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Multiplicity of Small or Large Energy Solutions for Kirchhoff–Schrödinger-Type Equations Involving the Fractional *p*-Laplacian in $\mathbb{R}^N$

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Abstract: We herein discuss the following elliptic equations: $\mathcal{M}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dx\,dy\Big)(-\Delta)^s_p u + V(x)|u|^{p-2}u = \lambda f(x,u)$ in $\mathbb{R}^N$, where $(-\Delta)^s_p$ is the fractional p-Laplacian defined by $(-\Delta)^s_p u(x) = 2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,dy$, $x \in \mathbb{R}^N$. Here, $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$, $V : \mathbb{R}^N \to (0,\infty)$ is a continuous function and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is the Carathéodory function. Furthermore, $\mathcal{M} : \mathbb{R}^+_0 \to \mathbb{R}^+$ is a Kirchhoff-type function. This study has two aims. One is to study the existence of infinitely many large energy solutions for the above problem via the variational methods. In addition, a major point is to obtain the multiplicity results of the weak solutions for our problem under various assumptions on the Kirchhoff function $\mathcal{M}$ and the nonlinear term f. The other is to prove the existence of small energy solutions for our problem, in that the sequence of solutions converges to 0 in the L^∞-norm.

Keywords: fractional p-Laplacian; Kirchhoff-type equations; fountain theorem; modified functional methods; Moser iteration method

1. Introduction

Significant attention has been focused on the study of fractional-type operators in view of the mathematical theory to some phenomena: the social sciences, quantum mechanics, continuum mechanics, phase transition phenomena, game theory, and Levy processes [1–5].

Herein, we discuss the results regarding the existence and multiplicity of nontrivial weak solutions for Kirchhoff-type equations

$$\mathcal{M}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dx\,dy\Big)(-\Delta)^s_p u + V(x)|u|^{p-2}u = \lambda f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $(-\Delta)^s_p$ is the fractional p-Laplacian operator defined by

$$(-\Delta)^s_p u(x) = 2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,dy$$

for $x \in \mathbb{R}^N$, with $0 < s < 1 < p < \infty$, $ps < N$, $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$, $V : \mathbb{R}^N \to (0,\infty)$ is a continuous function and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is the Carathéodory function. Furthermore, $\mathcal{M} \in C(\mathbb{R}^+_0, \mathbb{R}^+)$ is a Kirchhoff-type function.

Considering the effects of the change in the length of the stings that occurred by transverse vibrations, Kirchhoff in [6] originally proposed the following equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right| dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

which is the generalization of the classical D'Alembert's wave equation.

Subsequently, most researchers have extensively studied Kirchhoff-type equations associated with the fractional p-Laplacian problems in various ways; see [7–14] and the references therein. The critical point theory, originally introduced in [15] is critical in obtaining the solutions to elliptic equations of the variational type. It is considered that one of the crucial aspects for assuring the boundedness of the Palais–Smale sequence of the Euler–Lagrange functional, which is important to apply the critical point theory, is the Ambrosetti and Rabinowitz condition ((AR)-condition, briefly) in [15].

(AR) There exist positive constants C and ζ such that $\zeta > p$ and

$$0 < \zeta \mathcal{F}(x,t) \le f(x,t)t \quad \text{for} \quad x \in \Omega \quad \text{and} \quad |t| \ge C,$$

where $\mathcal{F}(x,t) = \int_0^t f(x,s)\,ds$ and Ω is a bounded domain in $\mathbb{R}^N$.

Most results for our problem (1) are to establish the existence of nontrivial solutions under the (AR)-condition; see [7,10,14,16] for bounded domains and [11] for a whole space $\mathbb{R}^N$. The (AR)-condition is natural and important to guarantee the boundedness of the Palais–Smale sequence; this condition, however, is too restrictive and gets rid of many nonlinearities. Many authors have attempted to eliminate the (AR)-condition for elliptic equations associated with the p-Laplacian; see [17–20] and also see [21–25] for the superlinear problems of the fractional Laplacian type.

In this regard, we show that problem (1) permits the existence of multiple solutions under various conditions weaker than the (AR)-condition. In particular, following ([17], Remark 1.8), there exist many examples that do not fulfill the condition of the nonlinear term f in [18,19,21,22,24–26]. Thus, motivated by these examples, the first aim of this paper is to demonstrate the existence of infinitely many large solutions for the problem above using the fountain theorem. One of novelties of this study is to obtain the multiplicity results for problem (1) when f contains mild assumptions different from those of [18,19,21,22,24–26] (see Theorem 1). The other is to demonstrate this result with sufficient conditions for the modified Kirchhoff function $\mathcal{M}$, and the assumption on f similar to that in [18,26] (see Theorem 2). As far as we are aware, none have reported such multiplicity results for our problem under the assumptions given in Theorem 2 of Section 2.

The second aim is to investigate that the existence of small energy solutions for problem (1), whose L^∞-norms converge to zero, depends only on the local behavior and assumptions on $f(x,t)$, and only sufficiently small t are required. Wang [27] initially investigated that nonlinear boundary value problems

$$\begin{cases} -\Delta u = \lambda |u|^{q-1} u + f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases}$$

admit a sequence of infinitely many small solutions where $0 < q < 1$, and the nonlinear term f was considered as a perturbation term. He employed global variational formulations and cut off techniques to obtain this existence result that is a local phenomenon and is forced by the sublinear term. Utilizing the argument in [27], Guo [28] showed that the p-Laplacian equations with indefinite concave nonlinearities have infinitely many solutions. In this regard, lots of authors have considered the results for the elliptic equations with nonlinear terms on a bounded domain in $\mathbb{R}^N$; see [29–31]. It is well known that the studies in [14,17,19,21,22,26,29,32,33] as well as our first primary result essentially demand some global conditions on $f(x,t)$ for t, such as oddness and behavior at infinity, for applying the fountain theorem to allow an infinite number of solutions. In contrast to these studies that yield large solutions in that they form an unbounded sequence, by modifying and extending the function $f(x,t)$ to a adequate function $\tilde{f}(x,t)$, the authors in [27–29] investigated the existence of small energy solutions to equations of the elliptic type. A natural question is whether the results in [27–31] may be extended to Equation (1). As is known, such a result for Kirchhoff–Schrödinger-type equations involving the non-local fractional p-Laplacian on the whole space $\mathbb{R}^N$ has not been much studied, although a given domain is bounded. In particular, no results are available even though the fractional p-Laplacian problems without Kirchhoff function $\mathcal{M}$ are considered, and we are only aware of paper [34] in this direction. In comparison with the papers [27–29], the main difficulty to obtain our second aim is to show the L^∞-bound of weak solutions for problem (1). We remark that the strategy for obtaining this multiplicity is to assign a regularity-type result based on the work of Drábek, Kufner, and Nicolosi in [35]. Furthermore, it is noteworthy that the conditions on $f(x,t)$ are imposed near zero; in particular, $f(x,t)$ is odd in t for a small t, and no conditions on $f(x,t)$ exist at infinity.

This paper is structured as follows. In Section 2, we state the basic results to solve the Kirchhoff-type equation, and review the well-known facts for the fractional Sobolev spaces. Moreover, under certain conditions on f, our problem admits a sequence of infinitely many large energy solutions of our problem (1) via the fountain theorem. Moreover, we assign the existence of nontrivial weak solutions for our problem with new conditions for the modified Kirchhoff function $\mathcal{M}$ and the nonlinear term f. In Section 3, we present the existence of small energy solutions for our problem in that the sequence of solutions converges to 0 in the L^∞-norm. Hence, we employ the regularity result on the L^∞-bound of a weak solution and the modified functional method.

2. Existence of Infinitely Many Large Energy Solutions

In this section, we recall some elementary concepts and properties of the fractional Sobolev spaces. We refer the reader to [4,36–38] for the detailed descriptions.

Suppose that

($\mathcal{V}1$) $V \in C(\mathbb{R}^N)$, $\inf_{x\in\mathbb{R}^N} V(x) > 0$.
($\mathcal{V}2$) $\text{meas}\{x \in \mathbb{R}^N : V(x) \le V_0\} < +\infty$ for all $V_0 \in \mathbb{R}$.

Let $0 < s < 1$ and $1 < p < +\infty$. We define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ by

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy < +\infty \right\},$$

endowed with the norm

$$\|u\|^p_{W^{s,p}(\mathbb{R}^N)} := |u|^p_{W^{s,p}(\mathbb{R}^N)} + \|u\|^p_{L^p(\mathbb{R}^N)} \quad \text{with} \quad |u|^p_{W^{s,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy.$$

Furthermore, we denote the basic function space $W(\mathbb{R}^N)$ by the completion of $C_0^\infty(\mathbb{R}^N)$ in $W^{s,p}(\mathbb{R}^N)$, equipped with the norm

$$\|u\|^p_{W(\mathbb{R}^N)} := |u|^p_{W^{s,p}(\mathbb{R}^N)} + \|u\|^p_{p,V} \quad \text{with} \quad \|u\|^p_{p,V} := \int_{\mathbb{R}^N} V(x)|u|^p\,dx.$$

Following a similar argument in [11,12], we can easily show that the space $W(\mathbb{R}^N)$ is a separable and reflexive Banach space.

We recall the continuous or compact embedding theorem in ([11], Lemma 1) and ([24], Lemma 2.1).

Lemma 1. *Let $0 < s < 1 < p < +\infty$ with $ps < N$. Then, there exists a positive constant $C = C(N,p,s)$ such that, for all $u \in W^{s,p}(\mathbb{R}^N)$,*

$$\|u\|_{L^{p_s^*}(\mathbb{R}^N)} \le C\,|u|_{W^{s,p}(\mathbb{R}^N)},$$

where $p_s^* = \frac{Np}{N-sp}$ *is the fractional critical exponent. Consequently, the space* $W^{s,p}(\mathbb{R}^N)$ *is continuously embedded in* $L^q(\mathbb{R}^N)$ *for any* $q \in [p, p_s^*]$*. Moreover, the space* $W^{s,p}(\mathbb{R}^N)$ *is compactly embedded in* $L^q_{loc}(\mathbb{R}^N)$ *for any* $q \in [p, p_s^*)$*.*

Lemma 2. *Let* $0 < s < 1 < p < +\infty$ *with* $ps < N$*. Suppose that the assumptions* $(\mathcal{V}1)$ *and* $(\mathcal{V}2)$ *hold. If* $r \in [p, p_s^*]$*, then the embeddings*

$$W(\mathbb{R}^N) \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$$

are continuous with $\|u\|^p_{W^{s,p}(\mathbb{R}^N)} \le C\|u\|^p_{W(\mathbb{R}^N)}$ *for all* $u \in W(\mathbb{R}^N)$*. In particular, there exists a constant* $K_r > 0$ *such that* $\|u\|_{L^r(\mathbb{R}^N)} \le K_r\|u\|_{W(\mathbb{R}^N)}$ *for all* $u \in W(\mathbb{R}^N)$*. If* $r \in [p, p_s^*)$*, then the embedding*

$$W(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$$

is compact.

