} = \frac{2}{\pi}(2s\sinh 2\pi s)^{\frac{1}{2}}\left|\frac{\Gamma(\frac{1}{2}+q+2si)}{\Gamma(\frac{1}{2}+2si)}\right|, \tag{6.16}$$

[7] β_l and β_r are simply left and right Euclidean evolution parameters, which should not be confused with the left and right temperatures.

[8] The Euclidean geometry of the 3d Janus two-sided black hole was constructed in [37], whose boundary of the thermal disk part is intrinsically two-sided involving different left and right Hamiltonians.

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with $Y_{q,s}(z) = \sqrt{\pi/z}\, W_{-q,2si}(z)$ where $W_{\kappa,\mu}(z)$ is the Whittaker function satisfying

$$\frac{d^2}{dz^2} W_{\kappa,\mu} + \left(-\frac{1}{4} + q z^{-1} + \left(\frac{1}{4} - \mu^2 \right) z^{-2} \right) W_{\kappa,\mu} = 0 \,. \tag{6.17}$$

The wavefunction is again scattering-normalized as

$$\int_{-\infty}^{\infty} d\chi\, \psi^*_{q,s}(\chi)\, \psi_{q,s'}(\chi) = \delta(s - s') \,. \tag{6.18}$$

In the scattering region of $\chi \to -\infty$, the wavefunction behaves as

$$\psi_{q,s} \to \frac{2^{4si}\, \Gamma(-4is)}{\sqrt{\pi}\, |\Gamma(-4is)|} \frac{\Gamma(\frac{1}{2} + q + 2is)}{|\Gamma(\frac{1}{2} + q + 2is)|} \left(e^{2is\chi} + R_q(s) e^{-2is\chi} \right) , \tag{6.19}$$

where the reflection amplitude is given by $R_q(s) = \frac{\Gamma(4is)}{\Gamma(-4is)} \frac{\Gamma(\frac{1}{2}+q-2is)}{\Gamma(\frac{1}{2}+q+2is)} 2^{-8si}$. On the other hand, in the forbidden region of $\chi \to \infty$, the wave function decays again doubly-exponentially as

$$\psi_{q,s} \to N_{q,s} \frac{\sqrt{\pi}}{(4e^{\chi})^{q+\frac{1}{2}}}\, e^{-2e^{\chi}} \,. \tag{6.20}$$

With the prescribed regularization in the above, one finds

$$Z_q\left(\frac{\beta}{2}, \frac{\beta}{2}\right) \propto \lim_{\chi_q^c \to \infty} W_q(\chi_q^c)\, w_q \int_0^{\infty} ds\, \rho_q(s)\, e^{-\beta \frac{s^2}{2\mathcal{C}}} \tag{6.21}$$

with

$$\rho_q(s) = \frac{\pi\, |\Gamma(\frac{1}{2}+q+2is)|^2}{|\Gamma(\frac{1}{2}+q)\, \Gamma(\frac{1}{2}+2is)|^2}\, s \sinh 2\pi s \,, \qquad W_q(\chi) = \frac{8\, \Gamma(\frac{1}{2}+q)^2\, e^{-4e^{\chi_c}}}{\pi^2\, (4e^{\chi})^{2q+1}\, w_q} \,. \tag{6.22}$$

Of course, at this point, one may freely adjust the redundant factor w_q. Since $W_q(\chi_q^c)$ is independent of s, we may drop this in the limit where χ_q^c goes to infinity. Hence the leading-order contribution of the two-sided function reads

$$Z_q^{(0)}\left(\frac{\beta}{2}, \frac{\beta}{2}\right) = w_q \int_0^{\infty} ds\, \rho_q(s)\, e^{-\beta \frac{s^2}{2\mathcal{C}}} \,, \tag{6.23}$$

where w_q is not determined at the moment.

Without insertion of any matter operators ($q = 0$ and $w_0 = 1$), the above expression agrees with pure JT result in (6.11), i.e. $Z_{q=0}^{(0)}(\frac{\beta}{2}, \frac{\beta}{2}) = Z(\beta)$; this also gives the partition function even in the presence of matter as was argued in [11]. For $q \neq 0$, the above involves an initial (or final) insertion of boundary matter operators leading to (two-sided) 2-point correlation functions in general.

