# Thermodynamics and Form Factors of Supersymmetric Integrable Field Theories 

Changrim Ahn<br>International Centre for Theoretical Physics<br>Strada Costiera 11<br>34014 Trieste, ITALY


#### Abstract

We study on-shell and off-shell properties of the supersymmetric sinh-Gordon and perturbed SUSY Yang-Lee models using the thermodynamic Bethe ansatz and form factors. Identifying the supersymmetric models with the Eight Vertex Free Fermion Model, we derive inversion relation for inhomogeneous transfer matrix and TBA equations and get correct UV results. We obtain two-point form factors of the trace of energy-momentum tensor using the Watson equations and their SUSY transformations. As an application, we compute the UV central charge using these form factors and spectral representation of the $C$ theorem.


## 1. Introduction

For 2D integrable field theories $S$-matrices are purely elastic, all incoming momenta are conserved and multi-particle scattering amplitudes are factorized into a product of two-particle $S$-matrices. These $S$-matrices, in turn, should satisfy Yang-Baxter equations which often determine the $S$-matrices completely along with unitarity and crossing symmetry [1]. The $S$-matrix provides essential tools to understand 2D field theories. First of all, the $S$-matrix gives information on the UV behaviour of the theory by relating the Casimir energy on the cylinder to the central charge of the corresponding UV conformal field theory (CFT) [2]. This program known as thermodynamic Bethe ansatz (TBA) [3] has provided consisitency checks for many factorizable scattering theories either with local lagrangians or without them such as perturbed CFTs [4].
$S$-matrix plays an important role in off-shell physics as well. It can be used to determine off-shell quantities such as correlation functions by computing the matrix elements of an operator on the basis of the on-shell particles. These objects known as form factors (FFs) may be computed exactly using only the $S$-matrices and particle spectrum (bound states) as input [5,6]. With exact FFs correlation functions are given by an infinite sum over intermediate on-shell states. This form factor approach has an advantage for the computation of correlation functions of massive integrable models that the infinite sum over all intermediate states converges very fast. For many cases, upto two-point FFs give quite accurate results on off-shell quantities [7-10]. Furthermore, the two-point FFs can be related to some exact non-perturbative informations of the underlying theories, such as the wave function renormalization $[5,12]$ and the UV central charges through the spectral representation of the $C$-theorem $[8,13,9]$. In this sense, without complete solutions of the FFs one can still extract non-perturbative off-shell informations from the FFs.

While the TBA analysis or the FF computation can be relatively simple for diagonal scattering theory, which has no mass degeneracy, non-diagonal scattering
theories entail much more complicacy. By nondiagonal we mean theories with different types of particles of the same mass for which the scattering of two particles can occur in more than one channels. Most of interesting 2D integrable field theories such as the soliton scattering theories, theories with internal gauge symmetries, and supersymmetric theories belong to this class.

For the non-diagonal theories, the equations for the TBA and FFs are expressed in terms of monodromy and transfer matrices. To solve the equations, one needs to diagonalize these matrices. It is remarkable that with some technical diffferences the same problem is often met in the study of the lattice models [14]. In the lattice model the Yang-Baxter equations are to be satisfied to construct infinite number of conserved charges through the commuting transfer matrices. Partition functions and free energies are expressed in terms of the eigenvalues of the transfer matrices. Due to this common feature, it is often quite useful to connect 2D field theories with lattice models.

There are two types of the models in the lattice and continuum which are connected with each other. The first one is so-called the vertex type; the states are assigned on the lines which form a lattice. For the square lattice, each vertex consists of four lines and assigned a Boltzmann weight depending on the four states of the lines [14]. These lines correspond to the world lines of incoming and outgoing particles in the scattering theories. While some of these vertex models are associated with field theories with local lagrangians, there remain many vertex-type lattice models still to be related to 2D integrable field theories.

The second type is the interacting-round-face models [15]. The Boltzmann weights are assigned on each vertex on the square lattice, depending on the heights of four faces. As a special case, if the heights are restricted, one obtains restricted solid-on-solid (RSOS) type of models. These wide class of the lattice models have been related to 2D CFTs. Due to the conformal invariance, the corresponding lattice models are at the criticality. Many exact results including correlation functions have been obtained using the CFT techniques. These identification can be con-
tinued in the off-critical region. Without the conformal symmetry, the off-critical RSOS models are associated with CFTs perturbed by relevent operators [16-19]. Again, $S$-matrices of the perturbed CFTs are given by the Boltzmann weights of the RSOS models.

The best known example is the relation between the six vertex model and the sine-Gordon (SG) model. The SG model has soliton and antisoliton spectrum and the $S$-matrix can be associated with the $R$-matrix of the $\widehat{\mathrm{s}}_{q}(2)$, affine quantum group [18]. The Boltzmann weights of the six vertex model are the same as the $S$ matrix elements after identifying the up and down arrows assigned on each vertex line with the soliton and antisoliton. In addition, quantum group reduction of the SG model corresponds to the RSOS lattice model obtained from the six vertex model. The TBA analysis of these models have been done by diagonalizing the inhomogeneous transfer matrices of the six vertex [21,22,23] and RSOS models [24].

The complete FFs of the SG model have been obtained by F. Smirnov using quantum inverse scattering methods, providing only known example with the complete FFs for nondiagonal theories. Based on this information, Smirnov found axioms for the FFs to satisfy [6]. Therefore, the problem to find complete FFs is reduced to solve these axioms for a given theory. However, solving these axiomatic equations completely is very difficult even for diagoanal scattering theories except a few simplest ones such as Ising, Yang-Lee, and sinh-Gordon models [8,7,9,11]. The problem becomes much more complicated for the nondiagonal cases. As an initial step to the problem, we will concentrate on two-point FFs. Two-point FFs can be determined relatively easily by diagonalizing $S$-matrix and evaluating the FFs using the Watson equations [5]. For the supersymmetric theories, details can be further simplified due to the SUSY relations between the FFs. As stressed before, the two-point FFs have many useful informations on the underlying theories.

In this paper, we want to apply these frameworks to the $N=1$ supersymmetric (SUSY) theories. The $S$-matrices of many SUSY models have been otained. These
$S$-matrices have the following factorized form $[19,26]$ :

$$
\begin{equation*}
S(\theta)=S_{\mathrm{S}}(\theta) \otimes S_{0}(\theta) \tag{1.1}
\end{equation*}
$$

where the first factor $S_{\mathrm{S}}$ carries the SUSY indices and commutes with the SUSY charges while the second one $S_{0}$ is the $S$-matrices of the models without the SUSY. So far, several SUSY integrable field theories and perturbed super CFTs are solved and their $S$-matrices are derived. Interesting aspect of the SUSY models is that these $S$-matrices commuting with SUSY charges are identified with Boltzmann weights of some lattice models.

For example, for the $N=1$ SUSY CFTs perturbed by the least relevent operator, $S_{\mathrm{S}}$, which commutes with SUSY charges with central extension due to the topological charges, is related to the RSOS weights corresponding to the tricritical Ising model [25]. For the $N=2$ SUSY models, the first factor is identified with the Boltzmann weights of the six vertex model [20,22]. These relations with lattice models are important not only for the lattice- field theory correspondence but for actual solutions of the models.
$N=1$ SUSY sine-Gordon (SSG) model has been solved in an unconventional way. Its soliton $S$-matrix has been derived from the results on the perturbed super CFTs by the least relevent operator [19]. The SUSY part of the SSG soliton $S$ matrix is given by the RSOS tricritical Ising model $S$-matrix while $S_{0}$ is ordinary sine-Gordon $S$-matrix. The $S$-matrices of the SSG bound states (breathers) have been derived from multi-soliton scattering amplitudes [27]. In particular, since the lightest bound states are forming a supermultiplet of the fundamental fields appearing in the SSG lagrangian, the lightest breather $S$-matrix of the SSG model can be analytically continued to get the $S$-matrix of supersymmetric sinh-Gordon (SShG) model. This $S$-matrix is identical to the one derived first by Shankar and Witten by explicitly requiring the commutativity with SUSY charges [28]. Besides, the SSG model with only the lightest breather in the spectrum can be understood as perturbed super CFTs, the SUSY Yang-Lee (SYL) model [26,27]; the simplest
nonunitary super CFT perturbed by the least relevent operator. This model includes only one supermultiplet of on-shell states and the $S$-matrix is identical with that of the SShG model. This $S$-matrix is our starting point.

These models with $N=1$ SUSY without a central extension will be identified with the general eight vertex models with an external field. If the Boltzmann weights of the general eight vertex model satisfy a 'free fermion' condition, the model is exactly solvable and the free energy was derived first from dimer method [29] and later diagonalizing the transfer matrix [30]. Also, this model has been identified with the general $X Y$-spin chain model with an magnetic field [31]. This relation with the lattice model will be very useful in our derivation of TBA equations for the SShG model. It turns out that the SShG model is at the critical point of the $X Y$-spin chain model.

We organize this paper in the following order. In next section, we write down the lagrangian of the SShG and SSG models and derive the energy-momentum tensor supermultiplet and their relations under the SUSY transformation. Also we present the $S$-matrices of the models. In sect.3, we review the basic formulae of TBA analysis which will be used in the next section where explicit derivations are explained. In sect.5, we construct the FFs of the SShG model using the Watson equations and SUSY relations of the energy-momentum tensor. With exact two-point FFs, we derive the UV central charge of the model using the spectral representation of the $C$-theorem.

## 2. $N=1$ SUSY Integrable Model and Factorizable $S$-Matrix

We present the energy-momentum tensor supermultiplet of the $N=1$ SSG and SShG model and the $S$-matrix of the theories.

### 2.1. Lagrangian and Energy-Momentum Tensor

We start with a langrangian of a general $N=1$ SUSY

$$
\begin{equation*}
\mathcal{L}(\Phi)=\frac{1}{4} \bar{D} \Phi D \Phi+\left.i W(\Phi)\right|_{\theta_{1} \theta_{2}} \tag{2.1}
\end{equation*}
$$

with a scalar superfield $\Phi$

$$
\begin{equation*}
\Phi(x, \theta)=\phi+i \bar{\theta} \psi+i \frac{1}{2} \bar{\theta} \theta F \tag{2.2}
\end{equation*}
$$

and $D$ and $\bar{D}$, the covariant derivatives

$$
\begin{equation*}
D_{\alpha}=\frac{\partial}{\partial \bar{\theta}^{\alpha}}+i\left(\gamma^{\mu} \theta\right)_{\alpha} \partial_{\mu} \tag{2.3}
\end{equation*}
$$

The Grassman variable $\theta$ is a Majorana spinor. ${ }^{\star}$. In terms of the component fields, one gets

$$
\begin{equation*}
\mathcal{L}(\Phi)=\frac{1}{2}\left(\partial_{\mu} \phi\right)^{2}+\frac{i}{2} \bar{\psi}\left[\not \partial+W^{\prime \prime}(\phi)\right] \psi+\frac{1}{2}\left[W^{\prime}(\phi)\right]^{2} . \tag{2.4}
\end{equation*}
$$

The SSG model is a particular case of Eq.(2.4) with the superpotential

$$
\begin{equation*}
W(\Phi)=\frac{m}{\beta^{2}} \cos (\beta \Phi) \tag{2.5}
\end{equation*}
$$

The SShG model is the same superpotential with the purely imaginary coupling constant $\beta=i \widehat{\beta}$. The $N=1$ SUSY algebra is generated by the conserved charges

[^0]$Q_{1}$ and $Q_{2}$
\[

$$
\begin{equation*}
Q_{1}^{2}=P_{+}, \quad Q_{2}^{2}=P_{-}, \quad \text { and } \quad\left\{Q_{1}, Q_{2}\right\}=0 \tag{2.6}
\end{equation*}
$$

\]

with the light-cone momenta defined as $P_{ \pm}=E \pm P$. These charges act on the component fields by

$$
\begin{array}{ll}
Q_{1} \phi=i \psi_{1}, & Q_{1} \psi_{1}=\partial_{+} \phi, \tag{2.7}
\end{array} Q_{1} \psi_{2}=F,
$$

with $F=-W^{\prime}(\phi)$.
Integrability of the SSG and SShG models is estabilished because they are equivatent to Toda theory based on the twisted super affine Lie algebra $C^{(2)}(2)$ [32-34]. The equations of motion of the SSG theory can be rewritten as super zero-curvature conditions. An infinite number of conserved charges at the classical level were derived [35] and checked to be preserved at the lowest order quantum corrections [36].