Definition 1. *Let* $0 < s < 1 < p < +\infty$*. We say that* $u \in W(\mathbb{R}^N)$ *is a weak solution of problem* (1) *if*

$$\mathcal{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)}) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy \tag{2}$$

$$+\int_{\mathbb{R}^N} V(x)\,|u|^{p-2}\,uv\,dx = \lambda \int_{\mathbb{R}^N} f(x,u)v\,dx$$

for any v *in* $W(\mathbb{R}^N)$*.*

We assume that the Kirchhoff function $\mathcal{M} : \mathbb{R}_0^+ \to \mathbb{R}^+$ satisfies the following conditions:

- $(\mathcal{M}1)$ $\mathcal{M} \in C(\mathbb{R}_0^+, \mathbb{R}^+)$ satisfies $\inf_{t\in\mathbb{R}_0^+} \mathcal{M}(t) \ge m_0 > 0$, where m_0 is a constant.
- $(\mathcal{M}2)$ There exists $\theta \in [1, \frac{N}{N-ps})$ such that $\theta\mathfrak{M}(t) \ge \mathcal{M}(t)t$ for any $t \ge 0$, where $\mathfrak{M}(t) := \int_0^t \mathcal{M}(\tau)d\tau$.

A typical example for $\mathcal{M}$ is given by $\mathcal{M}(t) = b_0 + b_1t^n$ with $n > 0, b_0 > 0$ and $b_1 \ge 0$.

Next, we consider the appropriate assumptions for the nonlinear term f. Let us denote $\mathcal{F}(x,t) = \int_0^t f(x,s)\,ds$ and let $\theta \in \mathbb{R}$ be given in $(\mathcal{M}2)$.

- $(\mathcal{F}1)$ $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition.
- $(\mathcal{F}2)$ There exist nonnegative functions $\rho \in L^{p'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\sigma \in L^{\frac{p_s^*}{p_s^*-q}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that
$$|f(x,t)| \le \rho(x) + \sigma(x)\,|t|^{q-1}, \quad q \in (\theta p, p_s^*)$$
for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.
- $(\mathcal{F}3)$ $\lim_{|t|\to\infty} \frac{\mathcal{F}(x,t)}{|t|^{\theta p}} = \infty$ uniformly for almost all $x \in \mathbb{R}^N$.
- $(\mathcal{F}4)$ There exist real numbers $c_0 > 0$, $r_0 \ge 0$, and $\kappa > \frac{N}{ps}$ such that
$$|\mathcal{F}(x,t)|^\kappa \le c_0\,|t|^{\kappa p}\,\mathfrak{F}(x,t)$$
for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $|t| \ge r_0$, where $\mathfrak{F}(x,t) = \frac{1}{\theta p}f(x,t)t - \mathcal{F}(x,t) \ge 0$.
- $(\mathcal{F}5)$ There exist $\mu > \theta p$ and $\varrho > 0$ such that
$$\mu\mathcal{F}(x,t) \le tf(x,t) + \varrho t^p$$
for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

For $u \in W(\mathbb{R}^N)$, the Euler–Lagrange functional $\mathcal{E}_\lambda : W(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$\mathcal{E}_\lambda(u) = A_{s,p}(u) - \lambda\Psi(u),$$

where

$$A_{s,p}(u) := \frac{1}{p}(\mathfrak{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u\|^p_{p,V}) \quad \text{and} \quad \Psi(u) := \int_{\mathbb{R}^N} \mathcal{F}(x,u)\,dx.$$

Then, it is easily verifiable that $A_{s,p} \in C^1(W(\mathbb{R}^N),\mathbb{R})$ and $\Psi \in C^1(W(\mathbb{R}^N),\mathbb{R})$. Therefore, the functional $\mathcal{E}_\lambda$ is Fréchet differentiable on $W(\mathbb{R}^N)$ and its (Fréchet) derivative is as follows:

$$\begin{aligned}\langle \mathcal{E}'_\lambda(u), v\rangle &= \langle A'_{s,p}(u) - \lambda\Psi'(u), v\rangle \\ &= \mathcal{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)}) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy \\ &\quad + \int_{\mathbb{R}^N} V(x)\,|u|^{p-2}\,uv\,dx - \lambda\int_{\mathbb{R}^N} f(x,u)v\,dx\end{aligned} \tag{3}$$

for any $u, v \in W(\mathbb{R}^N)$. Following Lemmas 2 and 3 in [11], the functional $A_{s,p}$ is weakly lower semi-continuous in $W(\mathbb{R}^N)$ and Ψ is weakly continuous in $W(\mathbb{R}^N)$.

We now show that the functional $\mathcal{E}_\lambda$ satisfies the Cerami condition ($(C)_c$-condition, briefly), i.e., for $c \in \mathbb{R}$, any sequence $\{u_n\} \subset W(\mathbb{R}^N)$ such that $\mathcal{E}_\lambda(u_n) \to c$ and $\|\mathcal{E}'_\lambda(u_n)\|_{W^*(\mathbb{R}^N)}(1 + \|u_n\|_{W(\mathbb{R}^N)}) \to 0$ as $n \to \infty$ has a convergent subsequence. Here, $W^*(\mathbb{R}^N)$ is a dual space of $W(\mathbb{R}^N)$. This plays a decisive role in establishing the existence of nontrivial weak solutions.

Lemma 3. *Let $0 < s < 1 < p < +\infty$ and $ps < N$. Assume that* ($\mathcal{V}$1), ($\mathcal{V}$2), ($\mathcal{M}$1), ($\mathcal{M}$2), *and* ($\mathcal{F}$1)–($\mathcal{F}$4) *hold. Then, the functional $\mathcal{E}_\lambda$ satisfies the $(C)_c$-condition for any $\lambda > 0$.*

Proof. For $c \in \mathbb{R}$, let $\{u_n\}$ be a $(C)_c$-sequence in $W(\mathbb{R}^N)$, that is,

$$\mathcal{E}_\lambda(u_n) \to c \quad \text{and} \quad \|\mathcal{E}'_\lambda(u_n)\|_{W^*(\mathbb{R}^N)}(1 + \|u_n\|_{W(\mathbb{R}^N)}) \to 0 \qquad \text{as} \quad n \to \infty, \tag{4}$$

which implies that

$$c = \mathcal{E}_\lambda(u_n) + o(1) \quad \text{and} \quad \langle \mathcal{E}'_\lambda(u_n), u_n\rangle = o(1), \tag{5}$$

where $o(1) \to 0$ is $n \to \infty$. If $\{u_n\}$ is bounded in $W(\mathbb{R}^N)$, it follows from the proceeding as in the proof of Lemma 6 in [11] that $\{u_n\}$ converges strongly to u in $W(\mathbb{R}^N)$. Hence, it suffices to verify that the sequence $\{u_n\}$ is bounded in $W(\mathbb{R}^N)$. However, we argue by contradiction and suppose that the conclusion is false, i.e., $\{u_n\}$ is a unbounded sequence in $W(\mathbb{R}^N)$. Therefore, we may assume that

$$\|u_n\|_{W(\mathbb{R}^N)} > 1 \quad \text{and} \quad \|u_n\|_{W(\mathbb{R}^N)} \to \infty, \quad \text{as} \quad n \to \infty. \tag{6}$$

Define a sequence $\{w_n\}$ by $w_n = u_n/\|u_n\|_{W(\mathbb{R}^N)}$. Then, it is clear that $\{w_n\} \subset W(\mathbb{R}^N)$ and $\|w_n\|_{W(\mathbb{R}^N)} = 1$. Hence, up to a subsequence (still denoted as the sequence $\{w_n\}$), we obtain $w_n \rightharpoonup w$ in $W(\mathbb{R}^N)$ as $n \to \infty$. Furthermore, by Lemma 2, we have

$$w_n(x) \to w(x) \text{ a.e. in } \mathbb{R}^N \text{ and } \quad w_n \to w \text{ in } L^r(\mathbb{R}^N) \quad \text{as} \quad n \to \infty \tag{7}$$

for $p \le r < p^*_s$. Owing to the condition (5), we have

$$c = \mathcal{E}_\lambda(u_n) + o(1) = \frac{1}{p}(\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}) - \lambda\int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\,dx + o(1).$$

Since $\|u_n\|_{W(\mathbb{R}^N)} \to \infty$ as $n \to \infty$, we assert that

$$\begin{aligned}\int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\,dx &= \frac{1}{\lambda p}(\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}) - \frac{c}{\lambda} + \frac{o(1)}{\lambda} \\ &\geq \frac{1}{\lambda p}(\frac{1}{\theta}\mathcal{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)})|u_n|^p_{W^{s,p}(\mathbb{R}^N)} + \|u_n\|^p_{p,V}) - \frac{c}{\lambda} + \frac{o(1)}{\lambda} \\ &\geq \frac{\min\{m_0/\theta, 1\}}{\lambda p}\|u_n\|^p_{W(\mathbb{R}^N)} - \frac{c}{\lambda} + \frac{o(1)}{\lambda} \to \infty \quad \text{as} \quad n \to \infty. \end{aligned} \tag{8}$$

The assumption ($\mathcal{F}3$) implies that there exists $t_0 > 1$ such that $\mathcal{F}(x,t) > |t|^{\theta p}$ for all $x \in \mathbb{R}^N$ and $|t| > t_0$. From the assumptions ($\mathcal{F}1$) and ($\mathcal{F}2$), there is a constant $\mathcal{C} > 0$ such that $|\mathcal{F}(x,t)| \leq \mathcal{C}$ for all $(x,t) \in \mathbb{R}^N \times [-t_0, t_0]$. Therefore, we can choose $\mathcal{C}_0 \in \mathbb{R}$ such that $\mathcal{F}(x,t) \geq \mathcal{C}_0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$; thus,

$$\frac{\mathcal{F}(x,u_n) - \mathcal{C}_0}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}} \geq 0, \tag{9}$$

for all $x \in \mathbb{R}^N$, and for all $n \in \mathbb{N}$. Set $\Omega = \{x \in \mathbb{R}^N : w(x) \neq 0\}$. By the convergence (7), we know that $|u_n(x)| = |w_n(x)|\,\|u_n\|_{W(\mathbb{R}^N)} \to \infty$ as $n \to \infty$ for all $x \in \Omega$. Therefore, it follows from the assumptions ($\mathcal{M}2$), ($\mathcal{F}3$), and the relation (6) that, for all $x \in \Omega$,

$$\begin{aligned}\lim_{n\to\infty} \frac{\mathcal{F}(x,u_n)}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}} &\geq \lim_{n\to\infty} \frac{\mathcal{F}(x,u_n)}{\mathfrak{M}(1)\,|u_n|^{\theta p}_{W^{s,p}(\mathbb{R}^N)} + \|u_n\|^p_{p,V}} \\ &\geq \lim_{n\to\infty} \frac{\mathcal{F}(x,u_n)}{\mathfrak{M}(1)\|u_n\|^{\theta p}_{W(\mathbb{R}^N)} + \|u_n\|^p_{W(\mathbb{R}^N)}} \\ &\geq \lim_{n\to\infty} \frac{\mathcal{F}(x,u_n)}{\mathfrak{M}(1)\|u_n\|^{\theta p}_{W(\mathbb{R}^N)} + \|u_n\|^{\theta p}_{W(\mathbb{R}^N)}} \\ &= \lim_{n\to\infty} \frac{\mathcal{F}(x,u_n)}{(\mathfrak{M}(1)+1)\|u_n\|^{\theta p}_{W(\mathbb{R}^N)}} \\ &= \lim_{n\to\infty} \frac{\mathcal{F}(x,u_n)}{(\mathfrak{M}(1)+1)\,|u_n(x)|^{\theta p}}\,|w_n(x)|^{\theta p} \\ &= \infty, \end{aligned} \tag{10}$$