With $q = 1$, for instance, one has

$$\rho_1 = (1 + 16s^2)\, s \sinh 2\pi s \tag{6.24}$$

which leads to

$$Z_{q=1}^{(0)}\left(\frac{\beta}{2}, \frac{\beta}{2}\right) = w_1 Z(\beta) \left(1 + 48 \frac{\mathcal{C}}{\beta} + 64\pi^2 \frac{\mathcal{C}^2}{\beta^2} \right) . \tag{6.25}$$

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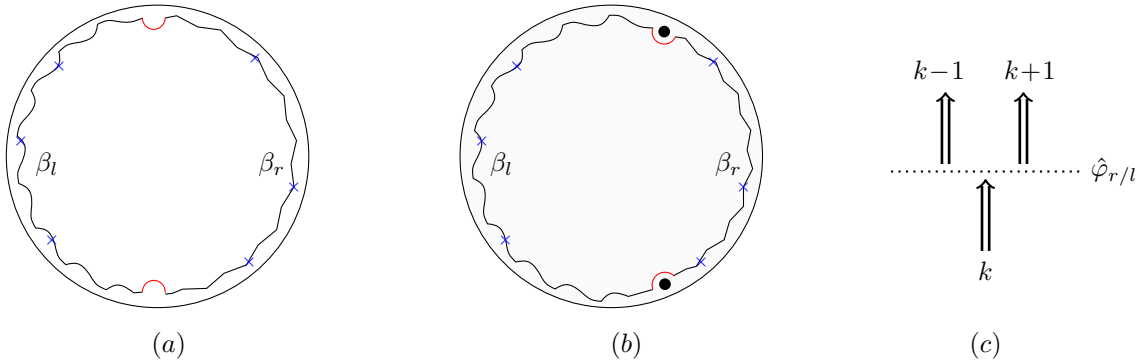

Figure 3. In (a) and (b), we draw the two-side evolution for JT theory with matter operators inserted along the left right evolutions. In these diagrams, the matter insertions are marked by cross symbols. (a) depicts the case where one starts from and ends with the matter vacuum state. (b) describes the case with general matter initial state. With a left or right insertion of matter operator, an $N_m = k$ state before the insertion turns into a linear combination of a $k+1$ and a $k-1$ state, which is depicted in (c).

Similarly, for the larger value of q, one may work out the zeroth-order contribution to the partition function. Adding the contribution from the first-order perturbation, one has

$$Z_q\left(\frac{\beta}{2},\frac{\beta}{2}\right) = Z_q^{(0)}\left(\frac{\beta}{2},\frac{\beta}{2}\right) e^{-\frac{\beta}{8C}\langle\vec{k}|(J_3^m)^2|\vec{k}\rangle} \tag{6.26}$$

where $\langle\vec{k}|(J_3^m)^2|\vec{k}\rangle = \frac{1}{2}\sum_{n=1}^{\infty}\left[n(n+1)k_nk_{n+1} + n^2k_n\right]$. Considering now a general matter initial state $\sum_{\vec{k}} c_{\vec{k}}|\vec{k}\rangle$, we need to fix the relative factor w_q. There seems no general principle to fix this relative factor because the $\chi \to \infty$ limit around the initial and final regularized surfaces is not well understood. Here we propose to set the relative factor $w_q = 1$ and to adjust $\chi^c_{\vec{k}}$ such that $W_q(\chi^c_{\vec{k}}) = W_0(\chi^c_{\vec{0}})$ in the $\chi^c_{\vec{k}} \to \infty$ limit. We then drop the overall factor $W_0(\chi^c_{\vec{0}})$ uniformly for any $|\vec{k}\rangle$. Then for the matter initial state $\sum_{\vec{k}} c_{\vec{k}}|\vec{k}\rangle$, the zeroth-order two-sided function becomes

$$Z_{(0)}\left(\frac{\beta}{2},\frac{\beta}{2}\right) = \sum_{\vec{k}} |c_{\vec{k}}|^2 \int_0^\infty ds\, \rho_{q_{\vec{k}}}(s)\, e^{-\beta\frac{s^2}{2C}} \tag{6.27}$$

where $q_{\vec{k}}$ denotes $\sum_{n=1}^{\infty} nk_n$. With the above prescription, one has $\rho_{q_{\vec{k}}}(s) \to \rho_0(s)$ in the $s \to 0$ limit; this also corresponds to fixing each density of states to that of pure JT theory in the zero temperature limit.

