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Existence of nontrivial weak solutions for p-biharmonic Kirchhoff-type equations

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Abstract

We are concerned with the following p-biharmonic equations:

$$\Delta_p^2 u + M\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u)\,dx\right)\operatorname{div}(\varphi(x,\nabla u)) + V(x)|u|^{p-2}u = \lambda f(x,u) \quad \text{in } \mathbb{R}^N,$$

where $2 < 2p < N$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$, the function $\varphi(x,v)$ is of type $|v|^{p-2}v$, $\varphi(x,v) = \frac{d}{dv}\Phi_0(x,v)$, the potential function $V : \mathbb{R}^N \to (0,\infty)$ is continuous, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. We study the existence of weak solutions for the problem above via mountain pass and fountain theorems.

MSC: 35J60; 35J92; 58E05

Keywords: p-biharmonic; Kirchhoff type; Variational method

1 Introduction

We are concerned with the following p-biharmonic equations:

$$\Delta_p^2 u + M\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u)\,dx\right)\operatorname{div}\bigl(\varphi(x,\nabla u)\bigr) + V(x)|u|^{p-2}u = \lambda f(x,u) \quad \text{in } \mathbb{R}^N, \tag{P}$$

where $2 < 2p < N$, $1 < p < p_* := \frac{Np}{N-2p}$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is a p-biharmonic operator, the function $\varphi(x,v)$ is of type $|v|^{p-2}v$, $\varphi(x,v) = \frac{d}{dv}\Phi_0(x,v)$, the potential function $V : \mathbb{R}^N \to (0,\infty)$ is continuous, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition.

The fourth-order differential equations arise in the study of deflections of elastic beams on nonlinear elastic foundations. Thus, they become very significant in engineering and physics. Many authors considered this type of equation in recent years, and we refer to [9, 13, 27] and the references therein. For this reason, the existence of solutions of p-biharmonic equations has been studied by several authors; see [6, 8, 12, 15, 21, 24, 30, 31, 34]. To obtain the existence and multiplicity results for the p-Laplace type operators, which generalize the usual p-Laplacian, the authors in [10, 28] considered the following condition:

$$d|v|^p \le \varphi(x,v)\cdot v \le p\Phi_0(x,v)$$

for all $x \in \mathbb{R}^N$ and $v \in \mathbb{R}^N$, and for some positive constant d; see also [14]. On the other hand, Kirchhoff in [20] initially proposed the following equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right| dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

which is a generalization of the classical D'Alembert's wave equation. Also, Woinowsky and Krieger [33] in the 1950s considered a stationary analogue of the evolution equation of Kirchhoff type, namely

$$u_{tt} + \Delta^2 u - M\bigl(\|\nabla u\|^2\bigr) \Delta u = f(x,u),$$

as a model for the deflection of an extensible beam on nonlinear foundations. Here, u denotes the displacement, f is the force that the foundations exert on the beam, and M models the effects of the small changes in the length of the beam (see, e.g., [3–5, 7] for the physics viewpoint model). In view of mathematics, many researchers have extensively studied the existence of weak solutions for the elliptic problem of Kirchhoff type in recent years (see, e.g., [11, 16, 18]). Based on these references, we consider the generalized elliptic equation (P) involving the p-biharmonic and generalized p-Laplacian of Kirchhoff type.

Since the seminal paper of Ambrosetti and Rabinowitz in [2], the existence of solutions for the elliptic problem has been studied by many researchers. A common feature of these works is that the following condition, which is originally due to Ambrosetti and Rabinowitz, is imposed on the nonlinearity f:

(AR) There exist positive constants m and ζ such that $\zeta > p$ and

$$0 < \zeta F(x,t) \le f(x,t)t \quad \text{for } x \in \Omega \text{ and } |t| \ge m,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$, and Ω is a bounded domain in $\mathbb{R}^N$.

The (AR) condition above is somewhat natural and important to guarantee the boundedness of Palais–Smale sequence of Euler–Lagrange functional for an elliptic equation, however, this condition is very restrictive and eliminates many nonlinearities. Thus, many researchers have tried to drop the (AR) condition for elliptic equations associated with the p-Laplacian; see, e.g., [1, 23, 25, 26, 29].

The purpose of this paper is to study the existence of weak solutions for problem (P) without assuming the (AR) condition, but imposing various assumptions for the divergence part φ and nonlinear term f. In particular, as observed by Remark 1.8 in [23], there are many examples which do not fulfill the condition of the nonlinear term f given in [1, 25, 26]. On the other hand, in case of the whole space $\mathbb{R}^N$, the main difficulty of this problem is the lack of compactness for the Sobolev theorem. In that sense, our study is to pursue two goals. First, we show the existence of nontrivial weak solutions for the problem above using the mountain pass theorem. To be precise, we prove the existence of weak solutions for problem (P) under Cerami condition, as a weak version of the Palais–Smale condition. Also, we try to do analysis using the properties of Kirchhoff function M and function φ. Second, we show the multiplicity of weak solutions to problem (P) via the fountain theorem. To the best of our knowledge, there were no such existence results for our problem in this situation.

2 Preliminaries

In this section, we briefly describe the framework for our problem. We assume that the potential $V \in C(\mathbb{R}^N)$ is a continuous function with

(V) $\inf_{x\in\mathbb{R}^N} V(x) > 0$, and $\operatorname{meas}\{x \in \mathbb{R}^N : V(x) \le K\} < +\infty$ for all $K \in \mathbb{R}$.

Also, we set $D^p(\mathbb{R}^N) = \{u \in L^{p_*}(\mathbb{R}^N) | \Delta u \in L^p(\mathbb{R}^N)\}$. Thus, we define the function space as follows:

$$X = \left\{u \in D^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|\Delta u|^p + |\nabla u|^p + V(x)|u|^p\right) dx < +\infty\right\}$$

equipped with the norm

$$\|u\|_X^p = \|\Delta u\|_{L^p(\mathbb{R}^N)}^p + \|\nabla u\|_{L^p(\mathbb{R}^N)}^p + \left\|V^{1/p}u\right\|_{L^p(\mathbb{R}^N)}^p.$$

For our problem, we first assume that $M : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following conditions:

(M1) $M \in C(\mathbb{R}^+)$ satisfies $\inf_{t\in\mathbb{R}^+} M(t) \ge m_0 > 0$, where m_0 is a constant.

(M2) There exists $\theta \in [1, \frac{N}{N-p})$ such that $\theta\mathcal{M}(t) = \theta\int_0^t M(\tau)\,d\tau \ge M(t)t$ for any $t \ge 0$.

A typical example for M is given by $M(t) = b_0 + b_1 t^n$ with $n > 0$, $b_0 > 0$, and $b_1 \ge 0$.