where we use the inequality $\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) \leq \mathfrak{M}(1)\,|u_n|^{\theta p}_{W^{s,p}(\mathbb{R}^N)}$. Hence, we obtain $\text{meas}(\Omega) = 0$. If $\text{meas}(\Omega) \neq 0$, according to relations (8)–(10) and Fatou's lemma, we deduce that

$$\begin{aligned}\frac{1}{\lambda} &= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\,dx}{\lambda \int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\,dx + c - o(1)} \\ &= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{p\mathcal{F}(x,u_n)}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx \\ &\geq \liminf_{n\to\infty} \int_{\Omega} \frac{p\mathcal{F}(x,u_n)}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx - \limsup_{n\to\infty} \int_{\Omega} \frac{p\mathcal{C}_0}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx \\ &= \liminf_{n\to\infty} \int_{\Omega} \frac{p(\mathcal{F}(x,u_n) - \mathcal{C}_0)}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx \\ &\geq \int_{\Omega} \liminf_{n\to\infty} \frac{p(\mathcal{F}(x,u_n) - \mathcal{C}_0)}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx \end{aligned}$$

$$= \int_\Omega \liminf_{n\to\infty} \frac{p\mathcal{F}(x,u_n)}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx - \int_\Omega \limsup_{n\to\infty} \frac{pC_0}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx$$
$$= \infty, \tag{11}$$

which yields a contradiction. Thus, $w(x) = 0$ for almost all $x \in \mathbb{R}^N$.

Observe that, for a sufficiently large n,

$$\begin{aligned} c+1 &\geq \mathcal{E}_\lambda(u_n) - \frac{1}{\theta p}\langle \mathcal{E}_\lambda'(u_n), u_n\rangle \\ &= \frac{1}{p}(\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}) - \lambda\int_{\mathbb{R}^N}\mathcal{F}(x,u_n)\,dx \\ &- \frac{1}{\theta p}(\mathcal{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)})|u_n|^p_{W^{s,p}(\mathbb{R}^N)} + \|u_n\|^p_{p,V}) + \frac{\lambda}{\theta p}\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx \\ &\geq \lambda\int_{\mathbb{R}^N}\mathfrak{F}(x,u_n)\,dx, \end{aligned} \tag{12}$$

where $\mathfrak{F}$ is given in ($\mathcal{F}$4). Let us define $\Omega_n(a,b) := \{x \in \mathbb{R}^N : a \leq |u_n(x)| < b\}$ for $a \geq 0$. By the convergence (7),

$$w_n \to 0 \quad \text{in} \quad L^r(\mathbb{R}^N) \quad \text{and} \quad w_n(x) \to 0 \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^N \quad \text{as} \quad n \to \infty \tag{13}$$

for $p \leq r < p_s^*$. Hence, from the relation (8), we obtain

$$0 < \frac{1}{\lambda p} \leq \limsup_{n\to\infty}\int_{\mathbb{R}^N} \frac{|\mathcal{F}(x,u_n)|}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx. \tag{14}$$

Meanwhile, from the assumptions ($\mathcal{M}$2), ($\mathcal{F}$2), the relation (13), and Lemma 2, we obtain

$$\begin{aligned} &\int_{\Omega_n(0,r_0)} \frac{\mathcal{F}(x,u_n)}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx \\ &\leq \int_{\Omega_n(0,r_0)} \frac{\rho(x)\,|u_n(x)| + \frac{\sigma(x)}{q}\,|u_n(x)|^q}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx \\ &\leq \frac{\|\rho\|_{L^{p'}(\mathbb{R}^N)}\|u_n\|_{L^p(\mathbb{R}^N)}}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}} + \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{\min\{1,m_0/\theta\}q}\int_{\Omega_n(0,r_0)} |u_n(x)|^{q-p}\,|w_n(x)|^p\,dx \\ &\leq \frac{\|\rho\|_{L^{p'}(\mathbb{R}^N)}\|u_n\|_{L^p(\mathbb{R}^N)}}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}} + \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{\min\{1,m_0/\theta\}q}r_0^{q-p}\int_{\mathbb{R}^N}|w_n(x)|^p\,dx \\ &\leq \frac{K_p\|\rho\|_{L^{p'}(\mathbb{R}^N)}\|u_n\|_{W(\mathbb{R}^N)}}{\min\{1,m_0/\theta\}\|u_n\|^p_{W(\mathbb{R}^N)}} + \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{\min\{1,m_0/\theta\}q}r_0^{q-p}\int_{\mathbb{R}^N}|w_n(x)|^p\,dx \\ &\leq \frac{C_1}{\min\{1,m_0/\theta\}\|u_n\|^{p-1}_{W(\mathbb{R}^N)}} + \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{\min\{1,m_0/\theta\}q}r_0^{q-p}\int_{\mathbb{R}^N}|w_n(x)|^p\,dx \to 0, \quad \text{as } n\to\infty, \end{aligned} \tag{15}$$

where C_1 is a positive constant, r_0 is given in ($\mathcal{F}$4), and we use the following inequality:

$$\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V} \geq \min\{1,m_0/\theta\}\|u_n\|^p_{W(\mathbb{R}^N)}.$$

We set $\kappa' = \kappa/(\kappa-1)$. Since $\kappa > N/ps$, we have $p < \kappa' p < p_s^*$. Hence, it follows from $(\mathcal{F}4)$, estimates (12) and (13) that

$$\begin{aligned}\int_{\Omega_n(r_0,\infty)} \frac{|\mathcal{F}(x,u_n)|}{\mathfrak{M}([u_n]^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx &\le \int_{\Omega_n(r_0,\infty)} \frac{|\mathcal{F}(x,u_n)|}{\min\{1,m_0/\theta\}\,|u_n(x)|^p}\,|w_n(x)|^p\,dx \\ &\le \frac{1}{\min\{1,m_0/\theta\}}\left\{\int_{\Omega_n(r_0,\infty)} \left(\frac{|F(x,u_n)|}{|u_n(x)|^p}\right)^{\kappa} dx\right\}^{\frac{1}{\kappa}} \left\{\int_{\Omega_n(r_0,\infty)} |w_n(x)|^{\kappa' p}\right\}^{\frac{1}{\kappa'}} \\ &\le \frac{c_0^{\frac{1}{\kappa}}}{\min\{1,m_0/\theta\}}\left\{\int_{\Omega_n(r_0,\infty)} \mathfrak{F}(x,u_n)\,dx\right\}^{\frac{1}{\kappa}} \left\{\int_{\mathbb{R}^N} |w_n(x)|^{\kappa' p}\right\}^{\frac{1}{\kappa'}} \\ &\le \frac{c_0^{\frac{1}{\kappa}}}{\min\{1,m_0/\theta\}}\left(\frac{c+1}{\lambda}\right)^{\frac{1}{\kappa}} \left\{\int_{\mathbb{R}^N} |w_n(x)|^{\kappa' p}\right\}^{\frac{1}{\kappa'}} \to 0, \quad \text{as } n\to\infty. \end{aligned} \tag{16}$$

Combining the relation (15) with the convergence (16), we have

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|\mathcal{F}(x,u_n)|}{\mathfrak{M}([u_n]^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx &= \int_{\Omega_n(0,r_0)} \frac{|\mathcal{F}(x,u_n)|}{\mathfrak{M}([u_n]^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx \\ &\quad + \int_{\Omega_n(r_0,\infty)} \frac{|\mathcal{F}(x,u_n)|}{\mathfrak{M}([u_n]^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx \to 0\end{aligned}$$

as $n\to\infty$, which contradicts inequality the convergence (14). The proof is completed. □

Lemma 4. *Let $0 < s < 1 < p < +\infty$ and $ps < N$. Assume that $(\mathcal{V}1)$, $(\mathcal{V}2)$, $(\mathcal{M}1)$, $(\mathcal{M}2)$, $(\mathcal{F}1)$–$(\mathcal{F}3)$, and $(\mathcal{F}5)$ hold. Then, the functional $\mathcal{E}_\lambda$ satisfies the $(C)_c$-condition for any $\lambda > 0$.*

Proof. For $c \in \mathbb{R}$, let $\{u_n\}$ be a $(C)_c$-sequence in $W(\mathbb{R}^N)$ satisfying (4). Then, relation (5) holds. As in the proof of Lemma 3, we only prove that $\{u_n\}$ is bounded in $W(\mathbb{R}^N)$. However, arguing by contradiction, suppose that $\|u_n\|_{W(\mathbb{R}^N)} \to \infty$ as $n\to\infty$. Let $v_n = u_n/\|u_n\|_{W(\mathbb{R}^N)}$. Then, $\|v_n\|_{W(\mathbb{R}^N)} = 1$ and $\|v_n\|_{L^r(\mathbb{R}^N)} \le K_r\|v_n\|_{W(\mathbb{R}^N)} = K_r$ for $p \le r \le p_s^*$ by the continuous embedding in Lemma 2. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in $W(\mathbb{R}^N)$ as $n\to\infty$; then, by compact embedding, $v_n \to v$ in $L^r(\mathbb{R}^N)$ for $p \le r < p_s^*$, and $v_n(x) \to v(x)$ for almost all $x \in \mathbb{R}^N$ as $n\to\infty$. By the assumption $(\mathcal{F}5)$, one obtains

$$\begin{aligned} c+1 &\ge \mathcal{E}_\lambda(u_n) - \frac{1}{\mu}\langle \mathcal{E}'_\lambda(u_n), u_n\rangle \\ &= \frac{1}{p}(\mathfrak{M}([u_n]^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}) - \lambda\int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\,dx \\ &\quad - \frac{1}{\mu}(\mathcal{M}([u_n]^p_{W^{s,p}(\mathbb{R}^N)})[u_n]^p_{W^{s,p}(\mathbb{R}^N)} + \|u_n\|^p_{p,V}) + \frac{\lambda}{\mu}\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx \\ &\ge \left(\frac{1}{\theta p} - \frac{1}{\mu}\right)\mathcal{M}([u_n]^p_{W^{s,p}(\mathbb{R}^N)})[u_n]^p_{W^{s,p}(\mathbb{R}^N)} + \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n\|^p_{p,V} - \frac{\lambda\varrho}{\mu}\int_{\mathbb{R}^N} |u_n(x)|^p\,dx \\ &\ge \min\{1,m_0\}\left(\frac{1}{\theta p} - \frac{1}{\mu}\right)\|u_n\|^p_{W(\mathbb{R}^N)} - \frac{\lambda\varrho}{\mu}\int_{\mathbb{R}^N} |u_n(x)|^p\,dx, \end{aligned} \tag{17}$$

which implies

$$1 \le \frac{\lambda\varrho\theta p}{\min\{1,m_0\}(\mu-\theta p)} \limsup_{n\to\infty} \|v_n\|^p_{L^p(\mathbb{R}^N)} = \frac{\lambda\varrho\theta p}{\min\{1,m_0\}(\mu-\theta p)}\|v\|^p_{L^p(\mathbb{R}^N)}. \tag{18}$$

Hence, it follows from the inequality (18) that $v \ne 0$. From the same argument as in Lemma 3, we can verify the relations (8)–(10), and hence yield the relation (11). Therefore, we arrive at a contradiction. Thus, $\{u_n\}$ is bounded in $W(\mathbb{R}^N)$. □

Next, based on the fountain theorem in ([39], Theorem 3.6), we demonstrate the infinitely many weak solutions for problem (1). Hence, we let X be a separable and reflexive Banach space. It is well known that there exists $\{e_n\} \subseteq X$ and $\{f_n^*\} \subseteq X^*$ such that

$$X = \overline{\text{span}\{e_n : n = 1, 2, \cdots\}}, \quad X^* = \overline{\text{span}\{f_n^* : n = 1, 2, \cdots\}},$$

and

$$\langle f_i^*, e_j \rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Let us denote $X_n = \text{span}\{e_n\}$, $Y_k = \bigoplus_{n=1}^{k} X_n$, and $Z_k = \overline{\bigoplus_{n=k}^{\infty} X_n}$. Then, we recall the fountain lemma.