In general, the evolution by $H_{l/r}$ preserves N_m and C_m quantum numbers as they commute with the left and right Hamiltonians. Thus, for instance, one may start from a matter initial state which belongs to a particular matter sector specified by the eigenvalues of N_m and C_m. Then, along the evolution by $H_{l/r}$, matter states stay within the initially prescribed sector.

As depicted in figures 3a and 3b, one may consider inserting matter operators along the boundary trajectories. Let us introduce the corresponding correlation function defined

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by

$$G_I^E(y_1^r,\cdots,y_{n_r}^r;y_1^l,\cdots,y_{n_l}^l)=\langle\prod_{k=1}^{n_r}\hat{\varphi}_r(-iy_k^r)\prod_{k'=1}^{n_l}\hat{\varphi}_l(-iy_{k'}^l)\rangle_I\,, \tag{6.28}$$

with $y_k^{r/l}\in[0,\beta_{r/l}]$. In this case, as a pair of the Euclidean times y_l and y_r evolves,[9] one will encounter insertions of left or right matter operators order by order. Now in between each successive encounters, let us focus on a state belonging to an $N_m=k$ sector with $k\geq 0$, where its evolution remains within the sector in between the encounters. This state will eventually encounter a left or right operator $\hat{\varphi}_{l/r}$ at a certain slice which is denoted by the dotted line in figure 3c. Right after the encounter, the $N_m=k$ state turns into a linear combination of $N_m=k-1$ and $k+1$ states where the $N_m=-1$ state (with $k=0$) does not exist and should be removed additionally. This process goes on with next encounters of operators. It then follows that any correlation function, defined with $|\Phi_I\rangle=|\Phi_F\rangle$ that belongs to a particular $N_m=k$ sector, vanishes if n_r+n_l is odd.

For the sake of illustration, let us consider the case where one starts from and ends with the matter vacuum state. Of course, for the partition function, the corresponding evolution stays within the matter vacuum sector. Now once we add matter operators along the evolution, the state no longer stays within the matter vacuum sector and mixing between sectors will occur as described in the above. The first nontrivial example is the two-point correlation function $G^E_{|\vec{0}\rangle}(y_1^r;y_1^l)$ where we further assume $\beta_r=\beta_l=\beta/2$ and $y_1^r<y_1^l$ for simplicity. Then one may evolve the system with $H_{\rm tot}$ with Euclidean time $0<y<\beta/2$. For $0<y<y_1^r$, the system remains within the matter vacuum sector. Then, for $y_1^r<y<y_1^l$, the state belongs to the $N_m=1$ sector. Finally for $y_1^l<y<\beta/2$, the evolution is restricted to the matter vacuum sector due to the final state condition. An explicit evaluation of this two-point function does not seem to be so straightforward. Neither is it clear to us how the above correlation functions are related to the conventional correlation functions in literatures [3, 34, 36, 38–41]. Further studies are required in this direction.

7 Conclusion

In this paper, we have presented the detailed canonical quantization of JT gravity coupled to a massless scalar field. Especially, we have identified the bulk matter charges J_i^m explicitly, and shown that the (matter) number operator N_m and the Casimir C_m commute with the boundary Hamiltonians $H_{l/r}$. This allows us to choose the simultaneous eigenstate of H_l, H_r, N_m and C_m in the two-sided Hilbert space. And then we computed some simultaneous eigenfunctions in the two-sided Hilbert space. In pure JT gravity, we reproduced the well-known eigenfunction given by a modified Bessel function. In [21], the two-sided version of the disk partition function was proposed by starting two-sided boundary evolution from an initial geodesic curve connecting two slightly separated boundary points in the bottom region of the disk and ending up with a final geodesic curve between again two slightly separated boundary points in the top region. From this definition of the two-sided partition

[9] The two-sided evolution here is very much dependent upon ways of slicing y_l and y_r. However the final answer should be independent of slicing, as dictated by any gravity theories.