The energy-momentum tensor supermultiplet can be expressed by [37],

$$
\begin{equation*}
J_{\alpha \mu}=\left[\left(\not \partial \Phi-W^{\prime}(\Phi)\right) \gamma_{\mu} D \Phi\right]_{\alpha} \tag{2.8}
\end{equation*}
$$

or in light-cone coordinates

$$
\begin{equation*}
J_{+}=\binom{D_{1} \Phi \partial_{+} \Phi}{-W^{\prime}(\Phi) \Phi D_{1} \Phi}, \quad J_{-}=\binom{-W^{\prime}(\Phi) \Phi D_{2} \Phi}{D_{2} \Phi \partial_{-} \Phi,} \tag{2.9}
\end{equation*}
$$

with $x_{ \pm}=\frac{1}{2}\left(x_{1} \pm x_{0}\right)$ and $\partial_{ \pm}=\partial_{1} \pm \partial_{0}$. In terms of the component currents,

$$
\begin{equation*}
J_{ \pm}=\binom{\Psi_{1 \pm}}{\Psi_{2 \pm}}+2 i\binom{\theta_{2} T_{+ \pm}}{\theta_{1} T_{- \pm}}+i \theta_{1} \theta_{2}\binom{\chi_{1 \pm}}{\chi_{2 \pm}} \tag{2.10}
\end{equation*}
$$

one gets the energy-momentum tensor of the SSG model

$$
\begin{align*}
& T_{++}=\frac{1}{2}\left[\left(\partial_{+} \phi\right)^{2}+i \psi_{1} \partial_{+} \psi_{1}\right], \quad T_{--}=\frac{1}{2}\left[\left(\partial_{-} \phi\right)^{2}-i \psi_{2} \partial_{-} \psi_{2}\right] \\
& T_{+-}=T_{-+}=\frac{1}{2} \frac{m^{2}}{\beta^{2}} \sin ^{2} \beta \phi-\frac{i m}{4} \bar{\psi} \psi \cos \beta \phi \tag{2.11}
\end{align*}
$$

and its superpartner

$$
\begin{array}{ll}
\Psi_{1+}=i \psi_{1} \partial_{+} \phi & \Psi_{2+}=-i \frac{m}{\beta} \psi_{1} \sin \beta \phi  \tag{2.12}\\
\Psi_{2-}=i \psi_{2} \partial_{-} \phi & \Psi_{1-}=-i \frac{m}{\beta} \psi_{2} \sin \beta \phi
\end{array}
$$

Including an appropriate normalization factor of $4 \pi$, we define the following notation for the SUSY energy-momentum tensor:

$$
\begin{equation*}
T=4 \pi T_{++}, \quad \bar{T}=4 \pi T_{--}, \quad \Theta=4 \pi T_{+-} \tag{2.13}
\end{equation*}
$$

and their SUSY partners,

$$
\begin{equation*}
T_{\mathrm{F}}=4 \pi \Psi_{1+}, \quad \bar{T}_{\mathrm{F}}=4 \pi \Psi_{2-}, \quad \Theta_{\mathrm{F}}=4 \pi \Psi_{1-}, \quad \bar{\Theta}_{\mathrm{F}}=4 \pi \Psi_{2+} \tag{2.14}
\end{equation*}
$$

They are related to each other by the SUSY transformation

$$
\begin{array}{lll}
Q_{1} T_{\mathrm{F}}=-2 i T & Q_{1} T=-\frac{1}{2} \partial_{+} T_{\mathrm{F}} & Q_{1} \Theta_{\mathrm{F}}=-2 i \Theta \quad Q_{1} \Theta=-\frac{1}{2} \partial_{+} \Theta_{\mathrm{F}}  \tag{2.15}\\
Q_{2} \bar{T}_{\mathrm{F}}=2 i \bar{T} \quad Q_{2} \bar{T}=-\frac{1}{2} \partial_{-} \bar{T}_{\mathrm{F}} \quad Q_{2} \bar{\Theta}_{\mathrm{F}}=2 i \Theta \quad Q_{2} \Theta=-\frac{1}{2} \partial_{-} \bar{\Theta}_{\mathrm{F}}
\end{array}
$$

### 2.2. On-Shell Particle States and $S$-Matrix

If the coupling constant of the SSG model in Eq.(2.5) becomes pure imaginary, we have a simplest $N=1$ SUSY field theory, namely the SShG model. Since the potential is not periodic, the soliton spectrum does not exist any more and the spectrum consists of only the fundamental particles appearing in the lagrangian, one scalar and fermion supermultiplet. We will denote on-shell states of these particles by $|b(\theta)\rangle$ and $|f(\theta)\rangle$ with a rapidity $\theta$ which is related to the momentum by $E=m \cosh \theta$ and $P=m \sinh \theta$.

The SUSY charges defined in Eq.(2.6) can act on on-shell states as (See sect.5.3)

$$
\begin{align*}
Q_{1}|f(\theta)\rangle & =\sqrt{m} e^{\theta / 2}|b(\theta)\rangle, \quad Q_{1}|b(\theta)\rangle=\sqrt{m} e^{\theta / 2}|f(\theta)\rangle,  \tag{2.16}\\
Q_{2}|f(\theta)\rangle & =-i \sqrt{m} e^{-\theta / 2}|b(\theta)\rangle, \quad Q_{2}|b(\theta)\rangle=i \sqrt{m} e^{-\theta / 2}|f(\theta)\rangle .
\end{align*}
$$

It is easy to see that this satisfies $N=1$ SUSY algebra, Eq.(2.6). Action of SUSY charges on multiparticle on-shell states can be easily worked out using this and the anticommutivity of $Q_{\alpha}$ and the fermion.

Exact $S$-matrix of the SShG model was derived using the Yang-Baxter equation, unitarity and crossing symmetry along with the commutativity of the SUSY charges and the $S$-matrix [28]. In the basis of two-particle on-shell states in the order of $\left|b_{1} b_{2}\right\rangle,\left|f_{1} f_{2}\right\rangle,\left|b_{1} f_{2}\right\rangle,\left|f_{1} b_{2}\right\rangle^{\star}$ the $S$-matrix has been obtained to be $\left(\theta=\theta_{1}-\theta_{2}\right):$

$$
S(\theta)=Y(\theta)\left(\begin{array}{cccc}
1+\frac{2 i \sin \alpha \pi}{\sinh \theta} & \frac{i \sin \alpha \pi}{\cosh \frac{\theta}{2}} & 0 & 0  \tag{2.17}\\
\frac{i \sin \alpha \pi}{\cosh \frac{\theta}{2}} & 1-\frac{2 i \sin \alpha \pi}{\sinh \theta} & 0 & 0 \\
0 & 0 & 1 & \frac{i \sin \alpha \pi}{\sinh \frac{\theta}{2}} \\
0 & 0 & \frac{i \sin \alpha \pi}{\sinh \frac{\theta}{2}} & 1
\end{array}\right)
$$

with an arbitrary constant $\alpha$ which will be related to the coupling constant $\beta$ of the SSG model in a moment. The prefactor $Y(\theta)$ is needed to make the $S$ matrix
$\star$ We use a short notation $\left|b_{1} b_{2}\right\rangle=\left|b\left(\theta_{1}\right) b\left(\theta_{2}\right)\right\rangle$ etc.
unitary and crossing symmetric. The following integral form will be useful later:

$$
\begin{equation*}
Y(\theta)=\frac{\sinh \frac{\theta}{2}}{\sinh \frac{\theta}{2}+i \sin (|\alpha| \pi)} \exp -\int_{0}^{\infty} \frac{d t}{t} \frac{\sinh (|\alpha| t) \sinh ((1-|\alpha|) t)}{\cosh ^{2} \frac{t}{2} \cosh t} \sinh \frac{\theta t}{\pi i} \tag{2.18}
\end{equation*}
$$

With $Y(\theta)=Y(i \pi-\theta)$ and a factor of $i$ arising in the crossing relation for $b b \rightarrow f f$ channel, the $S$-matrix of Eq.(2.17) is crossing symmetric.

To determine the constant $\alpha$ we should refer to another derivation of the SSG breather $S$-matrix. Using the SSG soliton $S$-matrix, one can compute four soliton scattering amplitudes. By taking bound state poles of the incoming and outgoing soliton-antisoliton pairs one can derive the $S$-matrices of the SSG breathers [27]. In particular the $S$-matrix of the lightest breathers is the $S$-matrix of the fundamental particles of the SSG and SShG models:

$$
\begin{align*}
S(\theta) & =Y(\theta) \cdot \mathcal{R}(\theta) \cdot S_{0}(\theta) \\
\text { where } \quad S_{0}(\theta) & =\frac{\sinh \theta+i \sin (2 \alpha \pi)}{\sinh \theta-i \sin (2 \alpha \pi)}, \\
\mathcal{R}(\theta) & =\left(\begin{array}{cccc}
1+\frac{2 i \sin \alpha \pi}{\sin h} & \frac{\sin \alpha \pi}{\cosh \frac{\theta}{2}} & 0 & 0 \\
\frac{\sin \alpha \pi}{\cosh \frac{\theta}{2}} & -1+\frac{2 \sin \alpha \pi}{\sinh \theta} & 0 & 0 \\
0 & 0 & 1 & \frac{i \sin \alpha \pi}{\sinh \frac{\theta}{2}} \\
0 & 0 & \frac{i \sin \alpha \pi}{\sinh \frac{\theta}{2}} & 1
\end{array}\right) . \tag{2.19}
\end{align*}
$$

The factor $S_{0}$ is the lightest breather $S$-matrix of the SG model. The constant $\alpha$ in Eq.(2.19) is given by the coupling constant of the SSG model [27]

$$
\begin{equation*}
\alpha=\frac{\gamma}{16 \pi}=\frac{\beta^{2} / 4 \pi}{1-\beta^{2} / 4 \pi} . \tag{2.20}
\end{equation*}
$$

For the SShG model with $\beta=i \widehat{\beta}$ ( $\widehat{\beta}$ real), this constant reduces to

$$
\begin{equation*}
\alpha=-\frac{\widehat{\beta}^{2} / 4 \pi}{1+\widehat{\beta}^{2} / 4 \pi}, \tag{2.21}
\end{equation*}
$$

and $-\frac{1}{2}<\alpha<0$.

Two $S$-matrices, Eqs.(2.17) and (2.19) are equivalent. The sign difference in the $f f \rightarrow f f$ channel is explained because all particles are considered as bosons in Eq.(2.19) by including the exchange factor -1 arising in $f f \rightarrow f f$ in the $S$-matrix element. In this convention, the crossing relation is satisfied without any extra factor because all particles are bosonic. Besides, for the SShG model with $\alpha<0$, the $S_{0}$ has no pole in the physical strip. Therefore, $S_{0}$ is nothing but a CDD factor and can be removed under the minimality assumption. For the SSG model, however, with a coupling in $0<\alpha<\frac{1}{2}\left(\beta^{2}<4 \pi / 3\right)$ the $S_{0}$ does have a bound state pole corresponding to the second breather.

For a complete description of the SSG model, one should include all the $S$ matrices of the solitons and breathers as was done in [27]. Depending on the values of the coupling constant of the SSG model, the spectrum of the bound states changes. In particular, if the coupling constant is in the range of $\frac{1}{2}<\frac{\gamma}{8 \pi}<1$, only the lightest bound states can exist along with the soliton and antisoliton in the spectrum. If the solitons are truncated from the theory keeping only the lightest bound states, the scattering theory becomes perturbed CFT by the least relevent operator. The UV CFT is the SUSY extension of the Yang-Lee model [26,27]. The $S$-matrix is given by Eq.(2.19).

## 3. Thermodynamic Bethe Ansatz

### 3.1. Diagonal TBA

The TBA computes the Casimir energy for a theory on a circle of length $R$ with $S$-matrices and particle spectrum as input data [38-42]. With a temperature $T=1 / R$ the configuration of minimizing free energy gives the ground state energy of the system, which is again related to the central charge of the underlying UV CFT by

$$
\begin{equation*}
E(R) \approx-\frac{\pi}{6 R}\left(C-12 \Delta_{\min }-12 \bar{\Delta}_{\min }\right) \tag{3.1}
\end{equation*}
$$

as $R \rightarrow 0($ or $T \rightarrow \infty) . \Delta_{\min }\left(\right.$ and $\left.\bar{\Delta}_{\min }\right)$ stands for the lowest conformal dimension allowed by the conformal theories. For unitary theories $\Delta_{\min }$ is zero for the identity operator while it is in general negative for nonunitary theories.

Consider $N+1$ particles in a box of length $L$ with periodic boundary condition (PBC). If we move the $(N+1)$-st particle of mass $m_{a}$ and rapidity $\theta_{k}$ exchanging with all the other particles and come back to the original configuration, we get the following PBC equation:

$$
\begin{equation*}
e^{i m_{a} L \sinh \theta_{k}} \prod_{i=1}^{N} S_{a a_{i}}\left(\theta_{k}-\theta_{i}\right)=1 \tag{3.2}
\end{equation*}
$$

where the index $a_{i}$ specifies species of the $i$-th particle. For the diagonal scattering theories, the product of $S$-matrices are just $C$-number and one can take logarithms on both sides to get

$$
\begin{equation*}
m_{a} L \sinh \theta_{k}+\sum_{i=1}^{N} \frac{1}{i} \ln S_{a a_{i}}\left(\theta_{k}-\theta_{i}\right)=2 \pi n_{k}, \tag{3.3}
\end{equation*}
$$

with an arbitrary integer $n_{k}$. These transcedental equations give solutions for $\left\{\theta_{i}\right\}$ for a given set of arbitrary integers $\left\{n_{k}\right\}$. Therefore, considering all possible integers, and as $N, L \rightarrow \infty$ the solutions form a band structure and one can
introduce a density of the rapidity states, $\rho_{a}(\theta)$, defined by the number of allowed rapidity states between $\theta$ and $\theta+d \theta$ divided by $d \theta$. Any $N$ rapidities of these states can be a solution of the PBC.