Lemma 5. *Let $\mathcal{X}$ be a real reflexive Banach space, $\mathcal{E} \in C^1(\mathcal{X}, \mathbb{R})$ satisfies the $(C)_c$-condition for any $c > 0$, and $\mathcal{E}$ is even. If for each sufficiently large $k \in \mathbb{N}$, there exist $\rho_k > \delta_k > 0$ such that the following conditions hold:*

(1) $b_k := \inf\{\mathcal{E}(u) : u \in Z_k, \|u\|_{\mathcal{X}} = \delta_k\} \to \infty \quad \text{as} \quad k \to \infty,$
(2) $a_k := \max\{\mathcal{E}(u) : u \in Y_k, \|u\|_{\mathcal{X}} = \rho_k\} \leq 0.$

Then, the functional $\mathcal{E}$ has an unbounded sequence of critical values, i.e., there exists a sequence $\{u_n\} \subset \mathcal{X}$ such that $\mathcal{E}'(u_n) = 0$ and $\mathcal{E}(u_n) \to \infty$ as $n \to \infty$.

Using Lemma 5, we demonstrate the existence of infinitely many nontrivial weak solutions for our problem.

Theorem 1. *Let $0 < s < 1 < p < +\infty$ and $ps < N$. Assume that $(\mathcal{V}1)$, $(\mathcal{V}2)$, $(\mathcal{M}1)$, $(\mathcal{M}2)$, and $(\mathcal{F}1)$–$(\mathcal{F}4)$ hold. If $f(x, -t) = -f(x, t)$ satisfies for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, then the functional $\mathcal{E}_\lambda$ has a sequence of nontrivial weak solutions $\{u_n\}$ in $W(\mathbb{R}^N)$ such that $\mathcal{E}_\lambda(u_n) \to \infty$ as $n \to \infty$ for any $\lambda > 0$.*

Proof. Clearly, $\mathcal{E}_\lambda$ is an even functional and satisfies the $(C)_c$-condition. Note that $W(\mathbb{R}^N)$ is a separable and reflexive Banach space. According to Lemma 5, it suffices to show that there exists $\rho_k > \delta_k > 0$ such that

(1) $b_k := \inf\{\mathcal{E}_\lambda(u) : u \in Z_k, \|u\|_{W(\mathbb{R}^N)} = \delta_k\} \to \infty \quad \text{as} \quad k \to \infty;$
(2) $a_k := \max\{\mathcal{E}_\lambda(u) : u \in Y_k, \|u\|_{W(\mathbb{R}^N)} = \rho_k\} \leq 0,$

for a sufficiently large k. We denote

$$\alpha_k := \sup_{u \in Z_k, \|u\|_{W(\mathbb{R}^N)} = 1} \left(\int_{\mathbb{R}^N} \frac{1}{q} |u(x)|^q \, dx \right), \quad \theta p < q < p_s^*.$$

Then, we know $\alpha_k \to 0$ as $k \to \infty$. Indeed, suppose to the contrary that there exist $\varepsilon_0 > 0$ and a sequence $\{u_k\}$ in Z_k such that

$$\|u_k\|_{W(\mathbb{R}^N)} = 1, \quad \int_{\mathbb{R}^N} \frac{1}{q} |u_k(x)|^q \, dx \geq \varepsilon_0$$

for all $k \geq k_0$. Since the sequence $\{u_k\}$ is bounded in $W(\mathbb{R}^N)$, there exists an element u in $W(\mathbb{R}^N)$ such that $u_k \rightharpoonup u$ in $W(\mathbb{R}^N)$ as $k \to \infty$, and

$$\langle f_j^*, u \rangle = \lim_{k \to \infty} \langle f_j^*, u_k \rangle = 0$$

for $j = 1, 2, \cdots$. Hence, $u = 0$. However, we obtain

$$\varepsilon_0 \leq \lim_{k\to\infty} \int_{\mathbb{R}^N} \frac{1}{q}|u_k(x)|^q\, dx = \int_{\mathbb{R}^N} \frac{1}{q}|u(x)|^q\, dx = 0,$$

which yields a contradiction.

For any $u \in Z_k$, it follows from assumptions ($\mathcal{M}2$), ($\mathcal{F}2$), and the Hölder inequality that

$$\begin{aligned}
\mathcal{E}_\lambda(u) &= \frac{1}{p}(\mathfrak{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u\|^p_{p,V}) - \lambda \int_{\mathbb{R}^N} \mathcal{F}(x,u)\, dx \\
&\geq \frac{1}{p}(\mathfrak{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u\|^p_{p,V}) - \lambda \int_{\mathbb{R}^N} |\rho(x)|\,|u(x)|\; dx - \lambda \int_{\mathbb{R}^N} \frac{|\sigma(x)|}{q}\,|u(x)|^q\; dx \\
&\geq \frac{1}{p}(\mathfrak{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u\|^p_{p,V}) - \lambda\|\rho\|_{L^{p'}(\mathbb{R}^N)}\|u\|_{L^p(\mathbb{R}^N)} - \frac{\lambda}{q}\|\sigma\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u(x)|^q\; dx \\
&\geq \frac{1}{p}(\mathfrak{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u\|^p_{p,V}) - \lambda C_2\|u\|_{W(\mathbb{R}^N)} - \frac{\lambda}{q}C_3\|u\|^q_{L^q(\mathbb{R}^N)} \\
&\geq \frac{\min\{1, m_0/\theta\}}{p}\|u\|^p_{W(\mathbb{R}^N)} - \lambda C_2\|u\|_{W(\mathbb{R}^N)} - \frac{\lambda}{q}\alpha_k^q C_4\|u\|^q_{W(\mathbb{R}^N)},
\end{aligned} \tag{19}$$

where C_2, C_3 and C_4 are positive constants. We choose $\delta_k = (2\lambda C_4\alpha_k^q / \min\{1, m_0/\theta\})^{1/(p-q)}$. Since $p < q$ and $\alpha_k \to 0$ as $k \to \infty$, we assert $\delta_k \to \infty$ as $k \to \infty$. Hence, if $u \in Z_k$ and $\|u\|_{W(\mathbb{R}^N)} = \delta_k$, then we deduce that

$$\mathcal{E}_\lambda(u) \geq \left(\frac{1}{p} - \frac{1}{q}\right)\delta_k^p - 2\lambda C_2\delta_k \to \infty \quad \text{as} \quad k \to \infty,$$

which implies the condition (1).

Next, suppose that condition (2) is not satisfied for some k. Then, there exists a sequence $\{u_n\}$ in Y_k such that

$$\|u_n\|_{W(\mathbb{R}^N)} > 1 \quad \text{and} \quad \|u_n\|_{W(\mathbb{R}^N)} \to \infty \text{ as } n \to \infty \quad \text{and} \quad \mathcal{E}_\lambda(u_n) \geq 0. \tag{20}$$

Let $w_n = u_n/\|u_n\|_{W(\mathbb{R}^N)}$. Then, it is obvious that $\|w_n\|_{W(\mathbb{R}^N)} = 1$. Since $\dim Y_k < \infty$, there exists $w \in Y_k \setminus \{0\}$ such that, up to a subsequence,

$$\|w_n - w\|_{W(\mathbb{R}^N)} \to 0 \quad \text{and} \quad w_n(x) \to w(x)$$

for almost all $x \in \mathbb{R}^N$ as $n \to \infty$. For $x \in \Omega := \left\{x \in \mathbb{R}^N : w(x) \neq 0\right\}$, we obtain $|u_n(x)| \to \infty$ as $n \to \infty$. Hence, it follows from the assumption ($\mathcal{F}3$) that

$$\lim_{n\to\infty} \frac{\mathcal{F}(x,u_n)}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}} \geq \lim_{n\to\infty} \frac{\mathcal{F}(x,u_n)}{(\mathfrak{M}(1)+1)\,|u_n(x)|^{\theta p}}\,|w_n(x)|^{\theta p} = \infty. \tag{21}$$

As shown in the proof of Lemma 3, we can choose $\mathcal{C}_1 \in \mathbb{R}$ such that

$$\frac{\mathcal{F}(x,u_n) - \mathcal{C}_1}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}} \geq 0 \tag{22}$$

for $x \in \Omega$. Considering the inequalities (21), (22) and Fatou's lemma, we assert by a similar argument to the inequality (10) that

$$\lim_{n\to\infty} \int_\Omega \frac{\mathcal{F}(x,u_n)}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\, dx \geq \liminf_{n\to\infty} \int_\Omega \frac{\mathcal{F}(x,u_n) - \mathcal{C}_1}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\, dx$$

$$\geq \int_{\Omega} \liminf_{n\to\infty} \frac{\mathcal{F}(x,u_n)}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx = \infty. \tag{23}$$

Therefore, using the relation (23), we have

$$\begin{aligned}\mathcal{E}_\lambda(u_n) &\leq \frac{1}{p}(\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}) - \lambda \int_{\Omega} \mathcal{F}(x,u_n)\,dx \\ &= \frac{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}{p}\left(1 - \lambda p \int_{\Omega} \frac{\mathcal{F}(x,u_n)}{\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}}\,dx\right) \to -\infty\end{aligned}$$

as $n \to \infty$, which yields a contradiction to the relation (20). The proof is complete. □

Remark 1. *Although we replaced* $(\mathcal{F}4)$ *with* $(\mathcal{F}5)$ *in the assumption of Theorem 1, we assert that the problem* (1) *admits a sequence of nontrivial weak solutions* $\{u_n\}$ *in* $W(\mathbb{R}^N)$ *such that* $\mathcal{E}_\lambda(u_n) \to \infty$ *as* $n \to \infty$*.*

Lastly, we investigate the existence of nontrivial weak solutions for our problem by replacing the assumptions $(\mathcal{F}4)$ and $(\mathcal{F}5)$ with the following condition, which is from the work of L. Jeanjean [40]:

$(\mathcal{F}6)$ There exists a constant $\nu \geq 1$ such that

$$\nu\hat{\mathfrak{F}}(x,t) \geq \hat{\mathfrak{F}}(x,st)$$

for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0,1]$, where $\hat{\mathfrak{F}}(x,t) = f(x,t)t - \theta p\mathcal{F}(x,t)$.