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function, the disk partition function was reproduced in the same reference. In the presence of matter, one may additionally arrange initial and final states including the matter part before and after the initial and final regularized curves, by which the bulk of the disk is affected in general. Thus we have introduced a two-sided correlation function in the presence of prescribed matter states. In particular, we tried to specify the prescribed states at the initial and final regularized curves generalizing the proposal of [21]. In JT gravity with a massless scalar, the eigenfunction of $H_{\rm tot}$ is shown to be given by a Wittaker function and the two-sided correlation function for $\beta_r = \beta_l = \beta/2$ is evaluated perturbatively for some simple initial states. One may additionally insert boundary matter operators along the two-sided evolution leading to the higher two-sided correlation functions. We have investigated some basic properties of these two-sided correlation functions.

The two-sided correlation functions we have introduced require the specific regularization procedure of initial (or final) state at the initial (or final) curve, which is not so well-motivated unfortunately. Instead one may provide some controlled initial (or final) state there and the resulting two-sided correlation functions may be directly related to the conventional correlation functions in [3, 34, 36, 38–41]. However the precise guiding principle to construct such initial (or final) state is lacking at this stage. Further investigations are required in this direction. In addition we have not considered the bulk wormhole contribution [11]. It would be interesting to include its effect at the level of the partition function and to consider the factorization issues.

Acknowledgments

We would like to thank Andreas Gustavsson for careful reading of the manuscript. CK thanks CCNY for hospitality where part of this work was done. DB was supported in part by NRF Grant RS-2023-00208011 and by Basic Science Research Program through NRF funded by the Ministry of Education (2018R1A6A1A06024977). C.K. was supported by NRF Grant 2022R1F1A1074051. S.-H.Y. was supported by NRF Grant 2021R1A2C1003644 and supported by Basic Science Research Program through the NRF funded by the Ministry of Education (NRF-2020R1A6A1A03047877).

A Representations of the matter charges

In this appendix, we show that the matter Hilbert space $\mathcal{H}_m$ is decomposed of negative discrete series of irreducible representations $\mathcal{D}_j^-$ of SL(2,**R**) (see [42] for a review of SL(2,**R**) representation). Introducing

$$J_\pm^m = J_2^m \pm i J_3^m, \tag{A.1}$$

we can rewrite the $\mathrm{SL}(2,\mathbf{R})$ algebra as

$$[J_1^m, J_\pm^m] = \pm J_\pm^m, \qquad [J_+^m, J_-^m] = -2J_1^m. \tag{A.2}$$

Thus $J_\pm^m$ may be considered as raising/lowering operators for the eigenstates of J_1^m. The Casimir operator C_m becomes

$$C_m = -(J_1^m)^2 - J_1^m + J_-^m J_+^m\,. \tag{A.3}$$

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In terms of the operators a and $a^\dagger$ in (4.1), $J^m_\pm$ are given by

$$J^m_+ = \sum_{n=1}^{\infty} \sqrt{n(n+1)}a^\dagger_n a_{n+1}\,,$$
$$J^m_- = \sum_{n=1}^{\infty} \sqrt{n(n+1)}a_n a^\dagger_{n+1}\,, \tag{A.4}$$