Therefore, Eq.(3.3) can be expressed by

$$
\begin{equation*}
2 \pi \rho_{a}(\theta)=m_{a} L \cosh \theta+\sum_{b} \int d \theta^{\prime} \rho_{b}^{1}\left(\theta^{\prime}\right) \phi_{a b}\left(\theta-\theta^{\prime}\right) \tag{3.4}
\end{equation*}
$$

where the $\rho_{a}^{1}(\theta)$ is the density of rapidity states which are actually occupied by the on-shell particles and

$$
\phi_{a b}(\theta)=\frac{1}{i} \frac{\partial}{\partial \theta} \ln S_{a b}(\theta)
$$

Introducing 'psuedo-energy' $\epsilon_{a}$ defined by

$$
\begin{equation*}
\frac{\rho_{a}^{1}(\theta)}{\rho_{a}(\theta)}=\frac{e^{-\epsilon_{a}(\theta)}}{1+e^{-\epsilon_{a}(\theta)}} \tag{3.5}
\end{equation*}
$$

one can express the ground state energy by

$$
\begin{equation*}
E(R)=-\sum_{a} m_{a} \int_{-\infty}^{\infty} \frac{d \theta}{2 \pi} \cosh \theta \ln \left(1+e^{-\epsilon_{a}(\theta)}\right) \tag{3.6}
\end{equation*}
$$

where $\epsilon_{a}$ is determined by the minimizing condition of the free energy:

$$
\begin{equation*}
\epsilon_{a}(\theta)=m_{a} R \cosh \theta-\sum_{b} \int_{-\infty}^{\infty} \frac{d \theta^{\prime}}{2 \pi} \phi_{a b}\left(\theta-\theta^{\prime}\right) \ln \left(1+e^{-\epsilon_{b}\left(\theta^{\prime}\right)}\right) . \tag{3.7}
\end{equation*}
$$

This TBA equation can be solved easily for the UV $(R \rightarrow 0)$ and IR $(R \rightarrow \infty)$ cases because the equations can be effectively described by simple algebraic equations.

### 3.2. Nondiagonal TBA

For the nondiagonal theories, the product of $S$-matrices in Eq.(3.2) is the monodromy matrix and the PBC equation can be expressed as

$$
\begin{align*}
& e^{i m_{a} L \sinh \theta} \mathcal{T}_{a a}\left(\theta \mid \theta_{1}, \ldots, \theta_{N}\right)=1, \quad \text { where } \\
& \mathcal{T}_{a b}\left(\theta \mid \theta_{1}, \ldots, \theta_{N}\right)_{\left\{a_{i}\right\}}^{\left\{a_{i}^{\prime}\right\}}=\sum_{\left\{\alpha_{i}\right\}} S_{a a_{1}}^{\alpha_{2} a_{1}^{\prime}}\left(\theta-\theta_{1}\right) S_{\alpha_{2} a_{2}}^{\alpha_{3} a_{2}^{\prime}}\left(\theta-\theta_{2}\right) \cdots S_{\alpha_{N} a_{N}}^{b a_{N}^{\prime}}\left(\theta-\theta_{N}\right) \tag{3.8}
\end{align*}
$$

If we add these equations for the index $a$, we can express it in terms of the transfer matrix

$$
\begin{equation*}
e^{i m L \sinh \theta} T\left(\theta \mid \theta_{1}, \ldots, \theta_{N}\right)=N_{c} \tag{3.9}
\end{equation*}
$$

where the integer $N_{c}$ is the number of colors and the transfer matrix $T \equiv \sum \mathcal{T}_{a a}$ acts on $V^{\otimes N}$. Precisely speaking, this is 'inhomogeneous' transfer matrix because it depends on each rapidity of in-coming particle states.

To derive the TBA equations, one must diagonalize the transfer matrix which is quite a difficult task due to the size of the matrix and the inhomogeneity. In the lattice models, many pretty ideas have been invented for the purpose [14]. Although these methods are, in principle, applicable only to homogeneous cases, some minor corrections make it possible to apply it to the inhomogeneous ones. Among these, two methods have been successfully applied to derive TBA equations. The first one is using the inversion relation of the transfer matrix. TBA equation for the RSOS model corresponding to $N=1$ SUSY soliton scattering theory has been obtained in this way [8]. Due to the periodic property of the transfer matrix and its inverted matrix, the eigenvalues are completely fixed by the location of 'zeroes'. These zeroes satisfy constraint equations in terms of the rapidities $\theta_{i}$ 's.

Another useful approach is based on the algebraic Bethe ansatz method. In this method, one construct the eigenstates in terms of the monodromy matrix element acting on the vacuum. The eigenvalues can be directly obtained with an additional contraint which makes the eigenstate ansatz to be actual eigenstate. For
the example, this method can be used for the SG model and $N=2$ SUSY theories [21-23].

In general, the eigenvalues of the transfer matrix have the form like

$$
\begin{equation*}
\Lambda\left(\theta \mid \theta_{1}, \ldots, \theta_{N}\right)=\prod_{k=1}^{N} f\left(\theta-x_{k}\right) \tag{3.10}
\end{equation*}
$$

with the zeroes $x_{k}$ of a function $f(x)(f(0)=0)$ which satisfy

$$
\begin{equation*}
\prod_{i=1}^{N} g\left(x_{k}-\theta_{i}\right)=\text { Const. } \tag{3.11}
\end{equation*}
$$

By taking an imaginary part of the logarithms of both sides of Eq.(3.8) and introducing pseudo-energies both for the real particle states and for the zeroes, one can find the TBA equations which are very similar with those of the diagonal TBA. Only difference is there is no mass term ('driving term') in the TBA equation for the density of the zeroes. This absence of the driving term makes a big difference in the analysis of the TBA equations.

### 3.3. Casimir Energy and UV Central Charge

The TBA equations in the UV limit $(R \rightarrow 0)$ can be easily solved because the pseudo-energies become independent of the rapidity around $\theta=0$. This plateaux extends upto $\theta \sim-\ln (m R)$. Therefore, the pseudo-energies become practically constant in the limit and the TBA equations can be reduced to the mere algebraic equations like

$$
\begin{equation*}
x_{a}=\prod_{b}\left(1+x_{b}\right)^{N_{a b}} \tag{3.12}
\end{equation*}
$$

with $x_{a}=\exp \left(-\epsilon_{a}(0)\right)$ and

$$
\begin{equation*}
N_{a b}=\frac{1}{2 \pi}\left[\phi_{a b}(\infty)-\phi_{a b}(-\infty)\right] . \tag{3.13}
\end{equation*}
$$

If all mass terms are non-zero, the $\epsilon_{a}(\infty)$ diverges like $\epsilon_{a}(\theta \rightarrow \infty) \sim m_{a} R \cosh \theta$ and $y_{a}=\exp \left(-\epsilon_{a}(\infty)\right)$ vanishes. If some of the driving terms are zero as is the
case for nondiagonal TBA, $y_{a^{\prime}}$ satisfies

$$
\begin{equation*}
y_{a^{\prime}}=\prod_{b^{\prime}}\left(1+y_{b^{\prime}}\right)^{N_{a^{\prime} b^{\prime}}} \tag{3.14}
\end{equation*}
$$

where $a^{\prime}$ denotes species with vanishing driving term. The ground state energy can be expressed compactly with these variables $x_{a}$ and $y_{a}$ by

$$
\begin{equation*}
E(m R) \sim-\frac{1}{\pi R} \sum_{a}\left[\mathcal{L}\left(\frac{x_{a}}{1+x_{a}}\right)-\mathcal{L}\left(\frac{y_{a}}{1+y_{a}}\right)\right] \tag{3.15}
\end{equation*}
$$

where $\mathcal{L}(x)$ is the Rogers dilogarithmic function

$$
\begin{equation*}
\mathcal{L}(x)=-\frac{1}{2} \int_{0}^{x} d t\left[\frac{\ln (1-t)}{t}+\frac{\ln t}{(1-t)}\right] \tag{3.16}
\end{equation*}
$$

From Eq.(3.1) the central charge of the UV CFT is given by

$$
\begin{equation*}
C-12\left(\Delta_{\min }+\bar{\Delta}_{\min }\right)=\frac{6}{\pi^{2}} \sum_{a}\left[\mathcal{L}\left(\frac{x_{a}}{1+x_{a}}\right)-\mathcal{L}\left(\frac{y_{a}}{1+y_{a}}\right)\right] . \tag{3.17}
\end{equation*}
$$

## 4. TBA for the $N=1$ SUSY models

In this section we diagonalize the inhomogeous transfer matrix associated with the SShG $S$-matrix. The essential observation is that Eq.(2.19) satisfies so-called 'free fermion' condition of the general eight vertex model with an external field. We will derive the TBA equation based on the inversion relation for the transfer matrix. We apply this TBA equation to both the SShG model and a SYL model perturbed by the least relevent operator and derive correct UV central charges.

### 4.1. Free Fermion Models

After the celebrating solution of the symmetric eight vertex model by Baxter, Fan and Wu obtained an exact expression of the free energy for the general eight vertex model with an external field if the Boltzmann weights satisfy some additional contraint, named the free fermion condition [29]. They called this model 'Free Fermion' model (FFM) although the name is slightly misleading. The model turned out to be highly non-trivial and interacting.

We start with the Boltzman weights of the general eight vertex model:

$$
R=\left(\begin{array}{cccc}
a_{+} & 0 & 0 & d  \tag{4.1}\\
0 & b_{+} & c & 0 \\
0 & c & b_{-} & 0 \\
d & 0 & 0 & a_{-}
\end{array}\right)
$$

for the following vertex configurations:


If $R(\theta)$ satisfies the Yang-Baxter equation and the free fermion condition

$$
\begin{equation*}
a_{+} a_{-}+b_{+} b_{-}=c^{2}+d^{2} \tag{4.3}
\end{equation*}
$$

and if the following combination of the Boltzman weights are independent of the
rapidity

$$
\begin{equation*}
\Gamma=\frac{2 c d}{a_{+} b_{-}+a_{-} b_{+}}, \quad h=\frac{a_{-}^{2}+b_{+}^{2}-a_{+}^{2}-b_{-}^{2}}{2\left(a_{+} b_{-}+a_{-} b_{+}\right)}, \tag{4.4}
\end{equation*}
$$

the transfer matrix $T$ commutues; $[T(u), T(v)]=0$. Due to this commutativity, there exist infinite number of conserved charges including a Hamiltonian of the corresponding one-dimensional spin-chain model. This Hamiltonian has been identified with that of the $X Y$-model with a magnetic field,

$$
\begin{equation*}
\mathcal{H}_{X Y}=-J \sum_{j=1}^{N}\left[\sigma_{j}^{+} \sigma_{j+1}^{-}+\sigma_{j}^{-} \sigma_{j+1}^{+}+\Gamma\left(\sigma_{j}^{+} \sigma_{j+1}^{+}+\sigma_{j}^{-} \sigma_{j+1}^{-}\right)-h \sigma_{j}^{z}\right] \tag{4.5}
\end{equation*}
$$

where $\sigma^{ \pm}=\frac{1}{2}\left(\sigma^{x} \pm i \sigma^{y}\right)$ with a conventional Pauli $\sigma^{i}$ matrices.
To identify the FFM with $N=1$ SShG model, we rewrite the $S$-matrix of the SShG model, Eq.(2.19), by rearranging the two-particle basis. In the order of $|b b\rangle,|b f\rangle,|f b\rangle,|f f\rangle$, the $R$-matrix of Eq.(2.19) becomes the general form of the FFM with

$$
\begin{equation*}
a_{ \pm}= \pm 1+\frac{2 i \sin \alpha \pi}{\sinh \theta}, \quad b_{ \pm}=1, \quad c=\frac{i \sin \alpha \pi}{\sinh \frac{\theta}{2}}, \quad d=\frac{\sin \alpha \pi}{\cosh \frac{\theta}{2}}, \tag{4.6}
\end{equation*}
$$

if we identify $\uparrow$ and $\rightarrow$ with $|b\rangle$ and $\downarrow$ and $\leftarrow$ with $|f\rangle^{\star}$.
It is an easy exercise to check these weights satisfy the free fermion condition Eq.(4.3). Also, the constants $\Gamma$ and $h$ becomes

$$
\begin{equation*}
\Gamma=\sin \alpha \pi, \quad h=-1 . \tag{4.7}
\end{equation*}
$$

Since $h=-1$ is a critical point of the $X Y$-model, the SShG model corresponds to the critical point of the general eight vertex model with free fermion condition and with vanishing modulus.

[^1]
### 4.2. Inversion Relation

Critical step to derive TBA equations for the SShG model is to find an inversion relation for the transfer matrix. We derive the following inversion relation in the Appendix A:

$$
\begin{align*}
& T\left(u \mid \theta_{1}, \ldots, \theta_{N}\right) T\left(u+i \pi \mid \theta_{1}, \ldots, \theta_{N}\right)=(-1)^{N} \times \\
& {\left[\prod_{i=1}^{N} M_{+}\left(u-\theta_{i}\right)+\prod_{i=1}^{N} M_{-}\left(u-\theta_{i}\right)+F\left(\prod_{i=1}^{N} F_{+}\left(u-\theta_{i}\right)+\prod_{i=1}^{N} F_{-}\left(u-\theta_{i}\right)\right)\right]} \tag{4.8}
\end{align*}
$$

where the fermion index operator $F$ is either +1 for the bosonic state or -1 for the fermionic one.