When the Kirchhoff function $\mathcal{M}$ is constant, and the condition $(\mathcal{F}6)$ with $\theta = 1$ holds, the author in [24] obtained the existence of at least one nontrivial weak solution for the superlinear problems of the fractional p-Laplacian, which is motivated by the works of [18,26].

To the best of our belief, such existence and multiplicity results are not available for the elliptic equation of the Kirchhoff type under the assumption $(\mathcal{F}6)$. Hence, we obtain the following lemma with the sufficient conditions for the modified Kirchhoff function $\mathcal{M}$ and the assumption $(\mathcal{F}6)$.

Lemma 6. *Let* $0 < s < 1 < p < +\infty$ *and* $ps < N$*. Assume that* $(\mathcal{V}1)$, $(\mathcal{V}2)$, $(\mathcal{M}1)$, $(\mathcal{M}2)$, $(\mathcal{F}1)$–$(\mathcal{F}3)$, *and* $(\mathcal{F}6)$ *hold. Furthermore, we assume that*

$(\mathcal{M}3)$ $\mathcal{H}(st) \leq \mathcal{H}(t)$ *for* $s \in [0,1]$*, where* $\mathcal{H}(t) = \theta\mathfrak{M}(t) - \mathcal{M}(t)t$ *for any* $t \geq 0$ *and* θ *is given in* $(\mathcal{M}2)$*.*

Then, the functional $\mathcal{E}_\lambda$ *satisfies the* $(C)_c$*-condition for any* $\lambda > 0$*.*

Proof. For $c \in \mathbb{R}$, let $\{u_n\}$ be a $(C)_c$-sequence in $W(\mathbb{R}^N)$ satisfying the convergence (4). Then, the relation (5) holds. By Lemma 3, we only prove that $\{u_n\}$ is bounded in $W(\mathbb{R}^N)$. Therefore, we argue by contradiction and suppose that the conclusion is false, i.e., $\|u_n\|_{W(\mathbb{R}^N)} > 1$ and $\|u_n\|_{W(\mathbb{R}^N)} \to \infty$ as $n \to \infty$. In addition, we define a sequence $\{\omega_n\}$ by $\omega_n = u_n/\|u_n\|_{W(\mathbb{R}^N)}$. Then, up to a subsequence (still denoted as the sequence $\{\omega_n\}$), we obtain $\omega_n \rightharpoonup \omega$ in $W(\mathbb{R}^N)$ as $n \to \infty$,

$$\omega_n(x) \to \omega(x) \text{ a.e. in } \mathbb{R}^N, \quad \omega_n \to \omega \text{ in } L^q(\mathbb{R}^N), \quad \text{and} \quad \omega_n \to \omega \text{ in } L^p(\mathbb{R}^N) \quad \text{as} \quad n \to \infty,$$

where $\theta p < q < p_s^*$.

We set $\Omega = \{x \in \mathbb{R}^N : \omega(x) \neq 0\}$. From the similar manner as in Lemma 3, we obtain $\text{meas}(\Omega) = 0$. Therefore, $\omega(x) = 0$ for almost all $x \in \mathbb{R}^N$. Since $\mathcal{E}_\lambda(tu_n)$ is continuous at $t \in [0,1]$, for each $n \in \mathbb{N}$, there exists $t_n \in [0,1]$ such that

$$\mathcal{E}_\lambda(t_n u_n) := \max_{t\in[0,1]} \mathcal{E}_\lambda(tu_n).$$

Let $\{\ell_k\}$ be a positive sequence of real numbers such that $\lim_{k\to\infty} \ell_k = \infty$ and $\ell_k > 1$ for any k. Then, it is clear that $\|\ell_k\omega_n\|_{W(\mathbb{R}^N)} = \ell_k > 1$ for any k and n. Fix k, since $\omega_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ as

$n \to \infty$, the continuity of the Nemytskii operator implies $\mathcal{F}(x, \ell_k \omega_n) \to 0$ in $L^1(\mathbb{R}^N)$ as $n \to \infty$. Hence, we assert

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \mathcal{F}(x, \ell_k \omega_n)\, dx = 0. \tag{24}$$

Since $\|u_n\|_{W(\mathbb{R}^N)} \to \infty$ as $n \to \infty$, we obtain $\|u_n\|_{W(\mathbb{R}^N)} > \ell_k$ for a sufficiently large n. Thus, we know by $(\mathcal{M}2)$ and the convergence (24) that

$$\begin{aligned}
\mathcal{E}_\lambda(t_n u_n) &\geq \mathcal{E}_\lambda\left(\frac{\ell_k}{\|u_n\|_{W(\mathbb{R}^N)}} u_n\right) = \mathcal{E}_\lambda(\ell_k \omega_n) \\
&= \frac{1}{p}(\mathfrak{M}(|\ell_k\omega_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|\ell_k\omega_n\|^p_{p,V}) - \lambda \int_{\mathbb{R}^N} \mathcal{F}(x, \ell_k\omega_n)\, dx \\
&\geq \frac{1}{p\theta}\mathcal{M}(|\ell_k\omega_n|^p_{W^{s,p}(\mathbb{R}^N)})|\ell_k\omega_n|^p_{W^{s,p}(\mathbb{R}^N)} + \frac{1}{p}\|\ell_k\omega_n\|^p_{p,V} - \lambda \int_{\mathbb{R}^N} \mathcal{F}(x, \ell_k\omega_n)\, dx \\
&\geq \frac{\min\{1, m_0\}}{p\theta}\|\ell_k\omega_n\|^p_{W(\mathbb{R}^N)} - \lambda \int_{\mathbb{R}^N} \mathcal{F}(x, \ell_k\omega_n)\, dx \\
&\geq \frac{\min\{1, m_0\}}{p\theta}\ell_k^p
\end{aligned}$$

for a large enough n. Then, letting $n, k \to \infty$, we get

$$\lim_{n\to\infty} \mathcal{E}_\lambda(t_n u_n) = \infty. \tag{25}$$

Since $\mathcal{E}_\lambda(0) = 0$ and $\mathcal{E}_\lambda(u_n) \to c$ as $n \to \infty$, it is obvious that $t_n \in (0,1)$, and $\langle \mathcal{E}'_\lambda(t_n u_n), t_n u_n\rangle = 0$. Therefore, owing to the assumptions $(\mathcal{M}3)$ and $(\mathcal{F}6)$, for all sufficiently large n, we deduce that

$$\begin{aligned}
\frac{1}{\nu}\mathcal{E}_\lambda(t_n u_n) &= \frac{1}{\nu}\mathcal{E}_\lambda(t_n u_n) - \frac{1}{p\theta\nu}\langle \mathcal{E}'_\lambda(t_n u_n), t_n u_n\rangle + o(1) \\
&= \frac{1}{p\nu}(\mathfrak{M}(|t_n u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|t_n u_n\|^p_{p,V}) - \frac{\lambda}{\nu}\int_{\mathbb{R}^N} \mathcal{F}(x, t_n u_n)\, dx \\
&\quad - \frac{1}{p\theta\nu}(\mathcal{M}(|t_n u_n|^p_{W^{s,p}(\mathbb{R}^N)})|t_n u_n|^p_{W^{s,p}(\mathbb{R}^N)} + \|t_n u_n\|^p_{p,V}) + \frac{\lambda}{p\theta\nu}\int_{\mathbb{R}^N} f(x, t_n u_n) t_n u_n\, dx + o(1) \\
&= \frac{1}{p\theta\nu}\mathcal{H}(t_n u_n) + \frac{1}{p\nu}\|t_n u_n\|^p_{p,V} - \frac{1}{p\theta\nu}\|t_n u_n\|^p_{p,V} + \frac{\lambda}{p\theta\nu}\int_{\mathbb{R}^N} \hat{\mathfrak{F}}(x, t_n u_n)\, dx + o(1) \\
&\leq \frac{1}{p\theta}\mathcal{H}(u_n) + \frac{1}{p}\|t_n u_n\|^p_{p,V} - \frac{1}{p\theta}\|t_n u_n\|^p_{p,V} + \frac{\lambda}{p\theta}\int_{\mathbb{R}^N} \hat{\mathfrak{F}}(x, u_n)\, dx + o(1) \\
&= \frac{1}{p}(\mathfrak{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u_n\|^p_{p,V}) - \lambda\int_{\mathbb{R}^N} \mathcal{F}(x, u_n)\, dx \\
&\quad - \frac{1}{p\theta}(\mathcal{M}(|u_n|^p_{W^{s,p}(\mathbb{R}^N)})|u_n|^p_{W^{s,p}(\mathbb{R}^N)} + \|u_n\|^p_{p,V}) + \frac{\lambda}{p\theta}\int_{\mathbb{R}^N} f(x, u_n) u_n\, dx + o(1) \\
&= \mathcal{E}_\lambda(u_n) - \frac{1}{p\theta}\langle \mathcal{E}'_\lambda(u_n), u_n\rangle + o(1) \to c \quad \text{as} \quad n \to \infty,
\end{aligned}$$

which contradicts the convergence (25). This completes the proof. □

We give an example regarding a function $\mathcal{M}$ with the assumptions $(\mathcal{M}1)$–$(\mathcal{M}3)$.

Example 1. *Let us see*

$$\mathcal{M}(t) = 1 + \frac{1}{e+t}, \quad t \geq 0.$$

Then, it is easily checked that this function $\mathcal{M}$ complies with the assumptions $(\mathcal{M}1)$–$(\mathcal{M}3)$.

Theorem 2. *Let $0 < s < 1 < p < +\infty$ and $ps < N$. Assume that $(\mathcal{V}1)$, $(\mathcal{V}2)$, $(\mathcal{M}1)$–$(\mathcal{M}3)$, $(\mathcal{F}1)$–$(\mathcal{F}3)$, and $(\mathcal{F}6)$ hold. If $f(x,-t) = -f(x,t)$ holds for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, then, for any $\lambda > 0$, the functional $\mathcal{E}_\lambda$ has a sequence of nontrivial weak solutions $\{u_n\}$ in $W(\mathbb{R}^N)$ such that $\mathcal{E}_\lambda(u_n) \to \infty$ as $n \to \infty$.*

Proof. The proof is essentially the same as that of Theorem 1. □

3. Existence of Infinitely Many Small Energy Solutions

In this section, we prove the existence of a sequence of small energy solutions for the problem (1) converging to zero in L^∞-norm based on the Moser bootstrap iteration technique in ([35], Theorem 4.1) (see also [34]). First, we state the following additional assumptions:

$(\mathcal{F}7)$ There exists a constant $s_0 > 0$ such that $p\mathcal{F}(x,t) - f(x,t)t > 0$ for all $x \in \mathbb{R}^N$ and for $0 < |t| < s_0$.
$(\mathcal{F}8)$ $\lim_{|t|\to 0} \frac{f(x,t)}{|t|^{p-2}t} = +\infty$ uniformly for all $x \in \mathbb{R}^N$.