Note that, for each pair of adjacent oscillators, $J^m_\pm$ shift the oscillator number by one to the left/right, respectively. In particular, J^m_+ annihilates states $|k\rangle \equiv |k000\cdots\rangle$ for any k,

$$J^m_+|k\rangle = 0. \tag{A.5}$$

Since

$$J^m_1|k\rangle = -k|k\rangle\,, \quad C_m|k\rangle = k(1-k)|k\rangle\,, \quad N_m|k\rangle = k|k\rangle\,, \tag{A.6}$$

we can identify $|k\rangle$ as the highest weight state of the representation $\mathcal{D}^-_k$ (with $N_m = k$) of SL(2,**R**) in negative discrete series. Then applying J^m_-, we obtain basis vectors of the representation $\{|l\rangle\}$ with $l = k, k+1, \ldots$ which are eigenstates of J^m_1 with $J^m_1|l\rangle = -l|l\rangle$. Normalized vectors are

$$|l\rangle \equiv \sqrt{\frac{(2k-1)!}{(k+l-1)!\,(l-k)!}}\,(J^m_-)^{l-k}|k\rangle\,. \tag{A.7}$$

Recall that the number operator N_m commutes with J^m_i's. Then, from (4.6), we see that $|l\rangle$ consists of the oscillator states $|\vec{k}\rangle$ with

$$\sum_{n=1}^{l-k+1} k_n = k\,, \qquad \sum_{n=1}^{l-k+1} nk_n = l\,. \tag{A.8}$$

Note that the upper limit of the summation range is limited by $l-k+1$. As l increases, there are more oscillator states involved to make a particular $|l\rangle$ state. For instance, for $l = k+1$ and $k+2$, we get

$$|k+1\rangle = \frac{1}{\sqrt{2k}}J^m_-|k\rangle = |k-1,1,0,0,\cdots\rangle\,,$$
$$|k+2\rangle = \frac{1}{\sqrt{4k(2k+1)}}(J^m_-)^2|k\rangle$$
$$= \sqrt{\frac{2(k-1)}{2k+1}}\,|k-2,2,0,0,\cdots\rangle + \sqrt{\frac{3}{2k+1}}\,|k-1,0,1,0,\cdots\rangle\,. \tag{A.9}$$

In this example, applying J^m_+ to two oscillator states in $|k+2\rangle$ would result in the same $|k-1,1,0,0,\cdots\rangle$ which is nothing but $|k+1\rangle$. This implies that the orthogonal combination

$$|\widetilde{k+2}\rangle = \sqrt{\frac{3}{2k+1}}\,|k-2,2,0,0,\cdots\rangle - \sqrt{\frac{2(k-1)}{2k+1}}\,|k-1,0,1,0,\cdots\rangle \tag{A.10}$$

with $k \geq 2$ should be annihilated by J^m_+, which can easily be checked. Then, we see that $|\widetilde{k+2}\rangle$ is the highest weight state of a new irreducible representation $\mathcal{D}^-_{k+2}$ (with $N_m = k$) which is obtained by applying J^m_- successively to $|\widetilde{k+2}\rangle$.

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It is clear to generalize this procedure. If applying J_-^m increases the number of oscillator states which participate in the linear combination, one would get highest weight states of new irreducible representations by considering the orthogonal linear combinations of the states. In this way, the matter Hilbert space $\mathcal{H}_m$ can be decomposed of negative discrete series of irreducible representations $\mathcal{D}_j^-$ of SL(2,$\mathbf{R}$).

B Gauge-fixing

In this appendix, we present some details on the gauge-fixing procedure. Though we use the commutator notation of quantum mechanics, it may be understood as the corresponding Poisson bracket in the context of classical mechanics. Note that, in the classical setup, the last terms in $J_2^{r/l}$ and $J_3^{r/l}$ in (2.15) do not appear. As mentioned in section 2, the condition $|\tau_r - \tau_l| < \pi$ will be assumed. Let us begin with

$$i\left[\tilde{J}_1,\, \frac{1}{2}(\tau_r+\tau_l)\right] = 1\,, \tag{B.1}$$

which allows us to fix the gauge, $\tau_r + \tau_l = 0$. Upon this gauge choice, we may see that

$$i\left[\tilde{J}_2,\, \frac{1}{2}(\tau_r-\tau_l)\right] = i[\tilde{J}_2,\, \pm\tau_{r/l}] = \cos\tau_r = \cos\tau_l\,. \tag{B.2}$$

Thus we may set $\tau_r = \tau_l = 0$ where we used the condition $|\tau_r - \tau_l| < \pi$. Now, with $\tau_r = \tau_l = 0$, we find

$$i[\tilde{J}_3,\, \pm\chi_{r/l}] = 1\,, \tag{B.3}$$

which allows us to fix the gauge $\chi_r - \chi_l = 0$. This completes our gauge-fixing procedure. Classically, starting with the relevant solutions in [20], one may work out the corresponding gauge transformations explicitly which lead to the fully gauge-fixed forms of solutions.