The functions appearing in Eq.(4.8) are expressed in terms of the Boltzmann weights as follows:

$$
\begin{align*}
& M_{+}=a_{+} a_{-}-d^{2}, \quad M_{-}=a_{+} a_{-}-c^{2} \\
& F_{+}=\sinh ^{2} \phi a_{+} b_{+}+\cosh ^{2} \phi a_{-} b_{-}-2 \sinh \phi \cosh \phi c d  \tag{4.9}\\
& F_{-}=-\cosh ^{2} \phi a_{+} b_{+}-\sinh ^{2} \phi a_{-} b_{-}+2 \sinh \phi \cosh \phi c d
\end{align*}
$$

and

$$
\begin{equation*}
\tanh (2 \phi)=\frac{2 c d}{a_{+} b_{+}+a_{-} b_{-}}=\sin \alpha \pi . \tag{4.10}
\end{equation*}
$$

Using Eq.(4.6) one can find

$$
\begin{align*}
& M_{+}=-\frac{\sinh \left(\frac{\theta}{2}+i \alpha \pi\right)}{\sinh \frac{\theta}{2}} \frac{\sinh \left(\frac{\theta}{2}-i \alpha \pi\right)}{\sinh \frac{\theta}{2}}, M_{-}=-\frac{\cosh \left(\frac{\theta}{2}+i \alpha \pi\right)}{\cosh \frac{\theta}{2}} \frac{\cosh \left(\frac{\theta}{2}-i \alpha \pi\right)}{\cosh \frac{\theta}{2}}, \\
& F_{+}=-\frac{\cosh \left(\frac{\theta}{2}+i \alpha \pi\right)}{\cosh \frac{\theta}{2}} \frac{\sinh \left(\frac{\theta}{2}-i \alpha \pi\right)}{\sinh \frac{\theta}{2}}, F_{-}=-\frac{\sinh \left(\frac{\theta}{2}+i \alpha \pi\right)}{\sinh \frac{\theta}{2}} \frac{\cosh \left(\frac{\theta}{2}-i \alpha \pi\right)}{\cosh \frac{\theta}{2}} . \tag{4.11}
\end{align*}
$$

From these expressions one can notice that under the change $u \rightarrow u+i \pi$

$$
M_{ \pm} \rightarrow M_{\mp} \quad \text { and } \quad F_{ \pm} \rightarrow F_{\mp}
$$

therefore, $T(u) T(u+\pi i)=T(u+\pi i) T(u+2 \pi i)$. This means

$$
\begin{equation*}
T\left(u \mid \theta_{1}, \ldots, \theta_{N}\right)=T\left(u+2 \pi i \mid \theta_{1}, \ldots, \theta_{N}\right) \tag{4.12}
\end{equation*}
$$

These matrix relations can be easily transformed to equations of the eigenvalues of the transfer matrices; $\Lambda\left(u \mid \theta_{1}, \ldots, \theta_{N}\right)$ is $2 \pi i$ symmetric function,

$$
\begin{equation*}
\Lambda\left(u \mid \theta_{1}, \ldots, \theta_{N}\right)=\Lambda\left(u+2 \pi i \mid \theta_{1}, \ldots, \theta_{N}\right) \tag{4.13}
\end{equation*}
$$

and the inversion relation is nicely factorized,

$$
\begin{align*}
& \Lambda\left(u \mid \theta_{1}, \ldots, \theta_{N}\right) \Lambda\left(u+\pi i \mid \theta_{1}, \ldots, \theta_{N}\right) \\
& =\left[\prod_{i=1}^{N} \frac{\cosh \left(\frac{u-\theta_{i}}{2}+i|\alpha| \pi\right)}{\cosh \left(\frac{u-\theta_{i}}{2}\right)}+F \prod_{i=1}^{N} \frac{\sinh \left(\frac{u-\theta_{i}}{2}+i|\alpha| \pi\right)}{\sinh \left(\frac{u-\theta_{i}}{2}\right)}\right]  \tag{4.14}\\
& \times\left[\prod_{i=1}^{N} \frac{\cosh \left(\frac{u-\theta_{i}}{2}-i|\alpha| \pi\right)}{\cosh \left(\frac{u-\theta_{i}}{2}\right)}+F \prod_{i=1}^{N} \frac{\sinh \left(\frac{u-\theta_{i}}{2}-i|\alpha| \pi\right)}{\sinh \left(\frac{u-\theta_{i}}{2}\right)}\right] .
\end{align*}
$$

### 4.3. Eigenvalues

Since $\Lambda(u)$ is a $2 \pi i$-periodic function with poles at $u=\theta_{k}$ and $u=\theta_{k}+i \pi$, it can be completely fixed by the location of zeroes on the strip in the complex plain of $-i \pi<\operatorname{Im}[u] \leq i \pi$ and $-\infty<\operatorname{Re}[u]<\infty$. Also from the fact that $\Lambda(u)$ becomes a constant as $u \rightarrow \infty$, we can find that

$$
\begin{equation*}
\Lambda\left(u \mid \theta_{1}, \ldots, \theta_{N}\right)=\text { Const. } \prod_{k=1}^{N}\left[\frac{\sinh \left(\frac{u-z_{k}^{+}}{2}\right)}{\sinh \left(\frac{u-\theta_{k}}{2}\right)} \frac{\sinh \left(\frac{u-z_{k}^{-}}{2}\right)}{\cosh \left(\frac{u-\theta_{k}}{2}\right)}\right] \tag{4.15}
\end{equation*}
$$

where the $2 N$ zeroes $\left\{z_{k}^{+}\right\}$and $\left\{z_{k}^{-}\right\}$located on the strip will be determined as functions of $\theta_{i}$ 's.

We defined the zeroes in the way that $z_{k}^{+}$and $z_{k}^{-}$come from the first and second factors of the RHS of Eq.(4.14), respectively. Therefore, they satisfy

$$
\begin{equation*}
\prod_{i=1}^{N} \frac{\tanh \left(\frac{z_{k}^{+}-\theta_{i}}{2}+i|\alpha| \pi\right)}{\tanh \left(\frac{z_{k}^{+}-\theta_{i}}{2}\right)}=-F, \quad \prod_{i=1}^{N} \frac{\tanh \left(\frac{z_{k}^{-}-\theta_{i}}{2}-i|\alpha| \pi\right)}{\tanh \left(\frac{z_{k}^{-}-\theta_{i}}{2}\right)}=-F \tag{4.16}
\end{equation*}
$$

The solutions of these equations can be written in terms of real variables $x_{k}$ in the following way:

$$
\begin{array}{ll}
z_{k}^{+}=x_{k}-i|\alpha| \pi, & x_{k}-i|\alpha| \pi+i \pi,  \tag{4.17}\\
z_{k}^{-}=x_{k}+i|\alpha| \pi, & x_{k}+i|\alpha| \pi-i \pi,
\end{array}
$$

where a real number $x_{k}$ satisfies

$$
\begin{equation*}
\prod_{i=1}^{N} \frac{\tanh \left(\frac{x_{k}-\theta_{i}}{2}-\frac{i|\alpha| \pi}{2}\right)}{\tanh \left(\frac{x_{k}-\theta_{i}}{2}+\frac{i|\alpha| \pi}{2}\right)}=-F . \tag{4.18}
\end{equation*}
$$

Out of four possible choices of $z_{k}^{ \pm}$for $k=1, \ldots, N$ from Eq.(4.17), only two choices are allowed. This can be understood easily if one considers the limit of $|\alpha| \rightarrow 0$. The Boltzmann weights are either +1 or -1 from Eq.(4.6). This means the transfer matrix is just a constant matrix without any dependence on the rapidities. Now from Eqs.(4.15) and (4.18), the only possibility for the eigenvalues of the transfer matrix to be independent of $\theta_{i}$ is when $\left\{x_{k}\right\}=\left\{\theta_{i}\right\}$ and when $z_{k}^{+}-z_{k}^{-}= \pm i \pi$ for all $k$. For example, if one choose $\left(z_{k}^{+}, z_{k}^{-}\right)=\left(\theta_{k}-i|\alpha| \pi, \theta_{k}+i|\alpha| \pi\right)$ pair for some $k$ as $|\alpha| \rightarrow 0$, the eigenvalue will get term like $\tanh \left(\frac{u-\theta_{k}}{2}\right)$. Obviously, this eigenvalue should be excluded for the constant transfer matrix. This leaves only two choices for the zeroes:

$$
\begin{equation*}
\left(z_{k}^{+}, z_{k}^{-}\right)=\left(x_{k}-i|\alpha| \pi, x_{k}+i|\alpha| \pi-i \pi\right) \quad \text { or } \quad\left(x_{k}-i|\alpha| \pi+i \pi, x_{k}+i|\alpha| \pi\right) . \tag{4.19}
\end{equation*}
$$

From the product form of Eq.(4.14), one notices that if we choose one pair of zeroes in Eq.(4.19) the other pair become zeroes of $\Lambda(u+i \pi)$. Since one can choose


Figure 1. The zeroes $z_{k}^{+}$on the complex $\theta$ plane.
the zeroes between the two possibilities for each $k(k=1, \ldots, N)$, we can construct $2^{N}$ different eigenvalues in this way. Also, one can convince that Eq.(4.15) satisfies Eq.(4.14) because if we dividing the RHS of Eq.(4.14) with $\Lambda(u) \Lambda(u+i \pi)$ using Eq.(4.15) the final expression has no poles and zeroes while it is bounded. This means the ratio should be a constant.

Using all these results, the eigenvalues are compactly expressed by

$$
\begin{equation*}
\Lambda(u)_{\epsilon_{1}, \ldots, \epsilon_{N}}=\text { Const. }\left[\frac{\prod_{k=1}^{N} \lambda_{\epsilon_{k}}\left(u-x_{k}\right)}{\prod_{i=1}^{N} \sinh \left(u-\theta_{i}\right)}\right], \quad \epsilon_{k}= \pm 1 \tag{4.20}
\end{equation*}
$$

with

$$
\begin{equation*}
\lambda_{\epsilon}(\theta)=\sinh \left(\frac{\theta}{2}+\epsilon \frac{i|\alpha| \pi}{2}\right) \cosh \left(\frac{\theta}{2}-\epsilon \frac{i|\alpha| \pi}{2}\right) . \tag{4.21}
\end{equation*}
$$

$\epsilon=+1$ corresponds to the first choice in Eq.(4.19) and $\epsilon=-1$ the second. (See Fig.1) The real zeroes $x_{k}$ are determined by Eq.(4.18).

### 4.4. Nondiagonal TBA

From Eq.(3.8) and Eq.(4.20), the PBC equation becomes

$$
\begin{equation*}
\frac{1}{2} e^{i m \sinh \theta} \prod_{i=1}^{N}\left[\frac{Y\left(\theta-\theta_{i}\right)}{\sinh \left(\theta-\theta_{i}\right)}\right] \prod_{k=1}^{N} \lambda_{\epsilon_{k}}\left(\theta \mid x_{1}, \ldots, x_{N}\right)=1 \tag{4.22}
\end{equation*}
$$

and the constraint equation (4.18) in the limit $N, L \rightarrow \infty$ are expressed by introducing the densities $\rho$ for the allowed states, $\rho^{1}$ for the occupied states, $P_{+}$for $\epsilon=+1$ zero-state, and $P_{-}$for $\epsilon=-1$. In terms of these densities, one gets

$$
\begin{align*}
& 2 \pi \rho(\theta)=m \cosh \theta \\
& +\int d \theta^{\prime}\left[\rho^{1}\left(\theta^{\prime}\right) \Phi_{Y}\left(\theta-\theta^{\prime}\right)+P_{+}\left(\theta^{\prime}\right) \Phi_{+}\left(\theta-\theta^{\prime}\right)+P_{-}\left(\theta^{\prime}\right) \Phi_{-}\left(\theta-\theta^{\prime}\right)\right]  \tag{4.23}\\
& 2 \pi P(\theta)=\int d \theta^{\prime} \rho^{1}\left(\theta^{\prime}\right) \Phi_{T}\left(\theta-\theta^{\prime}\right)
\end{align*}
$$

where

$$
\begin{align*}
& \Phi_{Y}(\theta)=\frac{\partial}{\partial \theta} \operatorname{Im} \ln \left[\frac{Y(\theta)}{\sinh \theta}\right], \quad \Phi_{ \pm}(\theta)=\frac{\partial}{\partial \theta} \operatorname{Im} \ln \lambda_{ \pm}(\theta) \\
& \Phi_{T}(\theta)=\frac{1}{i} \frac{\partial}{\partial \theta} \ln \left[\frac{\tanh \left(\frac{\theta}{2}-\frac{i|\alpha| \pi}{2}\right)}{\tanh \left(\frac{\theta}{2}+\frac{i|\alpha| \pi}{2}\right)}\right] . \tag{4.24}
\end{align*}
$$

Using Eq.(4.21) and $\lambda_{-}=\left(\lambda_{+}\right)^{*}$, one can easily show that the kernels are related by

$$
\begin{equation*}
\Phi_{T}(\theta)=2 \Phi_{+}(\theta)=-2 \Phi_{-}(\theta)=\frac{1}{i} \frac{\partial}{\partial \theta} \ln \left[\frac{\sinh \theta-i \sin |\alpha| \pi}{\sinh \theta+i \sin |\alpha| \pi}\right], \tag{4.25}
\end{equation*}
$$

which is nothing but the kernel of the sinh-Gordon model. We will denote this kernel by $\Phi$. Also, we can eliminate $P_{-}$from the first equation of (4.23) using the
second equation and $P=P_{+}+P_{-}$to rewrite it as

$$
\begin{equation*}
2 \pi \rho(\theta)=m \cosh \theta+\int d \theta^{\prime}\left[\rho^{1}\left(\theta^{\prime}\right)\left[\Phi_{Y}-\frac{1}{2} \Phi * \Phi\right]\left(\theta-\theta^{\prime}\right)+P_{+}\left(\theta^{\prime}\right) \Phi\left(\theta-\theta^{\prime}\right)\right] \tag{4.26}
\end{equation*}
$$

where we introduce a convolution defined by

$$
[f * g](\theta)=\int_{-\infty}^{\infty} \frac{d \theta^{\prime}}{2 \pi} f\left(\theta-\theta^{\prime}\right) g\left(\theta^{\prime}\right)
$$