Next, we assume that $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function with the continuous derivative with respect to v of the mapping $\Phi_0 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $\Phi_0 = \Phi_0(x, v)$, that is, $\varphi(x, v) = \frac{d}{dv}\Phi_0(x, v)$. Suppose that φ and Φ_0 satisfy the following assumptions:

(J1) The equality

$$\Phi_0(x, \mathbf{0}) = 0$$

holds for almost all $x \in \mathbb{R}^N$.

(J2) There are a nonnegative function $a \in L^{p'}(\mathbb{R}^N)$ and a nonnegative constant b such that

$$\left|\varphi(x, v)\right| \le a(x) + b|v|^{p-1}$$

holds for almost all $x \in \mathbb{R}^N$ and for all $v \in \mathbb{R}^N$. Here, p' is a conjugate number of p.

(J3) The relations

$$d|v|^p \le \varphi(x, v) \cdot v \quad \text{and} \quad d|v|^p \le p\Phi_0(x, v)$$

hold for all $x \in \mathbb{R}^N$ and $v \in \mathbb{R}^N$, where d is a positive constant.

(J4) $\Phi_0(x, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \mathbb{R}^N$.

(J5) The relation

$$p\Phi_0(x, v) - \varphi(x, v) \cdot v \ge 0$$

holds for all $x \in \mathbb{R}^N$ and all $v \in \mathbb{R}^N$.

Let us define the functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\Delta u|^p\,dx + \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla u)\,dx\right) + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p\,dx.$$

It is not difficult to prove that the functional $\Phi \in C^1(X,\mathbb{R})$, and its Fréchet derivative is given by

$$\begin{aligned}\langle \Phi'(u), v\rangle &= \int_{\mathbb{R}^N} |\Delta u|^{p-2}\Delta u \Delta v\,dx + M\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u)\,dx\right)\int_{\mathbb{R}^N} \varphi(x,\nabla u)\cdot \nabla v\,dx \\ &\quad + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\,dx.\end{aligned}$$

We give some examples satisfying assumptions (J1)–(J5).

Example 2.1

(1) Let us consider the following functions:

$$\varphi(x,v) = |v|^{p-2}v \quad \text{and} \quad \Phi_0(x,v) = \frac{|v|^p}{p}$$

for $v \in \mathbb{R}^N$ and $x \in \mathbb{R}^N$. Then it is obvious that assumptions (J1)–(J5) hold.

(2) Suppose that $a \in L^{2p'}(\mathbb{R}^N)$, and there is a positive constant a_0 such that $a(x) \geq a_0$ for almost all $x \in \mathbb{R}^N$. We consider

$$\varphi(x,t) = \left(a(x) + t^2\right)^{\frac{p-2}{2}} t \quad \text{and} \quad \Phi_0(x,t) = \frac{1}{p}\left[\left(a(x) + t^2\right)^{\frac{p}{2}} - a(x)^{\frac{p}{2}}\right]$$

for $t \in \mathbb{R}$, where $p \geq 2$ for all $x \in \mathbb{R}^N$. Then assumptions (J1)–(J5) hold.

By analogous arguments as in [19, 22], the following lemma is easily checked, and thus we omit the proof. That is, the operator Φ' is a mapping of type (S_+).

Lemma 2.2 *Assume that* (V), (M1), (M2), *and* (J1)–(J4) *hold. Then the functional* $\Phi : X \to \mathbb{R}$ *is convex and weakly lower semicontinuous on* X. *Moreover, the operator* Φ' *is a mapping of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0$, *then* $u_n \to u$ *in* X *as* $n \to \infty$.

Denoting $F(x,t) = \int_0^t f(x,s)\,ds$, for the number θ given in (M2), we assume that

(F1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\cdot,t)$ is measurable for all $t \in \mathbb{R}$ and $f(x,\cdot)$ is continuous for almost all $x \in \mathbb{R}^N$.

(F2) There exist nonnegative functions $\rho \in L^{q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\sigma \in L^\infty(\mathbb{R}^N)$ such that

$$\left|f(x,t)\right| \leq \rho(x) + \sigma(x)|t|^{q-1}, \quad q \in (\theta p, p_*)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

(F3) There exists $\delta > 0$ such that

$$F(x,t) \leq 0 \quad \text{for } x \in \mathbb{R}^N \text{ and } |t| < \delta.$$

(F4) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{\theta p}} = \infty$ uniformly for almost all $x \in \mathbb{R}^N$.

(F5) There exist $c_0 \geq 0$, $r_0 \geq 0$, and $\kappa > \frac{N}{p}$ such that

$$\left|F(x,t)\right|^{\kappa} \leq c_0 |t|^{\kappa p} \mathfrak{F}(x,t)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $|t| \geq r_0$, where $\mathfrak{F}(x,t) = \frac{1}{\theta p} f(x,t)t - F(x,t) \geq 0$.

(F6) There exist $\mu > \theta p$ and $\varrho > 0$ such that

$$\mu F(x,t) \leq tf(x,t) + \varrho t^p$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Next, we give some examples with respect to assumptions (F1)–(F6).

Since assumption (F5) is weaker than the following assumption, namely that

$$\frac{f(x,t)}{|t|^{\theta-2}t} \quad \text{is increasing for } t > 0 \text{ and decreasing for } t < 0 \tag{2.1}$$

for any $x \in \mathbb{R}^N$, we check that the following example satisfies assumption (F5) by applying condition (2.1).

Example 2.3 Let us consider

$$f(x,t) = |t|^{q-2} t \log\left(1 + |t|\right)$$

for all $t \in \mathbb{R}$. It is clear that function f satisfies assumptions (F1)–(F4). Since the following ratio, namely

$$\frac{f(x,t)}{|t|^{p-2}t} = \frac{|t|^{q-2} t \log(1 + |t|)}{|t|^{p-2}t} = |t|^{q-p} \log\left(1 + |t|\right),$$

is increasing for $t > 0$ and decreasing for $t < 0$ if $q > p = \theta$, it follows that assumption (F5) holds.

The following example can be found in [23] for the case of p-Laplace operator.

Example 2.4 Consider the following function:

$$f(x,t) = |t|^{p-2} t\left(4|t|^3 + 2t \sin t - 4 \cos t\right).$$

Then this function satisfies conditions (F2), (F6), but not the (AR) condition.

