Cosmological model with a traversable wormhole

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In this paper the Friedmann-Robertson-Walker model with a traversable wormhole is considered. It is shown that total matter including cosmic and wormhole matter cannot be exotic, while matter is exotic in the static and inflating wormhole spacetimes. This implies that it is not necessary for total matter to violate the energy conditions in the cosmological model with a traversable wormhole. Assume that the matter is divided into two parts: the cosmic part that depends on time only and the wormhole part that depends on space only. The time development and the spatial dependences of the scale factor, matter, and the wormhole shape function are also obtained. [S0556-2821(96)04912-0]

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Recently, research on Lorentzian wormholes has been increasing because of its attraction in physics and its usability. If there exist traversable wormholes [1] in our universe in spite of some unsolved problems, the detectability or the usability of them may be one of the interesting issues. It was suggested that a wormhole can be identified by the gravitational lensing effect [2]. It might be applied usefully to various usages, for example, interstellar travel [1], time travel [3,4], the inspection of the black hole interior [5], etc.

Normally, a wormhole originates from the nontrivial topological structure of spacetime in the Planckian era of the early universe. It should be enlarged into human size by a certain mechanism to be used in practical purposes. Roman [6] tried to enlarge the wormhole by the inflation of the universe. During the inflation era, the wormhole had also inflated with the universe. Some problems still remain to be solved in his scenario. It needs exotic matter, which violates all known energy conditions to maintain the shape of a traversable wormhole during inflation.

There are also some questions as to what the universe would be after inflation. Thus in this paper we shall try to examine the next era after inflation, and investigate the exotic property of the matter which makes up the spacetime. For spacetime, the wormhole spacetime is considered in the Friedmann-Robertson-Walker (FRW) universe. The time development of the scale factor and matter, and the spatial dependences of matter and the wormhole shape function are also checked in this paper.

Before examining the wormhole in the FRW universe, it is necessary to review the static Morris-Thorne- (MT-) type wormhole [1]. The static wormhole is given by

$$ds^2=-e^{2\Phi(r)}dt^2+\frac{dr^2}{1-b(r)/r}+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{1}$$

The arbitrary functions $\Phi(r)$ and $b(r)$ are lapse and wormhole shape functions, respectively. The function $b(r)$ controls the shape of the wormhole. The metric is spherically symmetric and static. This model requires the exotic property of the matter consisting of the wormhole which is to be traversable.

The metric element of the wormhole in an FRW universe is given by

$$ds^2=-e^{2\Phi(r)}dt^2+R^2(t)\left[\frac{dr^2}{1-kr^2-b(r)/r}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right]. \tag{2}$$

This is constructed by combining simply two spacetime metrics: static wormhole spacetime and FRW spacetime. This combination is analogous to the case of Schwarzschild–de Sitter spacetime by the Schwarzschild and de Sitter metrics. Here $R(t)$ is the scale factor of the universe and k is the sign of the curvature of spacetime. When the functions $b(r)$ and $\Phi(r)\to 0$, the spacetime metric, Eq. (2), becomes the FRW metric. As $R(t)\to$ const and $k\to 0$, it approaches the static wormhole metric, Eq. (1).

With the metric, Eq. (2), the Einstein equations are

$$8\pi T_{tt}=3\frac{\dot{R}^2}{R^2}+\frac{e^{2\Phi}}{R^2}\left(3k+\frac{b'}{r^2}\right), \tag{3}$$

$$8\pi T_{rr}=-\frac{e^{-2\Phi}(2R\ddot{R}+\dot{R}^2)+[k+b/r^3+2\Phi'(-1/r+kr+b/r^2)]}{(1-kr^2-b/r)}, \tag{4}$$

$$8\pi T_{tr}=-2\Phi'\frac{\dot{R}}{R}, \tag{5}$$

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$$8\pi T_{\theta\theta}=r^2e^{-2\Phi}(-2R\ddot{R}-\dot{R}^2)-kr^2+\frac{b}{2r}-\frac{b'}{2}+\Phi'\left(r-2kr^3-\frac{b}{2}-\frac{rb'}{2}\right)+(\Phi'^2+\Phi'')(r^2-kr^4-rb). \tag{6}$$

A prime denotes differentiation with respect to r and an overdot denotes differentiation with respect to t. The components of the stress-energy tensors are denoted by

$$T_{\hat{t}\hat{t}}=\rho(r,t),\ \ T_{\hat{r}\hat{r}}=-\tau(r,t),\ \ T_{\hat{t}\hat{r}}=-f(r,t),\ \ T_{\hat{\theta}\hat{\theta}}=P(r,t). \tag{7}$$

The quantities $\rho(r,t)$, $\tau(r,t)$, $f(r,t)$, and $P(r,t)$ are, respectively, the mass energy density, radial tension per unit area, energy flow in the (outward) radial direction, and lateral pressure as measured by observers stationed at constant r,θ,ϕ [6]. The stress-energy tensor has an off-diagonal component, due to the time dependence of R and/or the space dependence of Φ.