Eqs.(4.23) and (4.26) have the same form as those met in diagonal TBA like Eq.(3.7), except that the second equation in Eq.(4.23) has no driving term. The TBA equations, therefore, can be written down as before,

$$
\begin{align*}
m R \cosh \theta & =\epsilon(\theta)+\left(\left[\Phi_{Y}-\frac{1}{2} \Phi * \Phi\right] * \ln \left[1+e^{-\epsilon}\right]\right)(\theta)+\left(\Phi * \ln \left[1+e^{-\mathcal{E}}\right]\right)(\theta) \\
0 & =\mathcal{E}(\theta)+\left(\Phi * \ln \left[1+e^{-\epsilon}\right]\right)(\theta) \tag{4.27}
\end{align*}
$$

in terms of the pseudoenergies $\epsilon$ and $\mathcal{E}$ defined by

$$
\begin{equation*}
\frac{\rho^{1}(\theta)}{\rho(\theta)}=\frac{e^{-\epsilon(\theta)}}{1+e^{-\epsilon(\theta)}}, \quad \frac{P_{+}(\theta)}{P(\theta)}=\frac{e^{-\mathcal{E}(\theta)}}{1+e^{-\mathcal{E}(\theta)}} \tag{4.28}
\end{equation*}
$$

### 4.5. Central Charges of the SShG model

In the UV limit, Eq.(4.27) reduces to simple algebraic equations of the variables $x=\exp [-\epsilon(0)], X=\exp [-\mathcal{E}(0)]$ as argued before. For the SShG model, the algebraic equations become

$$
\begin{equation*}
x=(1+x)^{a}(1+X)^{b}, \quad X=(1+x)^{b} \tag{4.29}
\end{equation*}
$$

with

$$
\begin{equation*}
a=\int_{-\infty}^{\infty} \frac{d \theta}{2 \pi}\left(\Phi_{Y}-\frac{1}{2} \Phi * \Phi\right)(\theta), \quad b=\int_{-\infty}^{\infty} \frac{d \theta}{2 \pi} \Phi(\theta) \tag{4.30}
\end{equation*}
$$

It is not difficult to compute these exponents using Eqs.(4.21) and (4.24),

$$
\int_{-\infty}^{\infty} \frac{d \theta}{2 \pi} \Phi_{Y}(\theta)=\frac{1}{2}, \quad \int_{-\infty}^{\infty} \frac{d \theta}{2 \pi} \Phi(\theta)=1
$$

and

$$
\begin{gather*}
\int_{-\infty}^{\infty} \frac{d \theta}{2 \pi}[\Phi * \Phi](\theta)=\left[\int_{-\infty}^{\infty} \frac{d \theta}{2 \pi} \Phi(\theta)\right]^{2}=1 \\
a=0 \quad \text { and } \quad b=1 \tag{4.31}
\end{gather*}
$$

Using these values, the solution of Eq.(4.29) can be found easily as

$$
\begin{equation*}
x=\infty \quad \text { and } \quad X=\infty \tag{4.32}
\end{equation*}
$$

One also needs the psuedo-energies as $\theta \rightarrow \infty$. Since the mass term for $\epsilon(\theta)$ is non-zero, $\epsilon$ diverges as $\theta \rightarrow \infty$, and $y=\exp [-\epsilon(\infty)]=0$. Then, from the second equation of (4.27), $Y=\exp [-\mathcal{E}(\infty)]=1$.

Using all these values in Eq.(3.17) the UV central charge becomes

$$
\frac{6}{\pi^{2}}\left[\mathcal{L}(1)+\mathcal{L}(1)-\mathcal{L}\left(\frac{1}{2}\right)\right]=\frac{3}{2}
$$

with $\Delta_{\min }=\bar{\Delta}_{\min }=0$ for the SShG model. This is correct UV central charge of the SShG model with a boson and a fermion.

### 4.6. Central Charges of the SUSY Yang-Lee Model

As explained in the previous section, one can truncate all solitons from the SSG multi-soliton Hilbert space to have only breathers. In particular, for the coupling constant $\alpha=\frac{1}{3}$, only the lightest breather and its superpartner can exist in the spectrum with the $S$-matrix given in Eq.(2.19) [27]. This is the SYL model perturbed by the least relevent operator.

The fundamental difference from the SShG $S$-matrix is that because of $\alpha>0$ $S_{0}(\theta)$ is no more CDD factor. It has a pole which is identified with the particle itself. If we denote the particles as $B$ and $F$, the bootstrap relations are

$$
B B(F F) \rightarrow B \rightarrow B B(F F), \quad B F(F B) \rightarrow F \rightarrow B F(F B) .
$$

Except this difference, all the TBA analysis of the SShG model is equally applicable to the SYL model.

The SYL conformal theory can be constructed as a coset CFT given by

$$
\frac{S U(2)_{K} \otimes S U(2)_{L}}{S U(2)_{K+L}}, \quad K=2 \quad \text { and } \quad L+2=\frac{2}{3} .
$$

The central charge of the model is $C=-\frac{21}{4}$. Due to the nonunitarity of the model, the lowest conformal dimension is not zero. To determine the lowest conformal dimension of the model, we refer to the general formula of the general coset theories. The primary fields of the coset theory have the conformal dimensions given by the following formula [43]:

$$
\begin{equation*}
\Delta_{r, s}^{l}=\frac{l(l+2)}{4(K+2)}-\frac{l^{2}}{4 K}+\frac{\left(r p^{\prime}-s p\right)^{2}-\left(p^{\prime}-p\right)^{2}}{4 K p p^{\prime}}, \quad L+2=\frac{p}{q}, \quad p^{\prime}=p+K q, \tag{4.33}
\end{equation*}
$$

with the restrictions,

$$
0 \leq l \leq K, \quad 1 \leq r \leq p-1, \quad 1 \leq s \leq p^{\prime}-1, \quad l=|r-s \bmod 2 K| .
$$

For the SYL model with the values of $K=2, p=2, p^{\prime}=8$, one finds that only $r=1, l=0,1,2$, and $s=1,2,3,5,6,7$ are allowed. The minimal conformal
dimension arises when $(8-2 s)^{2}$ is minized, i.e. with $s=3$. Therefore, the minimal conformal dimension of the SYL model is $\Delta_{\min }=\Delta_{1,3}^{2}=-\frac{1}{4}$.

Now we compute the Casimir energy of the model using TBA. Notice that the only change from the SShG TBA is that the kernel $\Phi_{Y}$ in Eq.(4.27) gets extra factor $-\frac{\partial}{\partial \theta} \ln S_{0}(\theta)$ due to $S_{0}$ in Eq.(2.19). With $\alpha>0$ this introduces extra -1 in the exponent $a$ in Eq.(4.29) to make $a=-1$. With this change the algebraic equations now become

$$
\begin{equation*}
x=\frac{1+X}{1+x}, \quad X=1+x \tag{4.34}
\end{equation*}
$$

and the solutions are $x=\sqrt{2}$ and $X=1+\sqrt{2}$. Inserting this into Eq.(3.17), one can get

$$
\frac{6}{\pi^{2}}\left[\mathcal{L}\left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)+\mathcal{L}\left(\frac{1+\sqrt{2}}{2+\sqrt{2}}\right)-\mathcal{L}\left(\frac{1}{2}\right)\right]=\frac{3}{4}
$$

which is exactly $C-24 \Delta_{\min }$ as required.

## 5. Form Factors of the Supersymmetric Theories

In this section, we derive FFs of the SShG model from the $S$-matrix of the model. Starting with the axioms for the FFs to satisfy, we write two-point FFs in terms of two factors. The first factor is determined by the properties of operators and contents of poles while the second factor is completely determined by the eigenvalues of the $S$-matrix. For the SShG model, we compute one-point FFs first to fix overall normalization. Then, two-point FFs will be derived using the Watson equations and SUSY relations of the FFs.

### 5.1. Form Factor Axioms

We start with axioms of the FFs [6]. Denoting $|a(\theta)\rangle$ as an on-shell particle state of type $a$ with a rapidity $\theta$, a matrix element of an Hermitian operator $\mathcal{O}$ between vacuum and in-coming states can be expressed by

$$
\begin{equation*}
F_{a_{1}, \ldots, a_{n}}^{\mathcal{O}}\left(\theta_{1}, \ldots, \theta_{n}\right)=\langle 0| \mathcal{O}(0)\left|a_{1}\left(\theta_{1}\right), \ldots, a_{n}\left(\theta_{n}\right)\right\rangle_{\mathrm{in}} \tag{5.1}
\end{equation*}
$$

with a normal ordering of rapidities, $\theta_{1}>\theta_{2}>\ldots>\theta_{n}$. This FF should satisfy the following axioms:

1. $F$ is analytic in each variable $\theta_{i j}=\theta_{i}-\theta_{j}$ inside the strip of $0 \leq \operatorname{Im}\left[\theta_{i j}\right] \leq 2 \pi$ except for simple poles. All other type of FFs can be reduced to the standard form like

$$
\begin{align*}
& \text { out }\left\langle b_{1}\left(\theta_{1}^{\prime}\right), \ldots, b_{m}\left(\theta_{m}^{\prime}\right)\right| \mathcal{O}(0)\left|a_{1}\left(\theta_{1}\right), \ldots, a_{n}\left(\theta_{n}\right)\right\rangle_{\text {in }} \\
= & C^{b_{1} b_{1}^{\prime}} \ldots C^{b_{m} b_{m}^{\prime}} F_{b_{1}^{\prime}, \ldots, b_{m}^{\prime}, a_{1}, \ldots, a_{n}}\left(\theta_{1}^{\prime}+i \pi, \ldots, \theta_{m}^{\prime}+i \pi, \theta_{1}, \ldots, \theta_{n}\right) \tag{5.2}
\end{align*}
$$

with the charge conjugation matrix $C$ arising in the crossing process.
2. Due to the normal ordering, any exchange of two rapidities should involve
a scattering process;

$$
\begin{align*}
& F_{a_{1}, \ldots, a_{i}, a_{i+1}, \ldots, a_{n}}\left(\theta_{1}, \ldots, \theta_{i}, \theta_{i+1}, \ldots, \theta_{n}\right) \\
= & (-1)^{f_{a_{i}} f_{a_{i+1}}} S_{a_{i} a_{i+1}}^{a_{i}^{\prime} a_{i+1}^{\prime}}\left(\theta_{i}-\theta_{i+1}\right) F_{a_{1}, \ldots, a_{i+1}, a_{i}, \ldots, a_{n}}\left(\theta_{1}, \ldots, \theta_{i+1}, \theta_{i}, \ldots, \theta_{n}\right), \tag{5.3}
\end{align*}
$$

with the phase factor arising from the exchange of particles ( $f_{a}=0$ for a boson or 1 for a fermion). ${ }^{\star}$
3. Relativistic invariance dictates

$$
\begin{equation*}
F_{a_{1}, \ldots, a_{n}}\left(\theta_{1}+\Lambda, \ldots, \theta_{n}+\Lambda\right)=e^{s \Lambda} F_{a_{1}, \ldots, a_{n}}\left(\theta_{1}, \ldots, \theta_{n}\right), \tag{5.4}
\end{equation*}
$$

where $s$ is the spin of the operator $\mathcal{O}$.
4. A $2 \pi i$ translation of one of rapidity is equivalent to

$$
\begin{equation*}
F_{a_{1}, \ldots, a_{n}}\left(\theta_{1}, \ldots, \theta_{n}+2 \pi i\right)=(-1)^{f_{a_{n}} \sum f_{a_{i}}} F_{a_{n}, a_{1}, \ldots, a_{n-1}}\left(\theta_{n}, \theta_{1}, \ldots, \theta_{n-1}\right) \tag{5.5}
\end{equation*}
$$

There are two origins of the poles. One is the annihilation pole and the other corresponds to the bound states. Existence of the poles give extra constraints on the FFs.
5. The annihilation pole arises in the channel of two incoming particles which are related to each other by the charge conjugation at $\theta=i \pi$. This gives

$$
\begin{align*}
&{ }^{\text {Res }}  \tag{5.6}\\
&{ }_{\theta^{\prime} \rightarrow \theta} F_{a_{1}, \ldots, a_{n}, \bar{a}, a}\left(\theta_{1}, \ldots, \theta_{n}, \theta^{\prime}+i \pi, \theta\right) \\
&= {\left[\mathbf{1}-\mathcal{T}_{a a}\left(\theta \mid \theta_{1}, \ldots, \theta_{n}\right)\right]_{a_{1}, \ldots, a_{n}}^{a_{1}^{\prime}, \ldots, a_{n}^{\prime}} F_{a_{1}^{\prime}, \ldots, a_{n}^{\prime}}\left(\theta_{1}, \ldots, \theta_{n}\right), }
\end{align*}
$$

using the monodromy matrix defined in Eq.(3.8). This equation relates a $n+2$ point to a $n$-point function.