Define the functional $\Psi : X \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^N} F(x,u)\,dx.$$

Then it is easy to check that $\Psi \in C^1(X,\mathbb{R})$ and its Fréchet derivative is

$$\left\langle \Psi'(u), v\right\rangle = \int_{\mathbb{R}^N} f(x,u) v\,dx$$

for any $u, v \in X$. Next we define the functional $I_\lambda : X \to \mathbb{R}$ by

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u).$$

Then it follows that the functional $I_\lambda \in C^1(X, \mathbb{R})$ and its Fréchet derivative is

$$\begin{aligned}\langle I'_\lambda(u), v\rangle &= \int_{\mathbb{R}^N} |\Delta u|^{p-2}\Delta u \Delta v\, dx + M\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla u)\, dx\right) \int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v\, dx \\ &\quad + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\, dx - \lambda \int_{\mathbb{R}^N} f(x, u)v\, dx\end{aligned}$$

for any $u, v \in X$.

In our setting, first of all, we need the following lemma. Using a similar argument as in [17, Lemma 3.2], we can see that the functionals Ψ and Ψ' are weakly strongly continuous on X. We give a detailed proof for the convenience of the reader.

Lemma 2.5 *Assume that* (V) *and* (F1)–(F2) *hold. Then* Ψ *and* Ψ' *are weakly strongly continuous on* X.

Proof See Appendix. □

3 Existence of weak solutions

Definition 3.1 We say that $u \in X$ is a weak solution of problem (P) if

$$\begin{aligned}&\int_{\mathbb{R}^N} |\Delta u|^{p-2}\Delta u \cdot \Delta v\, dx + M\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla u)\, dx\right) \int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v\, dx \\ &\quad + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\, dx - \lambda \int_{\mathbb{R}^N} f(x, u)v\, dx = 0\end{aligned}$$

for any $v \in X$.

The following result is used to show that the energy functional I_λ satisfies the geometric conditions of the mountain pass theorem.

Lemma 3.2 *Assume that* (V), (M1), (M2), (J1)–(J3), *and* (F1)–(F4) *hold. Then the geometric conditions in the mountain pass theorem hold, i.e.*,

(1) $u = 0$ *is a strict local minimum for* $I_\lambda(u)$,

(2) $I_\lambda(u)$ *is unbounded from below on* X.

Proof By assumption (F3), $u = 0$ is a strict local minimum for $I_\lambda(u)$. Next we claim that condition (2) holds. Assumption (F4) implies that for any $K_0 > 0$, there exists a constant $\delta > 0$ such that

$$F(x, t) \geq K_0|t|^{\theta p} \tag{3.1}$$

for $|t| > \delta$ and for almost all $x \in \mathbb{R}^N$. Note that for $t > 1$, we can easily check that $\mathcal{M}(t) \leq \mathcal{M}(1)t$. For any $v \in X \setminus \{0\}$, from assumptions (J2), (J3) and relation (3.1), we have

$$\begin{aligned}
I_\lambda(tv) &= \frac{1}{p}\int_{\mathbb{R}^N} |t\Delta v|^p\,dx + \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x, t\nabla v)\,dx\right) \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|t|^p|v|^p\,dx - \lambda\int_{\mathbb{R}^N} F(x,tv)\,dx \\
&\leq \frac{1}{p}\int_{\mathbb{R}^N} |t\Delta v|^p\,dx + \mathcal{M}(1)\left(\int_{\mathbb{R}^N} \Phi_0(x, t\nabla v)\,dx\right)^\theta \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|t|^p|v|^p\,dx - \lambda\int_{\mathbb{R}^N} F(x,tv)\,dx \\
&\leq \frac{1}{p}|t|^p\|v\|_X^p + \mathcal{M}(1)\left(\int_{\mathbb{R}^N} a(x)|t\nabla v| + \frac{b}{p}|t\nabla v|^p\,dx\right)^\theta - \lambda\int_{\mathbb{R}^N} F(x,tv)\,dx \\
&\leq |t|^{\theta p}\left(\frac{1}{p}\|v\|_X^p + \mathcal{M}(1)\left(\int_{\mathbb{R}^N} a(x)|\nabla v| + \frac{b}{p}|\nabla v|^p\,dx\right)^\theta - \lambda K_0\int_{\mathbb{R}^N} |v|^{\theta p}\,dx\right)
\end{aligned}$$

for sufficiently large $t > 1$. If K_0 is large enough, then we assert that $I_\lambda(tv) \to -\infty$ as $t \to \infty$. Hence we conclude that the functional I_λ is unbounded from below. This completes the proof. □

With the aid of Lemmas 2.2 and 2.5, we prove that the energy functional I_λ satisfies the Cerami condition $(C)_c$ condition, for short, i.e., for $c \in \mathbb{R}$, any sequence $\{u_n\} \subset X$ such that

$$I_\lambda(u_n) \to c \quad \text{and} \quad \left\|I'_\lambda(u_n)\right\|_{X^*}\left(1 + \|u_n\|_X\right) \to 0 \quad \text{as } n \to \infty$$

has a convergent subsequence. This plays a key role in obtaining the existence of a nontrivial weak solution for the given problem.

Lemma 3.3 *Assume that* (V), (M1), (M2), (J1)–(J5), *and* (F1)–(F5) *hold. Then the functional I_λ satisfies the $(C)_c$ condition for any $\lambda > 0$.*

Proof For $c \in \mathbb{R}$, let $\{u_n\}$ be a $(C)_c$-sequence in X, that is,

$$I_\lambda(u_n) \to c \quad \text{and} \quad \left\|I'_\lambda(u_n)\right\|_{X^*}\left(1 + \|u_n\|_X\right) \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

This says that

$$c = I_\lambda(u_n) + o(1) \quad \text{and} \quad \left\langle I'_\lambda(u_n), u_n\right\rangle = o(1), \tag{3.3}$$

where $o(1) \to 0$ as $n \to \infty$. It follows from Lemmas 2.2 and 2.5 that Φ' and Ψ' are mappings of type (S_+). Since I'_λ is of type (S_+) and X is reflexive, it suffices to prove that the sequence $\{u_n\}$ is bounded in X. We argue by contradiction. Suppose that the sequence $\{u_n\}$ is unbounded in X. Then we may assume that $\|u_n\|_X > 1$ and $\|u_n\|_X \to \infty$ as $n \to \infty$. Define a sequence $\{w_n\}$ by $w_n = u_n/\|u_n\|_X$. It is clear that $\{w_n\} \subset X$ and $\|w_n\|_X = 1$. Hence, up to a subsequence still denoted by $\{w_n\}$, we obtain $w_n \rightharpoonup w$ in X as $n \to \infty$ and note that

$$w_n(x) \to w(x) \quad \text{a.e. in } \mathbb{R}^N \quad \text{and} \quad w_n \to w \quad \text{in } L^s\left(\mathbb{R}^N\right) \text{ as } n \to \infty \tag{3.4}$$