The Einstein equations for $\Phi(r)=0$, zero tidal force as seen by a stationary observer, are

$$8\pi T_{tt}=3\frac{\dot{R}^2}{R^2}+\frac{3k}{R^2}+\frac{b'}{R^2r^2}, \tag{8}$$

$$8\pi T_{rr}=-\frac{1}{1-kr^2-b(r)/r}\left(2R\ddot{R}+\dot{R}^2+k+\frac{b}{r^3}\right), \tag{9}$$

$$8\pi T_{\theta\theta}=-r^2\left(2R\ddot{R}+\dot{R}^2+k+\frac{b'}{2r^2}-\frac{b}{2r^3}\right). \tag{10}$$

In this case, the energy flux term vanishes, and so the stress-energy tensor can be diagonalized. If $R\sim e^{\chi t}$ and $k=0$, as is in the inflation case, these equations (8)–(10) become Roman's [6].

At a constant time t and $\theta=\pi/2$, the metric is written as

$$ds^2=\frac{R^2dr^2}{1-kr^2-b(r)/r}+R^2r^2d\phi^2$$
$$=d\overline{z}^2+d\overline{r}^2+\overline{r}^2d\phi^2=\frac{d\overline{r}^2}{1-\overline{b}/\overline{r}}+\overline{r}^2d\phi^2. \tag{11}$$

Then

$$\frac{d\overline{z}}{d\overline{r}}=\pm\left(\frac{\overline{r}}{\overline{b}}-1\right)^{-1/2}=\frac{dz}{dr}, \tag{12}$$

$$\overline{b}=Rb+kRr^3, \tag{13}$$

$$\overline{r}=Rr. \tag{14}$$

To maintain the shape of the traversable wormhole, the following $d^2\overline{r}/d\overline{z}^2$ term must be positive:

$$\frac{d^2\overline{r}}{d\overline{z}^2}=\frac{1}{R}\,\frac{b-b'r-2kr^3}{2(b+kr^3)^2}=\frac{\overline{b}-\overline{b}'\overline{r}}{2\overline{b}^2}>0. \tag{15}$$

As the universe expands this term $d^2\overline{r}/d\overline{z}^2$ approaches zero; that is, the shape of the wormhole is gradually straightened out. To test the positivity of the term, the exotic function ζ needs to be introduced:

$$\zeta\equiv\frac{\tau-\rho}{|\rho|}$$
$$=\frac{1}{8\pi|\rho|}\left[2\frac{\ddot{R}}{R}-2\frac{\dot{R}^2}{R^2}+\frac{2(b+kr^3)^2}{Rr^3}\,\frac{(b-b'r-2kr^3)}{2R(b+kr^3)^2}\right]$$
$$=\frac{1}{4\pi|\rho|}\left[\left(\frac{\ddot{R}}{R}-\frac{\dot{R}^2}{R^2}\right)+\frac{(b+kr^3)^2}{Rr^3}\,\frac{d^2\overline{r}}{d\overline{z}^2}\right]. \tag{16}$$

Here the exotic function is calculated for the case of $\Phi=0$. Since $d^2\overline{r}/d\overline{z}^2$ should be positive, the sign of the first term $A\equiv\ddot{R}/R-\dot{R}^2/R^2$ of Eq. (16) plays an important role in deciding the sign of ζ. In the static wormhole case ($A=0$), ζ must be positive or $\tau>\rho$, which is exotic matter.

The FRW cosmolgical model usually has various types according to the values of k and/or equations of state. For normal matter, such as radiation ($P=\frac{1}{3}\rho$) and dust ($P=0$), A is always negative for any k. When R is a power function of t, for example, $R\sim t^n(n>0)$, then $A\sim-n/t^2<0$ always. When the equation of state is $P=-\rho$ (vacuum energy) and $k=0$, then R inflates exponentially and $A=0$. This implies that the matter is exotic, $\tau-\rho>0$ as in the static wormhole case. However, power-law inflation [7] can also exhibit normal nonexotic behavior ($A<0$).