Because problem (1) includes the potential term and the nonlinear term f is slightly different from that of [35], a more complicated analysis has to be carefully performed when we apply the bootstrap iteration argument.

Proposition 1. *Assume that $(\mathcal{V}1)$, $(\mathcal{M}1)$, and $(\mathcal{F}1)$–$(\mathcal{F}2)$ hold. If u is a weak solution of the problem (1), then $u \in L^r(\mathbb{R}^N)$ for all $r \in [p_s^*, \infty]$.*

Proof. Suppose that u is non-negative. For $K > 0$, we define

$$v_K(x) = \min\{u(x), K\}$$

and choose $v = v_K^{mp+1}$ $(m \geq 0)$ as a test function in the equality (2). Then, $v \in W(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and it follows from the equality (2) that

$$\begin{aligned} &\mathcal{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)}) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v_K^{mp+1}(x)-v_K^{mp+1}(y))}{|x-y|^{N+ps}}\,dxdy \\ &\quad + \int_{\mathbb{R}^N} V(x)\,|u|^{p-2}\,uv_K^{mp+1}\,dx = \lambda \int_{\mathbb{R}^N} f(x,u)v_K^{mp+1}\,dx. \end{aligned} \tag{26}$$

The left-hand side of the relation (26) can be estimated as follows:

$$\begin{aligned} &\mathcal{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)}) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v_K^{mp+1}(x)-v_K^{mp+1}(y))}{|x-y|^{N+ps}}\,dxdy \\ &\quad + \int_{\mathbb{R}^N} V(x)\,|u|^{p-2}\,uv_K^{mp+1}\,dx \\ &\geq m_0 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-1}\left|v_K^{mp+1}(x)-v_K^{mp+1}(y)\right|}{|x-y|^{N+ps}}\,dxdy + \int_{\mathbb{R}^N} V(x)v_K^{(m+1)p}\,dx \\ &\geq m_0C_5 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_K^{m+1}(x)-v_K^{m+1}(y)|^p}{|x-y|^{N+ps}}\,dxdy + \int_{\mathbb{R}^N} V(x)v_K^{(m+1)p}\,dx \\ &\geq \min\{m_0C_5, 1\}\|v_K^{m+1}\|^p_{W(\mathbb{R}^N)} \\ &\geq \min\{m_0C_5, 1\}C_6 \left(\int_{\mathbb{R}^N} |v_K|^{(m+1)p_s^*}\,dx\right)^{\frac{p}{p_s^*}} \end{aligned} \tag{27}$$

for some positive constants C_5 and C_6. Using the assumption $(\mathcal{F}2)$, the Hölder inequality and the relation (27), the right-hand side of the relation (26) can be estimated:

$$\lambda \int_{\mathbb{R}^N} f(x,u)v_K^{mp+1}\,dx \leq \lambda \int_{\mathbb{R}^N} |f(x,u)||u|^{mp+1}\,dx$$

$$
\begin{aligned}
&\leq \lambda \int_{\mathbb{R}^N} \rho(x)|u|^{mp+1} + \sigma(x)|u|^{mp+q}\,dx \\
&\leq \lambda \int_{\mathbb{R}^N} \rho(x)(|u|^{mp+p} + |u|^{m+1})\,dx \\
&\quad + \lambda \left(\int_{\mathbb{R}^N} \sigma^{\gamma_1}(x)\,dx\right)^{\frac{1}{\gamma_1}} \left(\int_{\mathbb{R}^N} |u|^{(m+1)p\gamma_1'}|u|^{(q-p)\gamma_1'}\,dx\right)^{\frac{1}{\gamma_1'}} \\
&\leq \lambda\|\rho\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u|^{(m+1)p}\,dx + \lambda\|\rho\|_{L^{p'}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} |u|^{(m+1)p}\,dx\right)^{\frac{1}{p}} \\
&\quad + \lambda \left(\int_{\mathbb{R}^N} \sigma^{\gamma_1}(x)\,dx\right)^{\frac{1}{\gamma_1}} \left(\int_{\mathbb{R}^N} |u|^{(m+1)\beta}\,dx\right)^{\frac{p}{\beta}} \left(\int_{\mathbb{R}^N} |u|^{(q-p)\gamma_1'\frac{\beta}{\beta-p\gamma_1'}}\,dx\right)^{\frac{\beta-p\gamma_1'}{\beta\gamma_1'}},
\end{aligned} \tag{28}
$$

where $\gamma_1 = \frac{p_s^*}{p_s^*-q}$, and $\beta = \frac{pp_s^*\gamma_1'}{p_s^*-(q-p)\gamma_1'}$. Obviously $\beta \leq p_s^*$, $1 < \frac{\beta}{p\gamma_1'}$, and $\frac{(q-p)\gamma_1'\beta}{\beta-p\gamma_1'} = p_s^*$, and hence the estimate (28) yields

$$
\begin{aligned}
\lambda \int_{\mathbb{R}^N} f(x,u)v_K^{mp+1}\,dx &\leq \lambda\|\rho\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u|^{(m+1)p}\,dx + \lambda\|\rho\|_{L^{p'}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} |u|^{(m+1)p}\,dx\right)^{\frac{1}{p}} \\
&\quad + \lambda \left(\int_{\mathbb{R}^N} \sigma^{\gamma_1}(x)\,dx\right)^{\frac{1}{\gamma_1}} \left(\int_{\mathbb{R}^N} |u|^{p_s^*}\,dx\right)^{\frac{\beta-p\gamma_1'}{\beta\gamma_1'}} \left(\int_{\mathbb{R}^N} |u|^{(m+1)\beta}\,dx\right)^{\frac{p}{\beta}}.
\end{aligned} \tag{29}
$$

It follows from relations (26), (27), (29), and the Sobolev inequality that there exists positive constants C_7, C_8 and C_9 (independent of K and $m > 0$) such that

$$
\left(\int_{\mathbb{R}^N} |v_K|^{(m+1)p_s^*}\,dx\right)^{\frac{p}{p_s^*}} \leq C_7 \int_{\mathbb{R}^N} |u|^{(m+1)p}\,dx + C_8 \left(\int_{\mathbb{R}^N} |u|^{(m+1)p}\,dx\right)^{\frac{1}{p}} + C_9 \left(\int_{\mathbb{R}^N} |u|^{(m+1)\beta}\,dx\right)^{\frac{p}{\beta}},
$$

which implies

$$
\|v_K\|^{(m+1)p}_{L^{(m+1)p_s^*}(\mathbb{R}^N)} \leq C_7\|u\|^{(m+1)p}_{L^{(m+1)p}(\mathbb{R}^N)} + C_8\|u\|^{m+1}_{L^{(m+1)p}(\mathbb{R}^N)} + C_9\|u\|^{(m+1)p}_{L^{(m+1)\beta}(\mathbb{R}^N)}. \tag{30}
$$

To apply the argument that is critical in L^∞-estimates, we first assume that $\|u\|_{L^{(m+1)p}(\mathbb{R}^N)} \geq 1$. From the estimate (30), we have

$$
\begin{aligned}
\|v_K\|^{(m+1)p}_{L^{(m+1)p_s^*}(\mathbb{R}^N)} &\leq C_7\|u\|^{(m+1)p}_{L^{(m+1)p}(\mathbb{R}^N)} + C_8\|u\|^{m+1}_{L^{(m+1)p}(\mathbb{R}^N)} + C_9\|u\|^{(m+1)p}_{L^{(m+1)\beta}(\mathbb{R}^N)} \\
&\leq (C_7 + C_8)\|u\|^{(m+1)p}_{L^{(m+1)p}(\mathbb{R}^N)} + C_9\|u\|^{(m+1)p}_{L^{(m+1)\beta}(\mathbb{R}^N)},
\end{aligned} \tag{31}
$$

which implies

$$
\|v_K\|_{L^{(m+1)p_s^*}(\mathbb{R}^N)} \leq C_{10}^{\frac{1}{(m+1)p}}\|u\|_{L^{(m+1)t}(\mathbb{R}^N)} \tag{32}
$$

for some positive constant C_{10} and for any positive constant K, where t is either p or β. The expression in the estimate (32) is a starting point for a bootstrap technique. Since $u \in W(\mathbb{R}^N)$, hence $u \in L^{p_s^*}(\mathbb{R}^N)$ and we can choose $m := m_1$ in the estimate (32) such that $(m_1+1)t = p_s^*$, *i.e.*, $m_1 = \frac{p_s^*}{t} - 1$. Then, we have

$$
\|v_K\|_{L^{(m_1+1)p_s^*}(\mathbb{R}^N)} \leq C_{10}^{\frac{1}{(m_1+1)p}}\|u\|_{L^{(m_1+1)t}(\mathbb{R}^N)} \tag{33}
$$

for any positive constant K. Owing to $u(x) = \lim_{K\to\infty} v_K(x)$ for almost every $x \in \mathbb{R}^N$, Fatou's lemma and the estimate (33) imply

$$
\|u\|_{L^{(m_1+1)p_s^*}(\mathbb{R}^N)} \leq C_{10}^{\frac{1}{(m_1+1)p}}\|u\|_{L^{(m_1+1)t}(\mathbb{R}^N)}. \tag{34}
$$

Thus, we can choose $m = m_2$ in the estimate (32) such that $(m_2+1)t = (m_1+1)p_s^* = \frac{(p_s^*)^2}{t}$. By repeating the similar manner, we obtain

$$\|u\|_{L^{(m_2+1)p_s^*}(\mathbb{R}^N)} \le C_{10}^{\frac{1}{(m_2+1)p}} \|u\|_{L^{(m_2+1)t}(\mathbb{R}^N)}.$$

By the mathematical induction, we have

$$\|u\|_{L^{(m_n+1)p_s^*}(\mathbb{R}^N)} \le C_{10}^{\frac{1}{(m_n+1)p}} \|u\|_{L^{(m_n+1)t}(\mathbb{R}^N)} \tag{35}$$

for any $n \in \mathbb{N}$, where $m_n + 1 = \left(\frac{p_s^*}{t}\right)^n$. It follows from relations (34) and (35) that

$$\|u\|_{L^{(m_n+1)p_s^*}(\mathbb{R}^N)} \le C_{10}^{\frac{1}{p}\sum_{j=1}^{n}\frac{1}{m_j+1}} \|u\|_{L^{p_s^*}(\mathbb{R}^N)}. \tag{36}$$

However, $\sum_{j=1}^{n}\frac{1}{m_j+1} = \sum_{j=1}^{n}\left(\frac{t}{p_s^*}\right)^j$ and $\frac{t}{p_s^*} < 1$. Hence, it follows from the estimate (36) that there exists a constant $C_{11} > 0$ such that

$$\|u\|_{L^{r_n}(\mathbb{R}^N)} \le C_{11}\|u\|_{L^{p_s^*}(\mathbb{R}^N)} \tag{37}$$

for $r_n = (m_n+1)p_s^* \to \infty$ when $n \to \infty$. An indirect argument concludes that

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C_{11}\|u\|_{L^{p_s^*}(\mathbb{R}^N)} \le C_{12}$$

for some constant $C_{12} > 0$. Meanwhile, we assume that $\|u\|_{L^{(m+1)p}(\mathbb{R}^N)} < 1$. From the relation (30), we have

$$\|v_K\|_{L^{(m+1)p_s^*}(\mathbb{R}^N)}^{(m+1)p} \le C_7 + C_8 + C_9\|u\|_{L^{(m+1)\beta}(\mathbb{R}^N)}^{(m+1)p} \le C_{13}\|u\|_{L^{(m+1)\beta}(\mathbb{R}^N)}^{(m+1)p},$$

which implies

$$\|v_K\|_{L^{(m+1)p_s^*}(\mathbb{R}^N)} \le C_{13}^{\frac{1}{(m+1)p}} \|u\|_{L^{(m+1)\beta}(\mathbb{R}^N)}$$

for some positive constant C_{13}. Repeating the iterations as in the arguments above, we derive $\|u\|_{L^\infty(\mathbb{R}^N)} \le C_{14}$ for some positive constant C_{14}.