for $1 < s < p_*$. According to assumptions (M1), (M2), (J3), and relation (3.3), we obtain that

$$\begin{aligned} c &= I_\lambda(u_n) + o(1) \\ &= \frac{1}{p}\int_{\mathbb{R}^N} |\Delta u_n|^p\,dx + \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx\right) \\ &\quad + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u_n|^p\,dx - \lambda\int_{\mathbb{R}^N} F(x,u_n)\,dx + o(1) \\ &\geq \frac{1}{p}\int_{\mathbb{R}^N} |\Delta u_n|^p\,dx + \frac{1}{\theta}M\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx\right)\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx \\ &\quad + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u_n|^p\,dx - \lambda\int_{\mathbb{R}^N} F(x,u_n)\,dx + o(1) \\ &\geq \frac{1}{p}\int_{\mathbb{R}^N} |\Delta u_n|^p\,dx + \frac{dm_0}{\theta p}\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx \\ &\quad + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u_n|^p\,dx - \lambda\int_{\mathbb{R}^N} F(x,u_n)\,dx + o(1) \\ &\geq \frac{\min\{1,dm_0\}}{\theta p}\|u_n\|_X^p - \lambda\int_{\mathbb{R}^N} F(x,u_n)\,dx + o(1). \end{aligned} \tag{3.5}$$

Since $\|u_n\|_X \to \infty$ as $n \to \infty$, we have

$$\int_{\mathbb{R}^N} F(x,u_n)\,dx \geq \frac{\min\{1,dm_0\}}{\theta p\lambda}\|u_n\|_X^p - \frac{c}{\lambda} + \frac{o(1)}{\lambda} \to \infty \quad \text{as } n \to \infty. \tag{3.6}$$

In addition, we assert that

$$\begin{aligned} I_\lambda(u_n) &= \frac{1}{p}\int_{\mathbb{R}^N} |\Delta u_n|^p\,dx + \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx\right) \\ &\quad + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u_n|^p\,dx - \lambda\int_{\mathbb{R}^N} F(x,u_n)\,dx \\ &\leq \frac{1}{p}\|u_n\|_X^p + \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx\right) - \lambda\int_{\mathbb{R}^N} F(x,u_n)\,dx. \end{aligned}$$

Combining this with relation (3.3), we obtain that

$$\frac{1}{p}\|u_n\|_X^p + \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx\right) \geq \lambda\int_{\mathbb{R}^N} F(x,u_n)\,dx + c - o(1)$$

for sufficiently large n. Assumption (F4) implies that there exists $t_0 > 1$ such that $F(x,t) > |t|^{\theta p}$ for all $x \in \mathbb{R}^N$ and $|t| > t_0$. From assumptions (F1) and (F2), there exists $\mathcal{C} > 0$ such that $|F(x,t)| \leq \mathcal{C}$ for all $(x,t) \in \mathbb{R}^N \times [-t_0,t_0]$. Therefore we can choose a real number $\mathcal{C}_0$ such that $F(x,t) \geq \mathcal{C}_0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, and thus

$$\frac{F(x,u_n) - \mathcal{C}_0}{\frac{1}{p}\|u_n\|_X^p + \mathcal{M}(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx)} \geq 0, \tag{3.7}$$

for all $x \in \mathbb{R}^N$ and for all $n \in \mathbb{N}$. Set $\Omega_1 = \{x \in \mathbb{R}^N : w(x) \neq 0\}$. By the convergence in (3.4), we know that

$$|u_n(x)| = |w_n(x)|\|u_n\|_X \to \infty \quad \text{as } n \to \infty$$

for all $x \in \Omega_1$. So then, it follows from assumptions (M2), (J2), (F4), and Hölder's inequality that, for all $x \in \Omega_1$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{F(x,u_n)}{\frac{1}{p}\|u_n\|_X^p + \mathcal{M}(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx)} \\
&\quad\geq \lim_{n\to\infty} \frac{F(x,u_n)}{\frac{1}{p}\|u_n\|_X^p + \mathcal{M}(1)(1+(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx)^\theta)} \\
&\quad\geq \lim_{n\to\infty} \frac{F(x,u_n)}{\frac{1}{p}\|u_n\|_X^p + \mathcal{M}(1)(1+(\int_{\mathbb{R}^N} a(x)|\nabla u_n| + \frac{b}{p}|\nabla u_n|^p\,dx)^\theta)} \\
&\quad\geq \lim_{n\to\infty} \frac{F(x,u_n)}{\frac{1}{p}\|u_n\|_X^p + \mathcal{M}(1)(1+(\|a\|_{L^{p'}(\mathbb{R}^N)}\|\nabla u_n\|_{L^p(\mathbb{R}^N)} + \frac{b}{p}\|\nabla u_n\|_{L^p(\mathbb{R}^N)}^p)^\theta)} \\
&\quad\geq \lim_{n\to\infty} \frac{F(x,u_n)}{(\frac{1}{p} + \mathcal{M}(1)(1+(\|a\|_{L^{p'}(\mathbb{R}^N)} + \frac{b}{p})^\theta))\|u_n\|_X^{\theta p}} \\
&\quad= \lim_{n\to\infty} \frac{F(x,u_n)}{(\frac{1}{p} + \mathcal{M}(1)(1+(\|a\|_{L^{p'}(\mathbb{R}^N)} + \frac{b}{p})^\theta))|u_n(x)|^{\theta p}} \big|w_n(x)\big|^{\theta p} \\
&\quad= \infty,
\end{aligned} \tag{3.8}$$

where we have used the inequality $\mathcal{M}(t) \leq \mathcal{M}(1)(1+t^\theta)$ for all $t \in \mathbb{R}^+$, since if $0 \leq t < 1$, then $\mathcal{M}(t) = \int_0^t M(\tau)\,d\tau \leq \mathcal{M}(1)$ and if $t > 1$, then $\mathcal{M}(t) \leq \mathcal{M}(1)t^\theta$. Hence we get that $\operatorname{meas}(\Omega_1) = 0$. Indeed, if $\operatorname{meas}(\Omega_1) \neq 0$, according to (3.6)–(3.8) and Fatou's lemma, we would obtain