The first term A in the cosmological model is always less than zero (except in the case of exponential inflation) while the second term is always positive. This raises the possibility that ζ might be negative due to the cosmological contribution A, even though the exotic contribution from the wormhole is always guaranteed to be positive.

We can find the zero point t_0 at which ζ is equal to zero. In Eq. (16), the term $d^2\overline{r}/d\overline{z}^2$ is proportional to $1/R(t)$ at or near the neck of the wormhole and the second term is proportional to t^{-2n}, where $n\leqslant 2/3$ for normal matter ($\gamma\geqslant 0$). Since the first term $A\sim-n/t^2<0$, the cosmological term dominates the wormhole term at early times ($t<t_0$), while wormhole matter dominates the cosmological term as the universe expands ($t>t_0$).

The Einstein equations (8)–(10) for $\Phi=0$ can be rewritten using Eq. (7) as

$$\rho(r,t)=\frac{1}{8\pi}\left[\frac{3(\dot{R}^2+k)}{R^2}+\frac{1}{R^2}\,\frac{b'}{r^2}\right], \tag{17}$$

$$\tau(r,t)=\frac{1}{8\pi}\left[2\frac{\ddot{R}}{R}+\frac{(\dot{R}^2+k)}{R^2}+\frac{1}{R^2}\,\frac{b}{r^3}\right], \tag{18}$$

$$P(r,t)=\frac{1}{8\pi}\left[-2\frac{\ddot{R}}{R}-\frac{(\dot{R}^2+k)}{R^2}+\frac{1}{2R^2}\left(\frac{b}{r^3}-\frac{b'}{r^2}\right)\right]. \tag{19}$$

The conservation law $T^{\mu}{}_{\nu;\mu}=0$ becomes

$$\dot{\rho}+(3\rho+2P-\tau)\frac{\dot{R}}{R}=0, \tag{20}$$

$$\tau'+(P+\tau)\frac{2}{r}=0. \tag{21}$$

Since ρ, τ, and P depend on both t and r, the following ansatz for matter parts readily helps us to solve Einstein's equations:

$$R^2(t)\rho(t,r)=R^2(t)\rho_c(t)+\rho_w(r), \tag{22}$$

$$R^2(t)\tau(t,r)=R^2(t)\tau_c(t)+\tau_w(r), \tag{23}$$

$$R^2(t)P(t,r)=R^2(t)P_c(t)+P_w(r). \tag{24}$$

The subscript c indicates the cosmological part and w indicates the wormhole part. Since the right-hand sides of the Einstein equations (17)–(19) consist of two parts, the matter part can be separated into cosmic matter and wormhole matter parts. The cosmic part depends on t only and the static wormhole part depends on r only. Because the cosmological part of the equations and matter should be isotropic, we require that $\tau_c(t)=-P_c(t)$.

With the ansatz Eq. (22)–(24), the Einstein's equations and the conservation laws are changed into the variables-separated forms as

$$R^2\rho_c(t)-\frac{3}{8\pi}(\dot{R}^2+k)=\frac{b'(r)}{8\pi r^2}-\rho_w(r)=l, \tag{25}$$

$$R^2\tau_c(t)-\frac{3}{8\pi}(2R\ddot{R}+\dot{R}^2+k)=\frac{b(r)}{8\pi r^3}-\tau_w(r)=m, \tag{26}$$

$$R^2P_c(t)+\frac{3}{8\pi}(2R\ddot{R}+\dot{R}^2+k)=\frac{1}{8\pi}\frac{1}{2}\left(\frac{b(r)}{r^3}-\frac{b'(r)}{r^2}\right)-P_w(r)=n, \tag{27}$$

$$\frac{R^3}{\dot{R}}\left[\dot{\rho}_c+(3\rho_c+2P_c-\tau_c)\frac{\dot{R}}{R}\right]=-\rho_w-2P_w+\tau_w=q=l-3m, \tag{28}$$

$$\tau_w'+(P_w+\tau_w)\frac{2}{r}=0, \tag{29}$$

where l, m, n, and q are the separation constants independent of both t and r. By the relationship $\tau_c=-P_c$ we have $n=-m$. If all the above-mentioned constants are zero, this is the trivial case. The form of the scale factor $R(t)$ becomes the same as that of FRW and $b(r)$ becomes the shape function of the static wormhole case. There are no intersection parts between the cosmological part and wormhole part. These separation constants play the role of linking the two parts.