If u changes sign, we set positive and negative parts as $u^+(x) = \max\{u(x),0\}$ and $u^-(x) = \min\{u(x),0\}$. Then, it is obvious that $u^+ \in W(\mathbb{R}^N)$ and $u^- \in W(\mathbb{R}^N)$. For each $K > 0$, we define $v_K(x) = \min\{u^+(x), K\}$. Taking again $v = v_K^{mp+1}$ as a test function in $W(\mathbb{R}^N)$, we obtain

$$\begin{aligned}
&\mathcal{M}(|u|_{W^{s,p}(\mathbb{R}^N)}^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v_K^{mp+1}(x)-v_K^{mp+1}(y))}{|x-y|^{N+ps}}\,dxdy \\
&\quad + \int_{\mathbb{R}^N} V(x)\,|u|^{p-2}\,uv_K^{mp+1}\,dx = \lambda\int_{\mathbb{R}^N} f(x,u)v_K^{mp+1}\,dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathcal{M}(|u^+|_{W^{s,p}(\mathbb{R}^N)}^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u^+(x)-u^+(y)|^{p-2}(u^+(x)-u^+(y))(v_K^{mp+1}(x)-v_K^{mp+1}(y))}{|x-y|^{N+ps}}\,dxdy \\
&\quad + \int_{\mathbb{R}^N} V(x)\,|u^+|^{p-2}\,u^+v_K^{mp+1}\,dx = \lambda\int_{\mathbb{R}^N} f(x,u^+)v_K^{mp+1}\,dx.
\end{aligned}$$

Proceeding with the similar way as above, we obtain $u^+ \in L^\infty(\mathbb{R}^N)$. Similarly, we obtain $u^- \in L^\infty(\mathbb{R}^N)$. Therefore, $u = u^+ + u^-$ is in $L^\infty(\mathbb{R}^N)$. The proof is complete. □

The following result can be found in [41].

Lemma 7. *Let $\mathcal{E} \in C^1(\mathcal{X}, \mathbb{R})$ where $\mathcal{X}$ is a Banach space. We assume that $\mathcal{E}$ satisfies the (PS)-condition, is even and bounded from below, and $\mathcal{E}(0) = 0$. If, for any $n \in \mathbb{N}$, there exist an n-dimensional subspace $\mathcal{X}_n$ and $\rho_n > 0$ such that*

$$\sup_{\mathcal{X}_n \cap S_{\rho n}} \mathcal{E} < 0,$$

where $S_\rho := \{u \in \mathcal{X} : \|u\|_{\mathcal{X}} = \rho\}$, then $\mathcal{E}$ possesses a sequence of critical values $c_n < 0$ satisfying $c_n \to 0$ as $n \to \infty$.

Based on the work of [27,29], we provide the following two lemmas.

Lemma 8. *Assume that $(\mathcal{V}1)$, $(\mathcal{M}1)$ and $(\mathcal{F}1)$–$(\mathcal{F}2)$ hold. Furthermore, we assume that $\mathfrak{M}(t) \le \mathcal{M}(t)t$ for any $t \ge 0$, where $\mathfrak{M}$ is given in $(\mathcal{M}2)$. Furthermore, if*

$$p\mathcal{F}(x,t) - f(x,t)t > 0 \tag{38}$$

for all $x \in \mathbb{R}^N$ and for $t \neq 0$. Then,

$$\mathcal{E}_\lambda(u) = 0 = \langle \mathcal{E}_\lambda'(u), u \rangle \quad \textit{if and only if} \quad u = 0.$$

Proof. Let $\mathcal{E}_\lambda(u) = \langle \mathcal{E}_\lambda'(u), u \rangle = 0$. Then,

$$\begin{aligned} 0 &= -\,p\mathcal{E}_\lambda(u) \\ &= -\,\mathfrak{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)}) - \int_{\mathbb{R}^N} V(x)|u|^p\,dx + \lambda p \int_{\mathbb{R}^N} \mathcal{F}(x,u)\,dx, \end{aligned} \tag{39}$$

and

$$\langle \mathcal{E}_\lambda'(u), u \rangle = \mathcal{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)})|u|^p_{W^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} V(x)\,|u|^p\,dx - \lambda \int_{\mathbb{R}^N} f(x,u)u\,dx = 0. \tag{40}$$

It follows from the relations (39) and (40) that

$$\int_{\mathbb{R}^N} \{p\mathcal{F}(x,u) - f(x,u)u\}\,dx \le 0.$$

Consequently, the assumption (38) implies $u = 0$. □

Lemma 9. *Assume that $(\mathcal{F}1)$–$(\mathcal{F}2)$ and $(\mathcal{F}7)$–$(\mathcal{F}8)$ are fulfilled. Then, there exist $0 < t_0 < \min\{s_0, 1\}/2$ and $\tilde{f} \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ such that $\tilde{f}(x,t)$ is odd in t and satisfies*

$$\tilde{\mathfrak{F}}(x,t) := p\tilde{\mathcal{F}}(x,t) - \tilde{f}(x,t)t \ge 0,$$

$$\tilde{\mathfrak{F}}(x,t) = 0 \quad \textit{iff} \quad t = 0 \quad \textit{or} \quad |t| \ge 2t_0,$$

where $\frac{\partial}{\partial t}\tilde{\mathcal{F}}(x,t) = \tilde{f}(x,t)$.

Proof. Let us define a cut-off function $\kappa \in C^1(\mathbb{R}, \mathbb{R})$ satisfying $\kappa(t) = 1$ for $|t| \le t_0$, $\kappa(t) = 0$ for $|t| \ge 2t_0$, $|\kappa'(t)| \le 2/t_0$, and $\kappa'(t)t \le 0$. Therefore, we define

$$\tilde{\mathcal{F}}(x,t) = \kappa(t)\mathcal{F}(x,t) + (1 - \kappa(t))\xi|t|^p \quad \text{and} \quad \tilde{f}(x,t) = \frac{\partial}{\partial t}\tilde{\mathcal{F}}(x,t), \tag{41}$$

where $\xi > 0$ is a constant. It is straightforward that

$$p\tilde{\mathcal{F}}(x,t) - \tilde{f}(x,t)t = \kappa(t)\mathfrak{F}(x,t) - \kappa'(t)t\mathcal{F}(x,t) + \kappa'(t)t\xi|t|^p,$$

where $\mathfrak{F}(x,t) := p\mathcal{F}(x,t) - f(x,t)t$. For $0 \le |t| \le t_0$ and $|t| \ge 2t_0$, the conclusion is as follows. Owing to $(\mathcal{F}8)$, we choose a sufficiently small $t_0 > 0$ such that $\mathcal{F}(x,t) \ge \xi t^p$ for $t_0 \le |t| \le 2t_0$. By assuming $\kappa'(t)t \le 0$, we obtain the conclusion. □

Now, with the aid of Proposition 1, and Lemmas 7 and 9, we are ready to prove the second primary result.

Theorem 3. *Assume that* $(\mathcal{V}1)$, $(\mathcal{M}1)$, $(\mathcal{F}1)$–$(\mathcal{F}2)$, *and* $(\mathcal{F}7)$–$(\mathcal{F}8)$ *hold. Moreover, assume that* $\mathfrak{M}(t) \le \mathcal{M}(t)t$ *for any* $t \ge 0$ *and* $f(x,t)$ *is odd in* t *for a small* t. *Then, there is a positive* λ^* *such that the problem* (1) *admits a sequence of weak solutions* $\{u_n\}$ *satisfying* $\|u_n\|_{L^\infty(\mathbb{R}^N)} \to 0$ *as* $n \to \infty$ *for every* $\lambda \in (0, \lambda^*)$.

Proof. We can modify and extend the given function $f(x,t)$ to $\tilde{f} \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfying all properties given in Lemma 9. First, we will show that $\tilde{\mathcal{E}}_\lambda := A_{s,p} - \lambda\tilde{\Psi}$ is coercive on $W(\mathbb{R}^N)$. Let $u \in W(\mathbb{R}^N)$ and $\|u\|_{W(\mathbb{R}^N)} > 1$. By Lemma 9, it is easily shown that $\tilde{\mathcal{E}}_\lambda \in C^1(W(\mathbb{R}^N), \mathbb{R})$ and is even on $W(\mathbb{R}^N)$. Moreover, it follows from $(\mathcal{F}2)$ that, for $|u(x)| \le 2t_0$, there exists a positive constant K_1 such that $\rho(x)\,|u| + K_1|u|^p \ge |F(x,u)|$.