[^2]6. The bound state pole of a $S$-matrix with a residue
\[

$$
\begin{equation*}
i_{\theta \rightarrow \theta_{c}}^{\operatorname{Res}} S_{a b}^{a^{\prime} b^{\prime}}(\theta)=-\Gamma_{a b}^{c} \Gamma_{c}^{a^{\prime} b^{\prime}} \tag{5.7}
\end{equation*}
$$

\]

the FF should satisfy

$$
\begin{equation*}
i_{\theta_{a}-\theta_{b} \rightarrow \theta_{c}} \quad F_{a_{1}, \ldots, a_{n}, a_{a}, a_{b}}\left(\theta_{1}, \ldots, \theta_{n}, \theta_{a}, \theta_{b}\right)=\Gamma_{a b}^{c} F_{a_{1}, \ldots, a_{n}, a_{c}}\left(\theta_{1}, \ldots, \theta_{n}, \theta_{c}\right) . \tag{5.8}
\end{equation*}
$$

### 5.2. Two-point Form Factors

The general axioms can be much simplified for two-point FFs. It can be written in terms of two factors like,

$$
\begin{equation*}
F_{a_{1} a_{2}}\left(\theta_{1}, \theta_{2}\right)=K_{a_{1} a_{2}}\left(\theta_{1}, \theta_{2}\right) F_{a_{1} a_{2}}^{\min }\left(\theta_{1}-\theta_{2}\right), \tag{5.9}
\end{equation*}
$$

where $F^{\mathrm{min}}$ satisfies the Watson equation without any pole,

$$
\begin{equation*}
F_{a_{1} a_{2}}^{\min }(\theta)=F_{a_{1} a_{2}}^{\min }(-\theta) S_{a_{1} a_{2}}^{a_{1}^{\prime} a_{2}^{\prime}}(\theta), \quad F_{a_{1} a_{2}}^{\min }(i \pi-\theta)=F_{a_{1} a_{2}}^{\min }(i \pi+\theta), \tag{5.10}
\end{equation*}
$$

and the prefactor $K_{a_{1} a_{2}}\left(\theta_{1}, \theta_{2}\right)$ has all the required poles and operator dependence. Note we omitted a phase factor from the fermion exchange operator treating all particles as bosonic.
$F^{\text {min }}$ can be determined from the following steps: In the basis which diagonalizes the $S$-matrix, Eq.(5.10) becomes a simple functional relation. Then, using an integral representation for the $i$-th eigenvalue of the $S$-matrix, one finds

$$
\begin{equation*}
S_{i}(\theta)=\exp \left[\int_{0}^{\infty} \frac{d t}{t} f_{i}(t) \sinh \frac{\theta t}{\pi i}\right] \longrightarrow F_{i}^{\min }=\exp \left[\int_{0}^{\infty} \frac{d t}{t} \frac{f_{i}(t)}{\sinh \theta} \sin ^{2} \frac{\widehat{\theta t}}{2 \pi}\right] \tag{5.11}
\end{equation*}
$$

where $\widehat{\theta}=i \pi-\theta$. Rotating back to the original on-shell two-particle states, one finds the $F^{\mathrm{min}}$.

The function $K\left(\theta_{1}, \theta_{2}\right)$ should be determined by the other axioms. Eq.(5.5) requires it be a symmetric function of the rapidities if $a_{1}=a_{2}$ and has a $i \pi$-pole if $\bar{a}_{1}=a_{2}$. The asymptotic behaviour under the rapidity translation is related to the spin of the operator by Eq.(5.4). The overall normalization of the FFs is fixed by the one-point function.

### 5.3. One-Point Form Factor of the SShG model

We work out one-point function of the SShG model to fix normalization of the general FFs. From the Fourier transformation of the elementary fields

$$
\begin{align*}
& \phi(\mathbf{x})=\int \frac{d k}{2 \pi} \frac{1}{\sqrt{2} k_{0}}\left[b_{k} e^{i \mathbf{k} \cdot \mathbf{x}}+b_{k}^{\dagger} e^{-i \mathbf{k} \cdot \mathbf{x}}\right] \\
& \psi(\mathbf{x})=\int \frac{d k}{2 \pi} \frac{1}{k_{0}}\left[f_{k} u(k) e^{i \mathbf{k} \cdot \mathbf{x}}+f_{k}^{\dagger} v(k) e^{-i \mathbf{k} \cdot \mathbf{x}}\right] \tag{5.12}
\end{align*}
$$

with $u^{*}=v$ and with the commuation relations

$$
\begin{equation*}
\left\{f_{k}, f_{k^{\prime}}^{\dagger}\right\}=2 \pi k_{0} \delta\left(k-k^{\prime}\right), \quad\left[b_{k}, b_{k^{\prime}}^{\dagger}\right]=2 \pi k_{0} \delta\left(k-k^{\prime}\right) \tag{5.13}
\end{equation*}
$$

On-shell SUSY is determined from the SUSY transformation of the elementary fields, Eq. (2.7) and $|b(\theta)\rangle=b^{\dagger}(\theta)|0\rangle$ and $|f(\theta)\rangle=f^{\dagger}(\theta)|0\rangle$ :

$$
\begin{align*}
Q_{1}|f(\theta)\rangle & =-i \frac{P_{+}}{\sqrt{2} v_{1}(\theta)}|b(\theta)\rangle & Q_{1}|b(\theta)\rangle & =i \sqrt{2} v_{1}(\theta)|f(\theta)\rangle  \tag{5.14}\\
Q_{2}|f(\theta)\rangle & =i \frac{P_{-}}{\sqrt{2} v_{2}(\theta)}|b(\theta)\rangle & Q_{2}|b(\theta)\rangle & =i \sqrt{2} v_{2}(\theta)|f(\theta)\rangle .
\end{align*}
$$

From $0=\langle 0| Q_{\alpha}[\phi(0)|f(\theta)\rangle]$, one gets

$$
\begin{equation*}
Q_{\alpha}|f(\theta)\rangle=-i \sqrt{2} u_{\alpha}(\theta)|b(\theta)\rangle, \tag{5.15}
\end{equation*}
$$

and comparing this with Eq.(5.14), one can find the spinors

$$
v_{1}(\theta)=-i \sqrt{\frac{m}{2}} e^{\theta / 2}, \quad v_{2}(\theta)=\sqrt{\frac{m}{2}} e^{-\theta / 2}
$$

This gives the SUSY transformation of on-shell states given above in Eq.(2.16).

Combining this and Eq.(5.12), we fix the one-point function as follows:

$$
\begin{equation*}
\langle 0| \phi(0)|b(\theta)\rangle=\frac{1}{\sqrt{2}}, \quad\langle 0| \psi_{\alpha}(0)|f(\theta)\rangle=\sqrt{\frac{m}{2}}\binom{i e^{\theta / 2}}{e^{-\theta / 2}} . \tag{5.16}
\end{equation*}
$$

### 5.4. Two-point form factors of the SShG model

We compute two-point FFs of the trace of energy-momentum tensors and their SUSY counterparts, $\Theta$ and $\Theta_{\mathrm{F}}\left(\bar{\Theta}_{\mathrm{F}}\right)$, given in Eqs.(2.13),(2.14). These operators are of particular interest for their role in the $C$-theorem. First, we derive SUSY relations between the FFs using Eq.(2.15).
From $0=\langle 0| Q_{\alpha}\left[\mathcal{O}\left|a_{1}\left(\theta_{1}\right) a_{2}\left(\theta_{2}\right)\right\rangle\right]$, one finds a relation

$$
\begin{equation*}
\langle 0| Q_{\alpha}[\mathcal{O}]\left|a_{1}\left(\theta_{1}\right) a_{2}\left(\theta_{2}\right)\right\rangle=-(-1)^{F(\mathcal{O})}\langle 0| \mathcal{O}\left|Q_{\alpha}\left[a_{1}\left(\theta_{1}\right) a_{2}\left(\theta_{2}\right)\right]\right\rangle, \tag{5.17}
\end{equation*}
$$

with $F$ is 1 for fermionic and 0 for bosonic. This gives the following relations between the FFs:

$$
\begin{array}{ll}
F_{b b}^{\Theta}=i \frac{\sqrt{m}}{2}\left[\sqrt{x_{1}} F_{f b}^{\bar{\Theta}_{\mathrm{F}}}+\sqrt{x_{2}} F_{b f}^{\bar{\Theta}_{\mathrm{F}}}\right], & F_{f f}^{\Theta}=i \frac{\sqrt{m}}{2}\left[\sqrt{x_{1}} F_{b f}^{\bar{\Theta}_{\mathrm{F}}}-\sqrt{x_{2}} F_{f b}^{\bar{\Theta}_{\mathrm{F}}}\right], \\
F_{b b}^{\Theta}=\frac{\sqrt{m}}{2}\left[\frac{1}{\sqrt{x_{1}}} F_{f b}^{\Theta_{\mathrm{F}}}+\frac{1}{\sqrt{x_{2}}} F_{b f}^{\Theta_{\mathrm{F}}}\right], & F_{f f}^{\Theta}=\frac{\sqrt{m}}{2}\left[\frac{-1}{\sqrt{x_{1}}} F_{b f}^{\Theta_{\mathrm{F}}}+\frac{1}{\sqrt{x_{2}}} F_{f b}^{\Theta_{\mathrm{F}}}\right], \tag{5.18}
\end{array}
$$

where each FF is a function of $\theta_{1}$ and $\theta_{2}$ or of $x_{i}=e^{\theta_{i}}$.
A special case of $\alpha \rightarrow 0$
It is useful to consider a case of $\alpha \rightarrow 0$ where the $S$-matrix is of diagonalized form of $(1,-1,1,1)$ from Eq.(2.19). Let's compute $F_{b b}^{\Theta}$ and $F_{f f}^{\Theta}$. In terms of the solution of the Watson equation,

$$
\begin{equation*}
F_{b b}^{\min }=1, \quad F_{f f}^{\min }=\sinh \frac{\theta}{2}, \tag{5.19}
\end{equation*}
$$

[^3]the FFs can be written as
\[

$$
\begin{equation*}
F_{b b}^{\Theta}\left(x_{1}, x_{2}\right)=K_{b b}\left(x_{1}, x_{2}\right), \quad \text { and } \quad F_{f f}^{\Theta}\left(x_{1}, x_{2}\right)=K_{f f}\left(x_{1}, x_{2}\right) \sinh \frac{\theta}{2} \tag{5.20}
\end{equation*}
$$

\]

Since $K$ should have the $i \pi$ pole (or at $\left.x_{1}=-x_{2}\right)$ and $K\left(\theta_{1}+\Lambda, \theta_{2}+\Lambda\right)=K\left(\theta_{1}, \theta_{2}\right)$ because the spin of $\Theta$ is zero, we can find

$$
\begin{equation*}
F_{b b}^{\Theta}\left(x_{1}, x_{2}\right)=2 \pi m^{2}, \quad \text { and } \quad F_{f f}^{\Theta}\left(x_{1}, x_{2}\right)=2 \pi m^{2} \sinh \frac{\theta}{2} \tag{5.21}
\end{equation*}
$$

Here we fixed the normalization factor as $\pi m^{2}$ by comparing with the perturbative computation using Eqs.

After finding these, one can derive the other FFs simply using Eq.(5.18) as follows:

$$
\begin{equation*}
F_{b f}^{\Theta_{\mathrm{F}}}\left(x_{1}, x_{2}\right)=2 \pi m^{3 / 2} \sqrt{x_{2}}, \quad F_{b f}^{\bar{\Theta}_{\mathrm{F}}}\left(x_{1}, x_{2}\right)=2 \pi m^{3 / 2} \frac{-i}{\sqrt{x_{2}}} \tag{5.22}
\end{equation*}
$$

One can check that the spins of $\Theta_{F}$ and $\bar{\Theta}_{F}$ can be found correctly as $\pm \frac{1}{2}$ under the rapidity translation. Also one can check that these FFs are consistent with pertrubative computation.

## For General $\alpha$

For general cases, we should diagonalize the $S$-matrix first. Whether we use Eq.(2.17) with the phase factor in Eq.(5.3) or Eq.(2.19) without such factor, we find the eigenvalues of $F=0$ sector ( $b b$ and $f f$ ) are complicated and hard to find the integral representations. Instead, we consider $F=-1$ ( $b f$ and $f b$ ) sector first. The $S$-matrix is easily diagonalized by the eigenvectors
with eigenvalues

$$
\begin{equation*}
S_{+}(\theta)=\exp \left[\int_{0}^{\infty} \frac{d t}{t} f_{+}(t) \sinh \frac{\theta t}{\pi i}\right], \quad S_{-}(\theta)=-\exp \left[\int_{0}^{\infty} \frac{d t}{t} f_{-}(t) \sinh \frac{\theta t}{\pi i}\right] \tag{5.24}
\end{equation*}
$$

with

$$
\begin{equation*}
f_{ \pm}(t)=\frac{(1-\cosh t)(1+\cosh ((1-2|\alpha|) t))}{\sinh ^{2} t} \pm \frac{\cosh ((1-2|\alpha|) t)}{\cosh t} \tag{5.25}
\end{equation*}
$$

From these integrals, $F^{\text {min }}$ in the basis of $|+\rangle$ and $|-\rangle$ can be obtained as

$$
\begin{equation*}
F_{+}^{\min }=\exp \left[\int_{0}^{\infty} \frac{d t}{t} \frac{f_{+}(t)}{\sinh t} \sin ^{2} \frac{\widehat{\theta} t}{2 \pi}\right], \quad F_{-}^{\min }=\cosh \frac{\widehat{\theta}}{2} \exp \left[\int_{0}^{\infty} \frac{d t}{t} \frac{f_{-}(t)}{\sinh t} \sin ^{2} \frac{\widehat{\theta} t}{2 \pi}\right] \tag{5.26}
\end{equation*}
$$

where we chose a normalization such that

$$
\begin{equation*}
F_{ \pm} \rightarrow 1 \quad \text { as } \quad \alpha \rightarrow 0 \tag{5.27}
\end{equation*}
$$

For numerical computations, we list expressions of $F_{ \pm}^{\min }$ which converge fast in the Appendix B.