$$\begin{aligned}
\frac{1}{\lambda} &= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(x,u_n)\,dx}{\lambda\int_{\mathbb{R}^N} F(x,u_n)\,dx + c - o(1)} \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\frac{1}{p}\|u_n\|_X^p + \mathcal{M}(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx)}\,dx \\
&\geq \liminf_{n\to\infty} \int_{\Omega_1} \frac{F(x,u_n)}{\frac{1}{p}\|u_n\|_X^p + \mathcal{M}(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx)}\,dx \\
&\quad - \limsup_{n\to\infty} \int_{\Omega_1} \frac{\mathcal{C}_0}{\frac{1}{p}\|u_n\|_X^p + \mathcal{M}(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx)}\,dx \\
&= \liminf_{n\to\infty} \int_{\Omega_1} \frac{F(x,u_n) - \mathcal{C}_0}{\frac{1}{p}\|u_n\|_X^p + \mathcal{M}(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx)}\,dx \\
&\geq \int_{\Omega_1} \liminf_{n\to\infty} \frac{F(x,u_n) - \mathcal{C}_0}{\frac{1}{p}\|u_n\|_X^p + \mathcal{M}(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx)}\,dx \\
&\geq \int_{\Omega_1} \liminf_{n\to\infty} \frac{F(x,u_n)}{\frac{1}{p}\|u_n\|_X^p + \mathcal{M}(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx)}\,dx \\
&\quad - \int_{\Omega_1} \limsup_{n\to\infty} \frac{\mathcal{C}_0}{\frac{1}{p}\|u_n\|_X^p + \mathcal{M}(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx)}\,dx \\
&= \infty,
\end{aligned} \tag{3.9}$$

which is a contradiction. Thus $w(x) = 0$ for almost all $x \in \mathbb{R}^N$. Using assumptions (M1)–(M2) and (J5), we get

$$\begin{aligned}
c+1 &\geq I_\lambda(u_n) - \frac{1}{\theta p}\langle I'_\lambda(u_n), u_n\rangle \\
&= \frac{1}{p}\int_{\mathbb{R}^N} |\Delta u_n|^p\,dx + \mathcal{M}\biggl(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx\biggr) \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u_n|^p\,dx - \lambda\int_{\mathbb{R}^N} F(x,u_n)\,dx \\
&\quad - \frac{1}{\theta p}\int_{\mathbb{R}^N} |\Delta u_n|^p\,dx - \frac{1}{\theta p}M\biggl(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx\biggr)\int_{\mathbb{R}^N} \varphi(x,\nabla u_n)\cdot\nabla u_n\,dx \\
&\quad - \frac{1}{\theta p}\int_{\mathbb{R}^N} V(x)|u_n|^p\,dx + \frac{1}{\theta p}\lambda\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx \\
&\geq \frac{1}{\theta}M\biggl(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx\biggr)\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx - \lambda\int_{\mathbb{R}^N} F(x,u_n)\,dx \\
&\quad - \frac{1}{\theta p}M\biggl(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx\biggr)\int_{\mathbb{R}^N} \varphi(x,\nabla u_n)\cdot\nabla u_n\,dx + \frac{1}{\theta p}\lambda\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx \\
&= \frac{1}{\theta}M\biggl(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx\biggr)\biggl(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx - \frac{1}{p}\int_{\mathbb{R}^N} \varphi(x,\nabla u_n)\cdot\nabla u_n\,dx\biggr) \\
&\quad + \lambda\int_{\mathbb{R}^N} \mathfrak{F}(x,u_n)\,dx \\
&\geq \lambda\int_{\mathbb{R}^N} \mathfrak{F}(x,u_n)\,dx
\end{aligned} \tag{3.10}$$

for n large enough. Let us define $\Omega_n(a,b) := \{x \in \mathbb{R}^N : a \leq |u_n(x)| < b\}$ for $a \geq 0$. The convergence in (3.4) means that

$$w_n \to 0 \quad \text{in } L^r(\mathbb{R}^N) \quad \text{and} \quad w_n(x) \to 0 \quad \text{a.e. in } \mathbb{R}^N \text{ as } n \to \infty \tag{3.11}$$

for $1 < r < p_*$. Hence by using (3.5) we get

$$0 < \frac{\min\{1, dm_0\}}{\lambda\theta p} \leq \limsup_{n\to\infty}\int_{\mathbb{R}^N} \frac{|F(x,u_n)|}{\|u_n\|_X^p}\,dx. \tag{3.12}$$

On the other hand, from assumption (F2) and relation (3.11), it follows that

$$\begin{aligned}
&\int_{\Omega_n(0,r_0)} \frac{F(x,u_n)}{\|u_n\|_X^p}\,dx \\
&\quad\leq \int_{\Omega_n(0,r_0)} \frac{\rho(x)|u_n(x)| + \frac{\sigma(x)}{q}|u_n(x)|^q}{\|u_n\|_X^p}\,dx \\
&\quad\leq \frac{C_1}{\|u_n\|_X^p}\|\rho\|_{L^{q'}(\mathbb{R}^N)}\|u_n\|_{L^q(\mathbb{R}^N)} \\
&\qquad + \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{q}\int_{\Omega_n(0,r_0)} \bigl|u_n(x)\bigr|^{q-p}\bigl|w_n(x)\bigr|^p\,dx \\
&\quad\leq \frac{C_1}{\|u_n\|_X^p}\|\rho\|_{L^{q'}(\mathbb{R}^N)}\|u_n\|_{L^q(\mathbb{R}^N)} + \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{q} r_0^{q-p}\int_{\mathbb{R}^N} \bigl|w_n(x)\bigr|^p\,dx
\end{aligned}$$

$$\leq \frac{C_2}{\|u_n\|_X^p}\|\rho\|_{L^{q'}(\mathbb{R}^N)}\|u_n\|_X + \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{q} r_0^{q-p}\int_{\mathbb{R}^N}\left|w_n(x)\right|^p dx$$
$$\leq \frac{C_3}{\|u_n\|_X^{p-1}} + \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{q} r_0^{q-p}\int_{\mathbb{R}^N}\left|w_n(x)\right|^p dx \to 0 \quad \text{as } n\to\infty \tag{3.13}$$

for some positive constants C_i ($i = 1, 2, 3$). Set $\kappa' = \kappa/(\kappa-1)$. Since $\kappa > N/p$, we get $1 < \kappa' p < p_*$. Hence, it follows from (F5), (3.10), and (3.11) that

$$\begin{aligned}
\int_{\Omega_n(r_0,\infty)} \frac{|F(x,u_n)|}{\|u_n\|_X^p}\,dx &\leq \int_{\Omega_n(r_0,\infty)} \frac{|F(x,u_n)|}{|u_n(x)|^p}\left|w_n(x)\right|^p dx \\
&\leq \left\{\int_{\Omega_n(r_0,\infty)}\left(\frac{|F(x,u_n)|}{|u_n(x)|^p}\right)^\kappa dx\right\}^{\frac{1}{\kappa}}\left\{\int_{\Omega(r_0,\infty)}\left|w_n(x)\right|^{\kappa' p}\right\}^{\frac{1}{\kappa'}} \\
&\leq c_0^{\frac{1}{\kappa}}\left\{\int_{\Omega_n(r_0,\infty)}\mathfrak{F}(x,u_n)\,dx\right\}^{\frac{1}{\kappa}}\left\{\int_{\mathbb{R}^N}\left|w_n(x)\right|^{\kappa' p}\right\}^{\frac{1}{\kappa'}} \\
&\leq c_0^{\frac{1}{\kappa}}\left(\frac{c+1}{\lambda}\right)^{\frac{1}{\kappa}}\left\{\int_{\mathbb{R}^N}\left|w_n(x)\right|^{\kappa' p}\right\}^{\frac{1}{\kappa'}} \to 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{3.14}$$