We set $\Omega_1 := \{x \in \mathbb{R}^N : |u(x)| \le t_0\}$, $\Omega_2 := \{x \in \mathbb{R}^N : t_0 \le |u(x)| \le 2t_0\}$, and $\Omega_3 := \{x \in \mathbb{R}^N : 2t_0 \le |u(x)|\}$, where t_0 is given in Lemma 9. From the relation (41) and the conditions of κ, we have

$$
\begin{aligned}
\tilde{\mathcal{E}}_\lambda(u) &:= \frac{1}{p}(\mathfrak{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u\|^p_{p,V}) - \lambda\int_{\mathbb{R}^N} \tilde{\mathcal{F}}(x,u)\,dx \\
&\ge \frac{\min\{1, m_0/\theta\}}{p}\|u\|^p_{W(\mathbb{R}^N)} - \lambda\int_{\Omega_1}\mathcal{F}(x,u)\,dx - \lambda\int_{\Omega_2}\{\kappa(u)\mathcal{F}(x,u) + (1-\kappa(u))\xi|u|^p\}\,dx - \lambda\int_{\Omega_3}\xi|u|^p\,dx \\
&\ge \frac{\min\{1, m_0/\theta\}}{p}\|u\|^p_{W(\mathbb{R}^N)} - \lambda\int_{\Omega_1\cup\Omega_2}\mathcal{F}(x,u)\,dx - \lambda\int_{\Omega_2\cup\Omega_3}\xi|u|^p\,dx \\
&\ge \frac{\min\{1, m_0/\theta\}}{p}\|u\|^p_{W(\mathbb{R}^N)} - \lambda\int_{\Omega_1\cup\Omega_2}\rho(x)|u|\,dx - \lambda\int_{\Omega_1\cup\Omega_2}K_1|u|^p\,dx - \lambda\int_{\Omega_2\cup\Omega_3}\xi|u|^p\,dx \\
&\ge \frac{\min\{1, m_0/\theta\}}{p}\|u\|^p_{W(\mathbb{R}^N)} - 2\lambda\|\rho\|_{L^{p'}(\mathbb{R}^N)}\|u\|_{L^p(\mathbb{R}^N)} - \lambda\,(K_1+\xi)\int_{\mathbb{R}^N}|u|^p\,dx \\
&\ge \frac{\min\{1, m_0/\theta\}}{p}\|u\|^p_{W(\mathbb{R}^N)} - \lambda\left(2C_{15}\|\rho\|_{L^{p'}(\mathbb{R}^N)} + K_1 + \xi\right)\|u\|^p_{W(\mathbb{R}^N)}
\end{aligned}
$$

for some positive constant C_{15}. If we set

$$
\lambda^* := \frac{1}{p(2C_{15}\|\rho\|_{L^{p'}(\mathbb{R}^N)} + K_1 + \xi)},
$$

then we deduce that for any $\lambda \in (0, \lambda^*)$, $\tilde{\mathcal{E}}_\lambda$ is coercive, that is, $\tilde{\mathcal{E}}_\lambda(u) \to \infty$ as $\|u\|_{W(\mathbb{R}^N)} \to \infty$.

Next, we claim that the functional $\tilde{\Psi}' : W(\mathbb{R}^N) \to W^*(\mathbb{R}^N)$, defined by

$$
\langle \tilde{\Psi}'(u), \varphi \rangle = \int_{\mathbb{R}^N} \tilde{f}(x,u)\varphi\,dx \quad \text{for any} \quad \varphi \in W(\mathbb{R}^N),
$$

is compact in $W(\mathbb{R}^N)$. Let us assume that $u_n \rightharpoonup u$ in $W(\mathbb{R}^N)$ as $n \to \infty$. Since the measures of Ω_2 and Ω_3 are finite, we can write $\Omega_2 = \widetilde{\Omega}_2 \cup N_2$ and $\Omega_3 = \widetilde{\Omega}_3 \cup N_3$, where $\widetilde{\Omega}_2$ and $\widetilde{\Omega}_3$ are bounded sets and N_2, N_3 are of measure zero. Let us denote $B_R(0) := \{x \in \mathbb{R}^N : |x| \le R\}$ contained in the bounded sets $\widetilde{\Omega}_2$ and $\widetilde{\Omega}_3$ for a sufficiently large $R \in \mathbb{N}$. Then, from the definition of $\tilde{f}(x,u)$, we have $\tilde{f}(x,u) = f(x,u)$ on $\mathbb{R}^N \setminus (\Omega_2 \cup \Omega_3)$. Thus, we deduce that for any $\varphi \in W(\mathbb{R}^N)$

$$
\sup_{\|\varphi\|_{W(\mathbb{R}^N)} \le 1} \left|\langle \tilde{\Psi}'(u_n) - \tilde{\Psi}'(u), \varphi\rangle\right| = \sup_{\|\varphi\|_{W(\mathbb{R}^N)} \le 1} \left|\int_{\mathbb{R}^N}(\tilde{f}(x,u_n) - \tilde{f}(x,u))\varphi\,dx\right|
$$

$$\leq \sup_{\|\varphi\|_{W(\mathbb{R}^N)}\leq 1} \left| \int_{B_R(0)} (\tilde{f}(x,u_n) - \tilde{f}(x,u))\varphi\, dx \right|$$
$$+ \sup_{\|\varphi\|_{W(\mathbb{R}^N)}\leq 1} \left| \int_{\mathbb{R}^N \setminus (B_R(0)\cup N_4 \cup N_5)} (f(x,u_n) - f(x,u))\varphi\, dx \right|. \tag{42}$$

Owing to Lemma 1, the compact embedding

$$W(\mathbb{R}^N) \hookrightarrow L^p(B_R(0)) \quad \text{implies} \quad u_n \to u \quad \text{in} \quad L^p(B_R(0)) \quad \text{as} \quad n \to \infty.$$

The above, together with the continuity of the Nemytskij operator with $\tilde{f}$ and acting from $L^p(B_R(0))$ into $L^{q'}(B_R(0))$, it is clearly shown that the first term on the right side of the inequality (42) tends to 0 as $n \to \infty$. For the second term in the inequality (42), we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N \setminus (B_R(0)\cup N_2 \cup N_3)} (f(x,u_n) - f(x,u))\varphi dx \right| \\
&\leq \int_{\mathbb{R}^N \setminus (B_R(0)\cup N_2 \cup N_3)} \sigma(x)(|u_n(x)|^{q-1} + |u(x)|^{q-1})\, |\varphi|\, dx \\
&\leq \|\sigma\|_{L^{\frac{p_s^*}{p_s^*-q}}(\mathbb{R}^N \setminus (B_R(0)\cup N_2 \cup N_3))} \left(\|u_n\|^{q-1}_{L^{p_s^*}(\mathbb{R}^N)} + \|u\|^{q-1}_{L^{p_s^*}(\mathbb{R}^N)} \right) \|\varphi\|_{L^{p_s^*}(\mathbb{R}^N)}.
\end{aligned}$$

From the assumption ($\mathcal{F}2$), for $\varepsilon > 0$, there exists $N(R) \in \mathbb{R}$ such that

$$\|\sigma\|_{L^{\frac{p_s^*}{p_s^*-q}}(\mathbb{R}^N \setminus (B_R(0)\cup N_2 \cup N_3))} < \varepsilon$$

for $R > N(R)$. As the sequence $\{u_n\}$ is bounded in $W(\mathbb{R}^N)$, according to Lemma 1, one has $\{u_n\}$ bounded in $L^{p_s^*}(\mathbb{R}^N)$. Thus,

$$\left| \int_{\mathbb{R}^N \setminus (B_R(0)\cup N_2 \cup N_3)} (f(x,u_n) - f(x,u))\varphi\, dx \right| \leq C_{16}\varepsilon \tag{43}$$

for a positive constant C_{16}. Owing to the estimate (43), we can deduce that

$$\int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))\varphi\, dx \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that $\tilde{\Psi}'$ is compact in $W(\mathbb{R}^N)$, as claimed.

Since the derivative of $\tilde{\Psi}$ is compact, it follows from the coercivity of $\tilde{\mathcal{E}}_\lambda$ that the functional $\tilde{\mathcal{E}}_\lambda$ satisfies the (PS)-condition. The weak lower semicontinuity and the coercivity of $\tilde{\mathcal{E}}_\lambda$ ensure that $\tilde{\mathcal{E}}_\lambda$ is bounded from below. To utilize Lemma 7, we only need to obtain for any $n \in \mathbb{N}$, a subspace X_n and $\rho_n > 0$ such that $\sup_{X_n \cap S_{\rho_n}} \tilde{\mathcal{E}}_\lambda < 0$. For any $n \in \mathbb{N}$, we obtain n independent smooth functions ϕ_i for $i = 1, \cdots, n$, and define $X_n := \text{span}\{\phi_1, ..., \phi_n\}$. Owing to Lemma 9, when $\|u\|_{W(\mathbb{R}^N)} < 1$, we have

$$\begin{aligned}
\tilde{\mathcal{E}}_\lambda(u) &= \frac{1}{p}(\mathfrak{M}(|u|^p_{W^{s,p}(\mathbb{R}^N)}) + \|u\|^p_{p,V}) - \lambda \int_{\mathbb{R}^N} \tilde{\mathcal{F}}(x,u)\, dx \\
&\leq \frac{1}{p}\|u\|^p_{W(\mathbb{R}^N)} - \lambda C_{17} \int_{\mathbb{R}^N} \mathcal{F}(x,u)\, dx,
\end{aligned}$$

for $C_{17} > 0$. Taking the assumption ($\mathcal{F}8$) into account, it follows that there exists $\delta_0 > 0$ such that $|t| < \delta_0$, which implies

$$\int_{\mathbb{R}^N} \mathcal{F}(x,t)\, dx \geq \frac{K_2}{p} \int_{\mathbb{R}^N} |t|^p\, dx \tag{44}$$

for a sufficiently large $K_2 > 0$. Using the inequality (44) and the fact that all norms on X_n are equivalent, we can choose a appropriate constant C_{17} and a small enough $\rho_n > 0$ to obtain

$$\sup_{X_n \cap S_{\rho n}} \tilde{\mathcal{E}}_\lambda < 0.$$

According to Lemma 7, we obtain a sequence $c_n < 0$ for $\tilde{\mathcal{E}}_\lambda$ satisfying $c_n \to 0$ when n goes to ∞. Then, for any $u_n \in W(\mathbb{R}^N)$ satisfying $\tilde{\mathcal{E}}_\lambda(u_n) = c_n$ and $\tilde{\mathcal{E}}'_\lambda(u_n) = 0$, $\{u_n\}$ is a (PS)-sequence of $\tilde{\mathcal{E}}_\lambda(u)$, and $\{u_n\}$ has a convergent subsequence. From Lemmas 8 and 9, we deduce that 0 is the only critical point with 0 energy, and the subsequence of $\{u_n\}$ has to converge to 0. Using an indirect argument, we show that $\{u_n\}$ has to converge to 0. Meanwhile, we obtain $u_n \in L^r(\mathbb{R}^N)$ for all $p_s^* \leq r \leq \infty$ owing to Proposition 1. Since $\|u_n\|_{L^\infty(\mathbb{R}^N)} \to 0$, by Lemma 9 again, we have $\|u_n\|_{L^\infty(\mathbb{R}^N)} \leq t_0$ for a large n. Thus, $\{u_n\}$ is a sequence of weak solutions of problem (1). This completes the proof. □

4. Conclusions

In summary, this paper is devoted to the study of weak solutions for Kirchhoff–Schrödinger-type equations involving the fractional p-Laplacian. In the first part of the present paper, under various assumptions on $\mathcal{M}$ and f, we show that our problem admits a sequence of the weak solutions whose energy functional converges to infinity. As we know, a typical example for Kirchhoff function $\mathcal{M}$ is $\mathcal{M}(t) = b_0 + b_1 t^n$ $(n > 0, b_0 > 0, b_1 \geq 0)$ and, based on this example, most results for the multiplicity of solutions are presented. From a different point of view, an infinite number of solutions is proved when $\mathcal{M}$ contains new conditions different from those studied in previous related works; see Example 1. The second part is to investigate the existence of small energy solutions for the given problem whose L^∞-norms converge to zero. As mentioned in the Introduction, the main difficulty is to show the L^∞-bound of weak solutions. Our approach is new to the fractional p-Laplacian problems even if we utilize the well known Moser bootstrap iteration method to overcome this. To the best of our knowledge, such results have not been studied much in these situations.

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