We can solve the conservation law, Eq. (28), with the equation of state $P_c=\gamma\rho_c$ for the cosmic part. The cosmic energy density ρ_c becomes

$$\rho_c=aR^{-3(1+\gamma)}+\frac{q}{(3\gamma+1)}R^{-2}=\begin{cases} aR^{-4}+\frac{q}{2}R^{-2} & \left(\gamma=\frac{1}{3}\right), \\ aR^{-3}+qR^{-2} & (\gamma=0), \\ a-\frac{q}{2}R^{-2} & (\gamma=-1), \end{cases} \tag{30}$$

where a is an integration constant.

If we put this relationship into the field equation, the solution of the scale factor R is

$$R\sim t^{2/3(1+\gamma)} \quad \text{for} \quad \gamma\neq-1, \tag{31}$$

which is the same result as the case of $l=0$ because the occurrence of the nonzero constants l and q in Eqs. (25) and (28) does not affect the development of R except the simple modification of k. Thus, for $\gamma\neq-1$,

$$\rho_c\sim t^{-4/3(1+\gamma)}+t^{-2}\sim\begin{cases} t^{-1}+t^{-2} & \left(\gamma=\frac{1}{3}\right), \\ t^{-4/3}+t^{-2} & (\gamma=0). \end{cases} \tag{32}$$

The second term due to the separation constant q diminishes faster than the first term.

When $\gamma=-1$, the scale factor R and the energy density ρ_c become

$$R=\sqrt{\frac{\alpha}{a}}\cosh\sqrt{\frac{8\pi a}{3}}t, \tag{33}$$

$$\rho_c=a-\frac{q}{2}\frac{a}{\alpha}\text{sech}^2\sqrt{\frac{8\pi a}{3}}t, \tag{34}$$

where $\alpha=l+q/2+3k/8\pi\neq0$. When $\alpha=0$, they are

$$R\sim e^{(8\pi/3)^{1/2}t}, \tag{35}$$

$$\rho_c\sim1-e^{-2(8\pi/3)^{1/2}t}. \tag{36}$$

As we see in the solutions (33)–(36) for $\gamma=-1$, the scale factor R increases exponentially and the cosmic energy density ρ_c becomes constant very rapidly. Remember that the energy density is constant in the usual exponential-type inflation without a wormhole.

Next the spatial distributions of the wormhole parts $b(r)$, $\rho_w(r)$, $\tau_w(r)$, and $P_w(r)$ can be found. Assume that $P_w=\beta\rho_w$, similar to the equation of state $P_c=\gamma\rho_c$. With this equation of state, the spatial distributions of them are shown as

$$\rho_w(r)=Cr^{-2(1+3\beta)/(1+2\beta)}, \tag{37}$$

$$\tau_w(r)=C(1+2\beta)r^{-2(1+3\beta)/(1+2\beta)}, \tag{38}$$

$$b(r)=8\pi C(1+2\beta)r^{1/(1+2\beta)}. \tag{39}$$

The constant C is the proper integration constant and other integration constants are given as zero from the asymptotic flatness. The restrictions of the constants, $\beta<-1/2$ and $C<0$, are due to the asymptotic flatness condition of the wormhole $\lim_{r\to 0}b/r=0$ [1,4]. With these restrictions, the signs of the matter and wormhole shape functions are determined. Thus $\rho_w<0$ and $P_w, \tau_w, b>0$, which shows the exotic property of wormhole matter.

The flareness condition, Eq. (15), requires that

$$b-b'r-2kr^3=16\pi\beta Cr^{1/(1+2\beta)}-2kr^3>0. \tag{40}$$

When $k=0$ or -1, this model satisfies the condition for the entire range of r. When $k=1$, r should be less than $r_0=(8\pi\beta C)^{(1+\beta)/(2+3\beta)}$ to satisfy the inequality (40). Of course, it means that the size of the throat b_0 should be less than r_0. The traversable wormhole shape is maintained only in the region where $b_0\leqslant r<r_0$.

In this paper the Friedmann-Robertson-Walker model with a wormhole was studied, which found the total matter to be nonexotic, while it is exotic in the static wormhole or inflating wormhole models. The evolutions and distributions of the scale factor, matter, and the wormhole shape function were also obtained by assuming that the matter is divided into two parts.

For further problems, we should investigate the wormhole's change and its role during the reheating stage. Moreover, we can study cosmological models with many wormholes as more realistic models.

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