Combining the estimates in (3.13) with (3.14), we have

$$\int_{\mathbb{R}^N}\frac{|F(x,u_n)|}{\|u_n\|_X^p}\,dx = \int_{\Omega_n(0,r_0)}\frac{|F(x,u_n)|}{\|u_n\|_X^p}\,dx + \int_{\Omega_n(r_0,\infty)}\frac{|F(x,u_n)|}{\|u_n\|_X^p}\,dx \to 0 \quad \text{as } n\to\infty,$$

which contradicts (3.12). This completes the proof. □

Using Lemma 3.3, we prove the existence of a nontrivial weak solution for our problem under the considered assumptions.

Theorem 3.4 *Assume that* (V), (M1), (M2), (J1)–(J5), *and* (F1)–(F5) *hold. Then problem* (P) *has a nontrivial weak solution for all* $\lambda > 0$.

Proof Note that $I_\lambda(0) = 0$. In view of Lemma 3.2, the geometric conditions in the mountain pass theorem are fulfilled. And also I_λ satisfies the $(C)_c$ condition for any $\lambda > 0$ by Lemma 3.3. Hence, problem (P) has a nontrivial weak solution for all $\lambda > 0$. This completes the proof. □

Next, under assumption (F6) instead of (F5), we show that I_λ satisfies the Cerami condition.

Lemma 3.5 *Assume that* (V), (M1), (M2), (J1)–(J5), (F1)–(F4), *and* (F6) *hold. Then the functional* I_λ *satisfies the* $(C)_c$ *condition for any* $\lambda > 0$.

Proof For $c \in \mathbb{R}$, let $\{u_n\}$ be a $(C)_c$-sequence in X satisfying (3.2). Following the proof of Lemma 3.3, we only prove that $\{u_n\}$ is bounded in X. To this end, arguing by contradiction, suppose that $\|u_n\|_X \to \infty$ as $n \to \infty$. Let $v_n = u_n/\|u_n\|_X$. Then $\|v_n\|_X = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ as $n \to \infty$ in X. Thus by an embedding theorem, for $1 < s < p_*$, we have

$$v_n \to v \quad \text{in } L^s\left(\mathbb{R}^N\right) \quad \text{and} \quad v_n(x) \to v(x) \quad \text{a.e. in } \mathbb{R}^N \text{ as } n \to \infty.$$

From (M1), (M2), (J5), and (F6), it follows that

$$
\begin{aligned}
c+1 &\geq I_\lambda(u_n) - \frac{1}{\mu}\langle I_\lambda'(u_n), u_n\rangle \\
&= \frac{1}{p}\int_{\mathbb{R}^N} |\Delta u_n|^p\,dx + \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx\right) \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u_n|^p\,dx - \lambda\int_{\mathbb{R}^N} F(x,u_n)\,dx \\
&\quad - \frac{1}{\mu}\int_{\mathbb{R}^N} |\Delta u_n|^p\,dx - \frac{1}{\mu}M\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u_n)\,dx\right)\int_{\mathbb{R}^N} \varphi(x,\nabla u_n)\cdot\nabla u_n\,dx \\
&\quad - \frac{1}{\mu}\int_{\mathbb{R}^N} V(x)|u_n|^p\,dx + \frac{\lambda}{\mu}\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx \\
&\geq \left(\frac{1}{p}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} |\Delta u_n|^p\,dx + m_0\left(\frac{1}{\theta p}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} \varphi(x,\nabla u_n)\cdot\nabla u_n\,dx \\
&\quad + \left(\frac{1}{p}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x)|u_n|^p\,dx - \frac{\lambda\varrho}{\mu}\int_{\mathbb{R}^N} |u_n|^p\,dx \\
&\geq \min\{dm_0,1\}\left(\frac{1}{\theta p}-\frac{1}{\mu}\right)\left(\int_{\mathbb{R}^N} |\Delta u_n|^p\,dx + \int_{\mathbb{R}^N} |\nabla u_n|^p\,dx + \int_{\mathbb{R}^N} V(x)|u_n|^p\,dx\right) \\
&\quad - \frac{\lambda\varrho}{\mu}\int_{\mathbb{R}^N} |u_n|^p\,dx \\
&\geq \min\{dm_0,1\}\left(\frac{1}{\theta p}-\frac{1}{\mu}\right)\|u_n\|_X^p - \frac{\lambda\varrho}{\mu}\int_{\mathbb{R}^N} |u_n|^p\,dx \quad \text{for large } n\in\mathbb{N}.
\end{aligned}
$$

This implies

$$
1 \leq \frac{\lambda\varrho\theta p}{\min\{dm_0,1\}(\mu-\theta p)}\limsup_{n\to\infty}\|v_n\|_{L^p(\mathbb{R}^N)}^p = \frac{\lambda\varrho\theta p}{\min\{dm_0,1\}(\mu-\theta p)}\|v\|_{L^p(\mathbb{R}^N)}^p. \tag{3.15}
$$

Hence, due to (3.15), we see that $v \neq 0$. From the same argument as in Lemma 3.3, we can show that the relations (3.6), (3.7), and (3.8) hold, and hence we conclude that relation (3.9) is true. Therefore we get a contradiction. Thus $\{u_n\}$ is bounded in X. This completes the proof. □

Next, applying the fountain theorem in [32, Theorem 3.6] with the oddity of f, we demonstrate infinitely many weak solutions for problem (P). To do this, let W be a reflexive and separable Banach space. Then there are $\{e_n\} \subseteq W$ and $\{f_n^*\} \subseteq W^*$ such that

$$
W = \overline{\operatorname{span}\{e_n : n = 1,2,\dots\}}, \qquad W^* = \overline{\operatorname{span}\{f_n^* : n = 1,2,\dots\}},
$$

and

$$
\langle f_i^*, e_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
$$

Let us denote $W_n = \operatorname{span}\{e_n\}$, $Y_k = \bigoplus_{n=1}^k W_n$, and $Z_k = \overline{\bigoplus_{n=k}^\infty W_n}$. In order to establish the existence result, we use the following Fountain theorem.