C Classical bulk solutions

Here we summarize classical bulk solutions of JT gravity. See [20] for more details. Under the vanishing boundary condition, the scalar equation (2.5) with $m = 0$ is solved by

$$\varphi = \sum_{n=1}^{\infty} \mathbf{a}_n \sin n\left(\mu + \frac{\pi}{2}\right)\cos n(\tau - \tau_n). \tag{C.1}$$

In the main text, we introduced complex coefficients a_n's by the relations

$$a_n \equiv \frac{\sqrt{n\pi}}{2} e^{in\tau_n}\,\mathbf{a}_n\,, \qquad a_n^\dagger \equiv \frac{\sqrt{n\pi}}{2} e^{-in\tau_n}\,\mathbf{a}_n\,. \tag{C.2}$$

Then, the above solution can be rewritten as (4.1).

Now, let us return to the dilaton field ϕ. As was shown in [20], the classical solution of the dilaton ϕ is obtained in the form of

$$\phi = \bar{\phi} L \frac{\cos\tau}{\cos\mu} + \sum_{m,n=1}^{\infty} \mathbf{a}_m \mathbf{a}_n \phi_{m,n}\,, \tag{C.3}$$

where the explicit expressions of $\phi_{m,n}$'s are given by

$$
\begin{aligned}
\phi_{n,n} &= \frac{(-1)^n n}{8(4n^2-1)}(2n\cos 2n\mu + \sin 2n\mu + \tan\mu)\cos 2n(\tau-\tau_n) - \frac{n^2}{4}(1+\mu\tan\mu) \\
\phi_{n,n+1} &= \phi_{n+1,n} \\
&= \frac{(-1)^{n+1}}{16(2n+1)\cos\mu}\left[(n+1)\sin 2n\mu + n\sin(n+2)\mu\right]\cos[(2n+1)\tau-(n+1)\tau_{n+1}-n\tau_n)] \\
&\quad - \frac{n(n+1)}{8\cos\mu}(\tan\mu+\mu)\cos[\tau-(n+1)\tau_{n+1}+n\tau_n], \\
\phi_{m,n} &= \frac{mn}{8\cos\mu}\Bigg[\cos[n(\tau-\tau_n)-m(\tau-\tau_m)]\left(\frac{\sin(n-m+1)(\mu+\frac{\pi}{2})}{(n-m+1)(n-m)} - \frac{\sin(n-m-1)(\mu+\frac{\pi}{2})}{(n-m-1)(n-m)}\right) \\
&\quad + \cos[n(\tau-\tau_n)+m(\tau-\tau_m)]\left(\frac{\sin(n+m+1)(\mu+\frac{\pi}{2})}{(n+m+1)(n-m)} - \frac{\sin(n+m-1)(\mu+\frac{\pi}{2})}{(n+m-1)(n-m)}\right)\Bigg].
\end{aligned}
\tag{C.4}
$$

The asymptotic behaviors of these solutions as $\mu \to \mu_c^{r/l}$ read as

$$
\begin{aligned}
\phi_{n,n} &= -\frac{n^2}{4}(1+\mu\tan\mu) + \mathcal{O}(\cos^2\mu)\,, \\
\phi_{n,n+1} = \phi_{n+1,n} &= -\frac{n(n+1)}{8}(\sin\mu + \mu\sec\mu)\cos[\tau-(n+1)\tau_{n+1}+n\tau_n] + \mathcal{O}(\cos^2\mu)\,,
\end{aligned}
$$

and all the remaining $\phi_{n,m} = \mathcal{O}(\cos^2\mu)$. This asymptotic form leads to (5.12) and the expressions for $Q_i^{r/l}$ in (5.11).

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