Now we consider FFs of $\Theta_{F}$ in the following form:

$$
\begin{equation*}
F_{+}^{\Theta_{\mathrm{F}}}\left(x_{1}, x_{2}\right)=K_{+}\left(x_{1}, x_{2}\right) F_{+}^{\min }, \quad F_{-}^{\Theta_{\mathrm{F}}}\left(x_{1}, x_{2}\right)=K_{-}\left(x_{1}, x_{2}\right) F_{-}^{\min } \tag{5.28}
\end{equation*}
$$

and similarly for $\bar{\Theta}_{\mathrm{F}}$ in terms of $\bar{K}_{ \pm}$. These $K_{ \pm}$and $\bar{K}_{ \pm}$can be determined from the spins of the operators and symmetric properties of the states under the exchange, $|+\rangle \rightarrow|+\rangle$ and $|-\rangle \rightarrow-|-\rangle$ under $x_{1} \leftrightarrow x_{2}$, as follows:

$$
\begin{equation*}
K_{ \pm}=A\left(\sqrt{x_{1}} \pm \sqrt{x_{2}}\right), \quad \bar{K}_{ \pm}=B\left(\frac{1}{\sqrt{x_{1}}} \pm \frac{1}{\sqrt{x_{2}}}\right) \tag{5.29}
\end{equation*}
$$

where constants $A, B$ can be determined by taking $\alpha \rightarrow 0$ limit and comparing with Eq.(5.22).

Now rotating back to the on-shell states one can find

$$
\begin{align*}
& F_{b f}^{\Theta_{\mathrm{F}}}\left(x_{1}, x_{2}\right)=2 \pi m^{3 / 2}\left[\sqrt{x_{1}} \frac{\left(F_{+}^{\min }-F_{-}^{\min }\right)}{2}+\sqrt{x_{2}} \frac{\left(F_{+}^{\min }+F_{-}^{\min }\right)}{2}\right] \\
& F_{b f}^{\bar{\Theta}_{\mathrm{F}}}\left(x_{1}, x_{2}\right)=-2 \pi i m^{3 / 2}\left[\frac{1}{\sqrt{x_{1}}} \frac{\left(F_{+}^{\min }-F_{-}^{\min }\right)}{2}+\frac{1}{\sqrt{x_{2}}} \frac{\left(F_{+}^{\min }+F_{-}^{\min }\right)}{2}\right] . \tag{5.30}
\end{align*}
$$

Also from Eq.(5.18) one can obtain other FFs

$$
\begin{align*}
& F_{b b}^{\Theta}\left(x_{1}, x_{2}\right)=2 \pi m^{2}\left[\frac{\left(F_{+}^{\min }+F_{-}^{\mathrm{min}}\right)}{2}+\frac{\left(F_{+}^{\min }-F_{-}^{\min }\right)}{2} \cosh \frac{\theta}{2}\right]  \tag{5.31}\\
& F_{f f}^{\Theta}\left(x_{1}, x_{2}\right)=2 \pi m^{2} \frac{\left(F_{+}^{\min }+F_{-}^{\min }\right)}{2} \sinh \frac{\theta}{2}
\end{align*}
$$

We checked these FFs using the first order perturbative computations. FFs for other component of energy-momentum tensor can be written down by just multiplying $P_{+} / P_{-}$to the above FFs.

## 6. Spectral Representation of $C$-Theorem

We compute the UV central charge of the SShG model using the two-point FFs computed in the previous section and the spectral representation of the $C$-Theorem. This provides a consistency check for the FFs and shows the fast convergence of the FFs expansions of correlation functions.

### 6.1. Spectral Sum Rule

The $C$-theorem, first introduced by A.B. Zamolodchikov, plays an important role in the study of off-critical models [44]. The $C$-function, describing a degree of freedom of the 2D models, connects smoothly two renormalization group (RG) fixed points as the length scale of the theory increases from UV limit to IR. For some specific models like the perturbed minimal CFTs by the least relevent operator with positive coefficient, the renormalization group (RG) flow connects two RG fixed points corresponding to two adjacent minimal CFTs [45]. This RG flow will end up at the massive point with $C=0$.

This theorem can be neatly expressed in the following integral of the two-point correlation function of the trace of energy-momentum tensor following Cardy [46]:

$$
\begin{align*}
& \Delta C=\frac{3}{4 \pi} \int_{|x|>\varepsilon} d^{2} x x^{2}\langle\Theta(x) \Theta(0)\rangle=\int_{0}^{\infty} d \mu C_{1}(\mu, \Lambda),  \tag{6.1}\\
& C_{1}(\mu, \Lambda)=\frac{6}{\pi^{2}} \frac{1}{\mu^{3}} \operatorname{Im}\left[\int d^{2} x e^{-i p \cdot x}\langle\Theta(x) \Theta(0)\rangle\right]_{p^{2}=-\mu^{2}}
\end{align*}
$$

Expanding the correlation function in terms of intermediate on-shell states, the spectral density function $C_{1}$ can be expressed in terms of the FFs by

$$
\begin{equation*}
\left.C_{1}(\mu, \Lambda)=\frac{12}{\mu^{3}} \sum_{\alpha}|\langle 0| \Theta(0)| \alpha\right\rangle\left.\right|^{2} \delta^{2}\left(q-p_{\alpha}\right) \tag{6.2}
\end{equation*}
$$

where $p_{\alpha}$ is the energy-momentum vector of the multi-particle state $\alpha$ and the vector $q$ is defined as $q=(\mu, 0)$.

For the massive theory, the sum rule of $\Delta C$ effectively gives the UV central charge since $C_{\text {IR }}$ vanishes. Although one needs the infinite number of the FFs to compute it rigorously, there are many evidences that the sum in Eq.(6.2) converges very fast for the massive theories [7-10]. With this observation, one can compute the UV central charge using the two-point FFs of $\Theta$ quite accurately. In next stage, we will compute this numerically using the FFs of the SShG model derived in the previous section.

### 6.2. Sum Rule for the SShG model

The two-point contribution to the sum rule becomes

$$
\begin{align*}
C^{(2)}= & \frac{12}{\mu^{3}} \int \frac{d \theta_{1} d \theta_{2}}{2(2 \pi)^{2}} \sum_{a_{1}, a_{2}}\left|F_{a_{1} a_{2}}^{\Theta}\left(\theta_{1}, \theta_{2}\right)\right|^{2} \\
& \times \delta\left(m \cosh \theta_{1}+m \cosh \theta_{2}-\mu\right) \delta\left(m \sinh \theta_{1}+m \sinh \theta_{2}\right)  \tag{6.3}\\
= & \frac{3}{8 \pi^{2} m^{4}} \int_{0}^{\infty} \frac{d \theta}{\cosh ^{4} \theta}\left[\left|F_{b b}^{\Theta}(\theta,-\theta)\right|^{2}+\left|F_{f f}^{\Theta}(\theta,-\theta)\right|^{2}\right],
\end{align*}
$$

with the FFs given in Eq.(5.31).
For the special case of $\alpha=0$ where the SShG model becomes free with a boson and fermion, one can insert Eq.(5.21) into Eq.(6.3) and using

$$
\int_{0}^{\infty} \frac{d \theta}{\cosh ^{4} \theta}=\frac{2}{3}, \quad \int_{0}^{\infty} d \theta \frac{\sinh ^{2} \theta}{\cosh ^{4} \theta}=\frac{1}{3}
$$

one can easily find $C=\frac{3}{2}$.
For the generic value of $\alpha$ we integrate numerically using the regularized expressions for the $F_{ \pm}$in Appendix B. Using these we list $\Delta C^{(2)}$ as for several values of the coupling constant in Table 1. This shows a good agreement with the UV central charge $C=\frac{3}{2}$. The convergence of the SShG model seems slow compared with the sinh-Gordon result [9]. This suggests in the SShG model one arrives

| $\frac{\beta^{2}}{4 \pi}$ | $\alpha$ | $\Delta \widehat{C}^{(2)}$ |
| :---: | :---: | :---: |
| $\frac{1}{999}$ | 0.001 | 0.9993 |
| $\frac{1}{199}$ | 0.005 | 0.9953 |
| $\frac{1}{99}$ | 0.01 | 0.9902 |
| $\frac{1}{49}$ | 0.02 | 0.9800 |
| $\frac{3}{97}$ | 0.03 | 0.9697 |
| $\frac{1}{19}$ | 0.05 | 0.9495 |
| $\frac{1}{9}$ | 0.1 | 0.9093 |

Table 1. The first two-particle form factor in the Sum Rule of $\Delta \widehat{C}=\frac{2}{3} \Delta C$.
strong coupling region earlier than the sinh-Gordon model as one can see from the fact that the limit of the SShG coupling constant is $\frac{\widehat{\beta}^{2}}{8 \pi}=\frac{1}{2}$ while $\frac{\widehat{\beta}^{2}}{8 \pi}=1$ in the sinh-Gordon model.

## 7. Conclusion

In this paper we obtained two results on the $N=1$ SUSY integrable models. The first one is the computation of the UV central charges from TBA method. The nondiagonal TBA of the SShG and SYL models has been rigorously derived from the essential observation that the $N=1$ SUSY models can be identified with the eight vertex free fermion models. These TBA equations produced correct UV central charges.

The second result is two-point FFs of the SShG model using the FF axioms. Here the difficulty arising from the nondiagonal scattering theories has been avoided from the SUSY relations of the FFs. The spectral representation of the $C$-theorem showed that two-point FFs can give good approximations in the infinite sum of the intermediate states even in nondiagonal theories.

Our results suggest some interesting directions to proceed further. Actually, we notice that wider class of $N=1$ scattering theories are belonging to the eight vertex FFMs which will be reported in separate publication [47]. The relationship between these SUSY models and the eight vertex FFM may have some deep structure because the FFMs seem to have interesting hidden symmetries [48]. In particular, it has been noticed recently that the FFMs have a hidden quantum group symmetry [49]. It would be interesting to see how this quantum group symmetry will be related to the $N=1$ supersymmetry in the trigonometric limit.

In this paper, we could not say much on the general FFs of the theories. The solution of the FF bootstrap equations are very difficult and are limited to only a few simplest diagonal theories. We can reduce, however, the nondiagonal bootstrap equations to the level of diagonal theories by diagonalizing the inhomogeneous transfer matrix. It will need some more work to solve these reduced bootstrap equations completely.

Acknowledgements:
The author wish to thank T. Eguchi, A. LeClair, G. Mussardo, M. Peskin, F.

Smirnov, P. Weisz for useful discussions.

## APPENDIX A

Inversion relation for the Free Fermion Model
We follow Felderhof to diagonalize the transfer matrix of the FFM [30]. We want to point out, first, the difference of our derivation from the lattice model computation. The first difference is that we want to diagonalize the inhomogeous transfer matrix. This difference often introduces much difficulty for the computation. However, this difficulty can be avoided by the second difference, which is that we are working with the FFM at the critical point. With this advantage, we can derive the inverse matrix of the FFM transfer matrix and, furthermore, express it using the original transfer matrix with slight change in the rapidity $u$.

It is convenient to reexpress the Boltzman weights Eq.(4.1) in terms of the $\sigma$-matrices,

$$
\begin{gather*}
R(\theta)=\left(\begin{array}{cc}
A(\theta) & B(\theta) \\
C(\theta) & D(\theta)
\end{array}\right),  \tag{A.1}\\
A=a_{+} \sigma^{+} \sigma^{-}+b_{+} \sigma^{-} \sigma^{+}, \quad B=d \sigma^{+}+c \sigma^{-},  \tag{A.2}\\
C=c \sigma^{+}+d \sigma^{-}, \quad D=b_{-} \sigma^{+} \sigma^{-}+a_{-} \sigma^{-} \sigma^{+} .
\end{gather*}
$$

Then, the transfer matrix becomes

$$
\begin{equation*}
T\left(u \mid \theta_{1}, \ldots, \theta_{N}\right)=\operatorname{Tr}_{2}\left[\prod_{i=1}^{N} R\left(u-\theta_{i}\right)\right] . \tag{A.3}
\end{equation*}
$$

Now we define new transfer matrix $T_{1}$ corresponding to new Boltzman weghts
defined by

$$
\begin{equation*}
a_{ \pm}^{1}=-b_{ \pm}, \quad b_{ \pm}^{1}=a_{ \pm}, \quad c^{1}=c, \quad \text { and } \quad d^{1}=-d . \tag{A.4}
\end{equation*}
$$

In the same way as before, one can express $T_{1}$ by

$$
T_{1}\left(u \mid \theta_{1}, \ldots, \theta_{N}\right)=\operatorname{Tr}_{2}\left[\prod_{i=1}^{N} R_{1}\left(u-\theta_{i}\right)\right], \quad R_{1}(\theta)=\left(\begin{array}{cc}
A_{1}(\theta) & B_{1}(\theta)  \tag{A.5}\\
C_{1}(\theta) & D_{1}(\theta)
\end{array}\right)
$$

where

$$
\begin{align*}
& A_{1}=-b_{+} \sigma^{+} \sigma^{-}+a_{+} \sigma^{-} \sigma^{+}, \quad B_{1}=-d \sigma^{+}+c \sigma^{-} \\
& C_{1}=c \sigma^{+}-d \sigma^{-}, \quad D_{1}=a_{-} \sigma^{+} \sigma^{-}-b_{-} \sigma^{-} \sigma^{+} \tag{A.6}
\end{align*}
$$

One can check that these new Boltzman weights again satisfy the free fermion condition Eq.(4.3).