Lemma 3.6 ([1, 32]) *Let X be a real reflexive Banach space, $I \in C^1(X,\mathbb{R})$ satisfies the $(C)_c$ condition for any $c > 0$ and I is even. If for each sufficiently large $k \in \mathbb{N}$, there exist $\rho_k > \delta_k > 0$ such that the following conditions hold*:

(1) $b_k := \inf\{I(u) : u \in Z_k, \|u\|_X = \delta_k\} \to \infty$ *as* $k \to \infty$;
(2) $a_k := \max\{I(u) : u \in Y_k, \|u\|_X = \rho_k\} \le 0$.

Then the functional I has an unbounded sequence of critical values, i.e., there exists a sequence $\{u_n\} \subset X$ such that $I'(u_n) = 0$ and $I(u_n) \to +\infty$ as $n \to +\infty$.

Theorem 3.7 *Assume that* (V), (M1), (M2), (J1)–(J5), *and* (F1)–(F5) *hold. If $\Phi_0(x,-v) = \Phi_0(x,v)$ holds for all $(x,v) \in \mathbb{R}^N \times \mathbb{R}^N$ and $f(x,-t) = -f(x,t)$ holds for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, then for any $\lambda > 0$, problem* (P) *possesses an unbounded sequence of nontrivial weak solutions $\{u_n\}$ in X such that $I_\lambda(u_n) \to \infty$ as $n \to \infty$.*

Proof It is obvious that I_λ is an even functional and satisfies the $(C)_c$ condition. It suffices to show that there exist $\rho_k > \delta_k > 0$ such that

(1) $b_k := \inf\{I_\lambda(u) : u \in Z_k, \|u\|_X = \delta_k\} \to \infty$ as $n \to \infty$;
(2) $a_k := \max\{I_\lambda(u) : u \in Y_k, \|u\|_X = \rho_k\} \le 0$,

for k large enough. Denote

$$\alpha_k := \sup_{u \in Z_k, \|u\|_X = 1} \|u\|_{L^q(\mathbb{R}^N)}.$$

Then we have $\alpha_k \to 0$ as $k \to \infty$. In fact, assume to the contrary that there exist $\varepsilon_0 > 0$ and a sequence $\{u_k\}$ in Z_k such that

$$\|u_k\|_X = 1 \quad \text{and} \quad \|u_k\|_{L^q(\mathbb{R}^N)} \ge \varepsilon_0$$

for all $k \ge k_0$. By the boundedness of the sequence $\{u_k\}$ in X, we can find an element $u \in X$ such that $u_k \rightharpoonup u$ in X as $n \to \infty$ and

$$\langle f_j^*, u \rangle = \lim_{k\to\infty} \langle f_j^*, u_k \rangle = 0$$

for $j = 1, 2, \dots$. Thus we deduce $u = 0$. However, we see that

$$\varepsilon_0 \le \lim_{k\to\infty} \|u_k\|_{L^q(\mathbb{R}^N)} = \|u\|_{L^q(\mathbb{R}^N)} = 0,$$

which is a contradiction.

For any $u \in Z_k$, we may suppose that $\|u\|_X > 1$. According to assumptions (M1), (M2), (J3), and (F2), we obtain that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{p}\int_{\mathbb{R}^N} |\Delta u|^p \,dx + \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u)\,dx\right) \\ &\quad + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p \,dx - \lambda \int_{\mathbb{R}^N} F(x,u)\,dx \\ &\ge \frac{1}{p}\int_{\mathbb{R}^N} |\Delta u|^p \,dx + \frac{1}{\theta} M\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla u)\,dx\right)\int_{\mathbb{R}^N} \Phi_0(x,\nabla u)\,dx \\ &\quad + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p \,dx - \lambda \int_{\mathbb{R}^N} \left|\rho(x)\right||u|\,dx - \lambda \int_{\mathbb{R}^N} \frac{|\sigma(x)|}{q}|u|^q \,dx \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\min\{1,dm_0\}}{\theta p}\left(\int_{\mathbb{R}^N}|\Delta u|^p\,dx+\int_{\mathbb{R}^N}|\nabla u|^p\,dx+\int_{\mathbb{R}^N}V(x)|u|^p\,dx\right)\\
&\quad-\lambda\int_{\mathbb{R}^N}\bigl|\rho(x)\bigr||u|\,dx-\lambda\int_{\mathbb{R}^N}\frac{|\sigma(x)|}{q}|u|^q\,dx\\
&\geq \frac{\min\{1,dm_0\}}{\theta p}\|u\|_X^p-2\lambda\|\rho\|_{L^{q'}(\mathbb{R}^N)}\|u\|_{L^q(\mathbb{R}^N)}-\frac{2\lambda}{q}\|\sigma\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N}|u|^q\,dx\\
&\geq \frac{\min\{1,dm_0\}}{\theta p}\|u\|_X^p-2\lambda C_4\|u\|_X-\frac{2\lambda}{q}\alpha_k^q C_5\|u\|_X^q,
\end{aligned}
$$

where C_4 and C_5 are positive constants. If we take

$$
\delta_k=\left(\frac{2\lambda C_5\alpha_k^q}{\min\{1,dm_0\}}\right)^{1/(p-q)},
$$

then $\delta_k\to\infty$ as $k\to\infty$ because $\theta p<q$ and $\alpha_k\to 0$ as $k\to\infty$. Hence, if $u\in Z_k$ and $\|u\|_X=\delta_k$, then we conclude that

$$
I_\lambda(u)\geq\min\{1,dm_0\}\left(\frac{1}{\theta p}-\frac{1}{q}\right)\delta_k^p-2\lambda C_4\delta_k\to\infty\quad\text{as } k\to\infty.
$$

This implies that condition (1) holds.

The proof of condition (2) proceeds analogously as in the proof of [1, Theorem 1.3]. For the reader's convenience, we give the proof. Assume that condition (2) is not true. Then for some k there exists a sequence $\{u_n\}$ in Y_k such that

$$
\|u_n\|_X\to\infty\quad\text{as } n\to\infty\quad\text{and}\quad I_\lambda(u_n)\geq 0. \tag{3.16}
$$

Set $w_n=u_n/\|u_n\|_X$. Note that $\|w_n\|_X=1$. Since $\dim Y_k<\infty$, there exists $w\in Y_k\setminus\{0\}$ such that, up to a subsequence,

$$
\|w_n-w\|_X\to 0\quad\text{and}\quad w_n(x)\to w(x)
$$

for almost all $x\in\mathbb{R}^N$ as $n\to\infty$. If $w(x)\neq 0$, then $|u_n(x)|\to\infty$ for all $x\in\mathbb{R}^N$ as $n\to\infty$. Hence we obtain by assumption (F4) that

$$
\lim_{n\to\infty}\frac{F(x,u_n(x))}{\|u_n\|_X^{\theta p}}=\lim_{n\to\infty}\frac{F(x,u_n(x))}{|u_n(x)|^{\theta p}}\bigl|w_n(x)\bigr|^{\theta p}=\infty
$$

for all $x\in\Omega_2:=\{x\in\mathbb{R}^N:w(x)\neq 0\}$. As in the proof of Lemma 3.3, we have

$$
\int_{\Omega_2}\frac{F(x,u_n(x))}{\|u_n\|_X^{\theta p}}\,dx\to\infty\quad\text{as } n\to\infty.
$$

Therefore, we conclude that

$$
\begin{aligned}
I_\lambda(u_n)&\leq\frac{1}{p}\|u_n\|_X^p+\mathcal{M}\left(\int_{\mathbb{R}^N}\Phi_0(x,\nabla u_n)\,dx\right)-\lambda\int_{\mathbb{R}^N}F(x,u_n)\,dx\\
&\leq\frac{1}{p}\|u_n\|_X^{\theta p}+\mathcal{M}(1)\left(1+\left(\|a\|_{L^{p'}(\mathbb{R}^N)}+\frac{b}{p}\right)^\theta\right)\|u_n\|_X^{\theta p}-\lambda\int_{\mathbb{R}^N}F(x,u_n)\,dx
\end{aligned}
$$

$$\begin{aligned}&\le \|u_n\|_X^{\theta p}\left(\frac{1}{p}+\mathcal{M}(1)\left(1+\left(\|a\|_{L^{p'}(\mathbb{R}^N)}+\frac{b}{p}\right)^{\theta}\right)-\lambda\int_{\Omega_2}\frac{F(x,u_n(x))}{\|u_n\|_X^p}\,dx\right)\\&\to -\infty \quad \text{as } n\to\infty,\end{aligned}$$

which contradicts (3.16). This completes the proof. □

Remark 3.8 Although we replace (F5) with (F6) in the assumptions of Theorem 3.7, we can show that problem (P) possesses an unbounded sequence of nontrivial weak solutions $\{u_n\}$ in X such that $I_\lambda(u_n)\to\infty$ as $n\to\infty$.

Appendix: Proof of Lemma 2.5

In this section, we give a proof of Lemma 2.5 for the reader's convenience. In fact, we consider that it is a well-known fact to researchers in this area.

Proof Let $\{u_n\}$ be a sequence in X such that $u_n \rightharpoonup u$ in X as $n\to\infty$. Then $\{u_n\}$ is bounded in X, and we know the embeddings $X\hookrightarrow L^p(\mathbb{R}^N)$ and $X\hookrightarrow L^q(\mathbb{R}^N)$ are compact for $p<q<p_*$. So we know that

$$u_n\to u \quad \text{in } L^p(\mathbb{R}^N) \quad \text{and} \quad u_n\to u \quad \text{in } L^q(\mathbb{R}^N) \text{ as } n\to\infty.$$

First we prove that Ψ is weakly strongly continuous in X. Let $u_n\to u$ in $L^p(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ as $n\to\infty$. By the convergence principle, there exist a subsequence $\{u_{n_k}\}$ such that $u_{n_k}(x)\to u(x)$ as $k\to\infty$ for almost all $x\in\mathbb{R}^N$ and a function $u_0\in L^p(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ such that $|u_{n_k}(x)|\le u_0(x)$ for all $k\in\mathbb{N}$ and for almost all $x\in\mathbb{R}^N$. Therefore from (F2), we deduce

$$\begin{aligned}\int_{\mathbb{R}^N}\left|F(x,u_{n_k})\right|dx&\le\int_{\mathbb{R}^N}\rho(x)\left|u_{n_k}(x)\right|+\frac{\sigma(x)}{q}\left|u_{n_k}(x)\right|^q dx\\&\le\|\rho\|_{L^{q'}(\mathbb{R}^N)}\|u_0\|_{L^q(\mathbb{R}^N)}+\|\sigma\|_{L^\infty(\mathbb{R}^N)}\|u_0\|_{L^q(\mathbb{R}^N)}^q.\end{aligned}$$

Since function f satisfies the Carathéodory condition by (F1), we obtain that $F(x,u_{n_k})\to F(x,u)$ as $k\to\infty$ for almost all $x\in\mathbb{R}^N$. Therefore, the Lebesgue convergence theorem tells us that

$$\int_{\mathbb{R}^N}F(x,u_{n_k})\,dx\to\int_{\mathbb{R}^N}F(x,u)\,dx$$

as $k\to\infty$, which says $\Psi(u_{n_k})\to\Psi(u)$ as $k\to\infty$. Thus Ψ is weakly strongly continuous in X.

Next, we show that Ψ' is weakly strongly continuous on X. By (F2) and Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\left|f(x,u_n)-f(x,u)\right|^{q'}dx&\le C_6\int_{\mathbb{R}^N}\left|f(x,u_n)\right|^{q'}+\left|f(x,u)\right|^{q'}dx\\&\le C_7\int_{\mathbb{R}^N}\left|\rho(x)\right|^{q'}+\left\||\sigma|^{q'}\right\|_{L^\infty(\mathbb{R}^N)}\left(|u_n|^q+|u|^q\right)dx \qquad \text{(A.1)}\end{aligned}$$

for some positive constants C_6, C_7, which implies that $|f(x,u_n)-f(x,u)|^{q'}\le g(x)$ for almost all $x\in\mathbb{R}^N$ and for some $g\in L^1(\mathbb{R}^N)$. Since $u_n\to u$ in $L^p(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ and almost all in

$\mathbb{R}^N$, it follows from (A.1) and the convergence principle that $f(x,u_n) \to f(x,u)$ for almost all $x \in \mathbb{R}^N$. Combining this with the Lebesgue convergence theorem, we have

$$\begin{aligned}\left\|\Psi'(u_n) - \Psi'(u)\right\|_{X^*} &= \sup_{\|v\|_X \le 1} \left|\left\langle \Psi'(u_n) - \Psi'(u), v\right\rangle\right| \\ &= \sup_{\|v\|_X \le 1} \int_{\mathbb{R}^N} \left|f(x,u_n) - f(x,u)\right| |v|\, dx \\ &\le \left(\int_{\mathbb{R}^N} \left|f(x,u_n) - f(x,u)\right|^{q'} dx\right)^{\frac{1}{q'}} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Therefore, we derive that $\Psi'(u_n) \to \Psi'(u)$ in X. This completes the proof. □

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Authors' contributions
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