Next step is to show that $T T_{1} \propto 1$. For this purpose, we multiply two matrices

$$
\begin{equation*}
T(u) T_{1}(u)=\operatorname{Tr}_{2}\left[\prod_{i=1}^{N} R_{i}\right] \operatorname{Tr}_{2}\left[\prod_{i=1}^{N} R_{1, i}\right]=\operatorname{Tr}_{2 \otimes 2}\left[\prod_{i=1}^{N} R_{i} \otimes R_{1, i}\right] . \tag{A.7}
\end{equation*}
$$

Defining the $4 \times 4$ matrix $R_{i} \otimes R_{1, i}$ as $S_{i}$, one can find a similarity transformation $S_{i}^{\prime}=X_{i} S_{i} X_{i}^{-1}$ where $S_{i}^{\prime}$ is of triangular form. The $X$ and $S^{\prime}$ are given by
$X=\left(\begin{array}{cccc}0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \cosh \phi & 0 & 0 & -\sinh \phi \\ -\sinh \phi & 0 & 0 & \cosh \phi \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{array}\right), \quad S^{\prime}=\left(\begin{array}{cccc}M_{+} & * & * & * \\ 0 & F_{-} \sigma^{z} & 0 & * \\ 0 & 0 & F_{+} \sigma^{z} & * \\ 0 & 0 & 0 & M_{-}\end{array}\right)$,
where $M_{ \pm}, F_{ \pm}$, and $\phi$ are given in Eqs.(4.9) and (4.10). We did not specify the unnecessary non-vanishing components $(*)$.

Most important observation is that $\tanh \phi$ becomes just a constant for the $N=1$ supersymmetric theory. This means one can make all the $S_{i}$ 's in the trace
of triangular form by the same similarity transformation $X$. Therefore, $T T_{1}=$ $\operatorname{Tr}_{4} \prod S_{i}^{\prime}$ and from Eq.(A.8) one can derive

$$
\begin{align*}
T(u) T_{1}(u)= & {\left[\prod_{i=1}^{N} M_{+}\left(u-\theta_{i}\right)+\prod_{i=1}^{N} M_{-}\left(u-\theta_{i}\right)\right.} \\
& \left.+F\left(\prod_{i=1}^{N} F_{+}\left(u-\theta_{i}\right)+\prod_{i=1}^{N} F_{-}\left(u-\theta_{i}\right)\right)\right] \tag{A.9}
\end{align*}
$$

with $F=\prod \sigma_{i}^{z}$ is either 1 (bosonic) or -1 (fermionic).
Now, consider a translation $u \rightarrow u+i \pi$. Under this the Boltzman weights of the SShG model change

$$
\begin{equation*}
a_{ \pm} \rightarrow-a_{\mp}, \quad b_{ \pm} \rightarrow b_{\mp}, \quad c \rightarrow d, \quad \text { and } \quad d \rightarrow-c . \tag{A.10}
\end{equation*}
$$

Again this satisfies the free fermion condition. Now, the transfer matrix with translated rapidity can be expressed in terms of $R_{2}(u-\theta)=R(u+i \pi-\theta)$ by,

$$
T(u+i \pi)=\operatorname{Tr}_{2}\left[\prod_{i=1}^{N} R_{2}\left(u-\theta_{i}\right)\right], \quad R_{2}=\left(\begin{array}{ll}
A_{2} & B_{2}  \tag{A.11}\\
C_{2} & D_{2}
\end{array}\right)
$$

where

$$
\begin{align*}
& A_{2}=-a_{+} \sigma^{+} \sigma^{-}+b_{+} \sigma^{-} \sigma^{+}, \quad B_{2}=-c \sigma^{+}+d \sigma^{-}= \\
& C_{2}=d \sigma^{+}-c \sigma^{-}, \quad D_{2}=b_{-} \sigma^{+} \sigma^{-}-a_{-} \sigma^{-} \sigma^{+} \tag{A.12}
\end{align*}
$$

From Eq.(A.6), one can notice that

$$
\begin{equation*}
A_{2}=-D_{1}, \quad B_{2}=-C_{1}, \quad, C_{2}=-B_{1}, \quad \text { and } \quad D_{2}=-A_{1} \tag{A.13}
\end{equation*}
$$

Considering the $R$-matrices as $2 \times 2$ matrices, the $R_{1}$ and $R_{2}$ are related by

$$
R_{2}=-\sigma^{x} R_{1} \sigma^{x}
$$

where $\sigma^{x}$ is the usual Pauli spin matrix. This gives

$$
\begin{equation*}
T\left(u+i \pi \mid \theta_{1}, \ldots, \theta_{N}\right)=(-1)^{N} T_{1}\left(u \mid \theta_{1}, \ldots, \theta_{N}\right) \tag{A.14}
\end{equation*}
$$

and from Eqs.(A.9) and (A.14), the inversion relation Eq.(4.8).

## APPENDIX B

Regularized Expression for the Form Factors
For the numerical computation we can rewrite $F_{ \pm}^{\text {min }}$ in Eq.(5.20) as follows:

$$
\begin{equation*}
F_{ \pm}^{\min }(\theta)=C_{ \pm}(\theta)\left[\prod_{k=1}^{n} G_{k}(\alpha, \theta)\left[H_{k}(\alpha, \theta)\right]^{ \pm 1}\right] \exp \left[\int_{0}^{\infty} \frac{d t}{t} \frac{f_{n}^{ \pm}(\alpha, t)}{\sinh t} \sin ^{2} \frac{\widehat{\theta t}}{2 \pi}\right] \tag{B.1}
\end{equation*}
$$

with $C_{+}=1, C_{-}(\theta)=\cosh \frac{\widehat{\theta}}{2}$ and

$$
\begin{align*}
& G_{k}(\alpha, \theta)=\frac{P_{k}(2|\alpha|+1, \theta)^{2} P_{k}(0, \theta)^{2}}{P_{k}(1, \theta)^{2} P_{k}(2|\alpha|, \theta) P_{k}(2|\alpha|+2, \theta)} \\
& P_{k}(x, \theta)=\left[\left(1+\frac{\widehat{\theta} /(2 \pi)}{(k+(1+x) / 2)}\right)\left(1+\frac{\widehat{\theta} /(2 \pi)}{(k+(1-x) / 2)}\right)\right]^{\frac{k(k+1)}{4}}  \tag{B.2}\\
& H_{k}(\alpha, \theta)=\left[1+\frac{\widehat{\theta} /(2 \pi)}{(2 k+(2|\alpha|+3) / 2)}\right]^{\frac{1}{2}}\left[1+\frac{\widehat{\theta} /(2 \pi)}{(2 k-(2|\alpha|-1) / 2)}\right]^{\frac{1}{2}}
\end{align*}
$$

and the exponents are given by

$$
f_{n}^{ \pm}(\alpha, t)=\frac{(1-\cosh t)(1+\cosh ((1-2|\alpha|) t)) D_{n}(t)}{2 \sinh ^{2} t} \pm \frac{\cosh ((1-2|\alpha|) t) e^{-4 n t}}{\cosh t}
$$

with $D_{n}(t)=\left[(n+1)(n+2)-2 n(n+2)+n(n+1) e^{-4 t}\right] e^{-2 n t}$.
If one choose $n=0$, this reduces to Eq.(5.26). For the fast convergence, one can increase $n$ although the final expression is independent of $n$.

## REFERENCES

1. A. B. Zamolodchikov and Al. B. Zamolodchikov, Annals. Phys. 120 (1979) 253
2. A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B241 (1984) 333
3. Al. B. Zamolodchikov, Nucl. Phys. B342 (1990) 695
4. A. B. Zamolodchikov, Int. Journ. of Mod. Phys. A4 (1989) 4235
5. M. Karowski and P.H. Weisz, Nucl. Phys. B139 (1978) 455; M. Karowski, Phys. Rep. 49 (1979) 229
6. F. A. Smirnov, Form Factors in Completely Integrable Models of Quantum Field Theory (World Scientific, Singapore, 1992) and references therein.
7. V.P. Yurov and Al.B. Zamolodchikov, Int. J. Mod. Phys. A4 (1989) 3419
8. Al. B. Zamolodchikov, Nucl. Phys. B348 (1991) 619
9. A. Fring, G. Mussardo and P. Simonetti, Nucl. Phys. B393 (1993) 413; A. Koubek and G. Mussardo, SISSA preprint, SISSA preprint ISAS-93-42
10. J.L. Cardy and G. Mussardo, UCSB and SISSA preprint UCSBTH-93-12, ISAS-93-75
11. O. Babelon and D. Bernard, Phys. Lett. B288 (1992) 113
12. C. Destri and H.J. De Vega, Nucl. Phys. B338 (1991) 251
13. D.Z. Freedman, J. Latorre and X. Vilasis, Mod. Phys. Lett. A6 (1991) 531
14. R. J. Baxter, Exactly solved models in statistical mechanics (Academic Press, New York, 1982)
15. G. E. Andrew, R. J. Baxter, and P. J. Forrester, J. Stat. Phys. 35 (1984) 193
16. A. LeClair, Phys. Lett. 230B (1989) 103; D. Bernard and A. LeClair, Nucl. Phys. B340 (1990) 721
17. F. A. Smirnov, Int. J. Mod. Phys. A4 (1989) 4213; N. Yu Reshetikhin and F. Smirnov, Comm. Math. Phys. 131 (1990) 157
18. D. Bernard and A. LeClair, Comm. Math. Phys. 142 (1991) 99
19. C. Ahn, D. Bernard and A. LeClair, Nucl. Phys. B336 (1990) 409
20. D. Bernard and A. LeClair, Phys. Lett. B247 (1990) 309
21. H.J. de Vega and M. Karowski, Nucl. Phys. B285 (1987) 619
22. P. Fendley and K. Intriligator, Nucl. Phys. B372 (1992) 533; B380 (1992) 265
23. P. Fendley and H. Saleur, Nucl. Phys. B388 (1992) 609
24. Al.B. Zamolodchikov, Nucl. Phys. B358 (1991) 497
25. A.B. Zamolodchikov, in Proc. Fields, strings and quantum gravity, Beijing, China, 1989
26. K. Schoutens, Nucl. Phys. B344 (1990) 665
27. C. Ahn, Nucl. Phys. B354 (1991) 57
28. R. Shankar and E. Witten, Phys. Rev. D17 (1978) 2134
29. C. Fan and F.Y. Wu, Phys. Rev. B2 (1970) 723
30. B.U. Felderhof, Physica 65 (1973) 421; 66 (1973) 279, 509
31. B. Sutherland, J. Math. Phys. 11 (1970) 3183
32. M. Chaichian and P. Kulish, Phys. Lett. B183 (1987) 169
33. O. Babelon and F. Langouche, Nucl. Phys. B290 [FS20] (1987) 603
34. M.A. Olshanesky, Comm. Math. Phys. 88 (1983) 63
35. S. Ferrara, L. Girardello, and S. Sciuto, Phys. Lett. B76 (1978) 303
36. S. Sengupta and P. Majumdar, Phys. Rev. D33 (1986) 3138
37. P. Di Vecchia and S. Ferrara, Nucl. Phys. B130 (1977) 93
38. T. Klassen and E. Melzer, Nucl. Phys. B338 (1990) 485; B350 (1991) 635
39. P. Christe and M.J. Martins, Mod. Phys. Lett. A5 (1990) 2189; M.J. Martins, Phys. Lett. 240B (1990) 404
40. Al. B. Zamolodchikov, Phys. Lett. B253 (1991) 391
41. H. Itoyama and P. Moxhay, Phys. Rev. Lett. 65 (1990) 2102
42. C. Ahn and S. Nam, Phys. Lett. B271 (1991) 329
43. S. Chung, E. Lyman, and H. Tye, Int. J. Mod. Phys. A7 (1992) 3339
44. A.B. Zamolodchikov, JETP Lett. 43 (1986) 702; Sov. J. Nucl. Phys. 46 (1987) 1090
45. A.W.W. Ludwig and J.L. Cardy, Nucl. Phys. B285 (1987) 687
46. J.L. Cardy, Phys. Rev. Lett. 60 (9188) 2709
47. C. Ahn, in preparation
48. V.V. Bazhanov and Yu.G. Stroganov, Theor. Math. Phys. 62 (1985) 253; 63 (1985) 519, 604
49. R. Cuerno, C. Gomez, E. Lopez-Manzanares, and G. Sierra, Madrid preprint IMAFF-2/93 (1993).

[^0]:    $\star$ Dirac matrices are $\gamma^{0}=\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right) \quad \gamma^{1}=\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$.

[^1]:    * The Boltzman weights in Eq.(4.2) become the $S$-matrix element if we adopt a convention that time flows from bottom-left to top-right $(\nearrow)$.

[^2]:    * We do not need this phase factor if we use Eq.(2.19) as the $S$-matrix instead of Eq.(2.17) since all particles are bosonic.

[^3]:    $\star$ This solution is not unique in the sense that one can multiply any even function of $\theta$ satisfying $f(i \pi+\theta)=f(i \pi-\theta)$. If we include these functions in the prefactor $K$, one can define $F^{\text {min }}$ uniquely.

