Wave functions for quantum black hole formation in scalar field collapse

Dongsu Bak,[1,*] Sang Pyo Kim,[2,†] Sung Ku Kim,[3,‡] Kwang-Sup Soh,[4,§] and Jae Hyung Yee[5,‖]
[1]*Department of Physics, University of Seoul, Seoul 130-743, Korea*
[2]*Department of Physics, Kunsan National University, Kunsan 573-701, Korea*
[3]*Department of Physics, Ewha Women's University, Seoul 120-750, Korea*
[4]*Department of Physics Education, Seoul National University, Seoul 151-742, Korea*
[5]*Department of Physics, Yonsei University, Seoul 120-749, Korea*
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We study quantum mechanically self-similar black hole formation by a collapsing scalar field and find the wave functions that give the correct semiclassical limit. In contrast with classical theory, the wave functions for black hole formation even in the supercritical case have not only incoming flux but also outgoing flux. From this result we compute the rate for black hole formation. In the subcritical case our result agrees with the semiclassical tunneling rate. Furthermore, we show how to recover the classical evolution of black hole formation from the wave function by defining the Hamilton-Jacobi characteristic function as $W=\hbar\ \mathrm{Im}\ln\psi$. We find that the quantum-corrected apparent horizon deviates from the classical value only slightly without any qualitative change even in the critical case.

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I. INTRODUCTION

Matter can strongly interact gravitationally to form black holes. The Schwarzschild, Reissner-Nordtröm, Kerr, and Kerr-Newman black holes are supposed to describe the end state of such gravitationally collapsing objects and are characterized by mass, charge, and angular momentum. Since these black holes are stationary, it would be physically interesting to understand the dynamical process for black hole formation that leads to the stationary black hole as the end state. Such an attempt toward understanding black hole formation process was done by Christodoulou who analytically studied a massless scalar field to prove that it either forms a black hole for the strong self-gravitational interaction or disperses for the weak self-gravitational interaction [1]. The gravitational collapse of the massless scalar field was confirmed in an early numerical simulation [2]. The most important results of gravitational collapse was discovered by Choptuik who found through numerical investigation that an initial Gaussian packet for the massless scalar field collapses self-similarly from spatial infinity either to form a black hole or to disperse back into spatial infinity depending on whether the parameter characterizing the wave packet is above or below a critical value, and that the black hole mass exhibits a power-law scaling behavior whose critical exponent is independent of the initial data [3]. Since then the critical phenomena of gravitational collapse in the spherically symmetric geometry have been found numerically in the perfect fluid [4–6], complex scalar [7–9], SU(2) Einstein-Yang-Mills [10], nonlinear σ [11], axion-dilaton [12,13], and gravitational wave models [14].

The simplest model for gravitational collapse is a massless scalar field evolving self-similarly in the spherical-symmetric geometry, and has been studied analytically [1] and numerically [2,3]. The analytic solutions [15] for that model together with self-similarity has been used to study the critical behavior [16–18]. Most of these works treat the classical aspects of gravitational collapse. Since the massless scalar field model predicts the type-II critical behavior, the mass of the black hole near the critical parameter can be arbitrarily small. So one may ask whether quantum effects play a significant role in the system. Hence, it would be physically interesting and important to study the quantum gravitational collapse. As a quantum-mechanical treatment of gravitational collapse, the wave function of a supercritical parameter for a quantum black hole was expressed in terms of those of the subcritical parameters and thus explains the black hole decay quantum mechanically [19]. Recently, we have also studied such quantum effects that, although classically the collapsing scalar field below the critical parameter does not form a black hole, quantum mechanically a black hole can be formed through tunneling process [20]. However, the interpretation of the wave functions in the supercritical and subcritical cases has not completely been settled yet [19,20], and a question has been raised quite recently how quantum effects can change the physics such as the black hole mass or critical exponent near the critical value [21].

It is the purpose of this paper to study quantum mechanically the black hole formation in the massless scalar field model and to investigate how quantum effects modify the classical picture of gravitational collapse. We use the Arnowitt-Deser-Misner (ADM) formulation to quantize the model and find analytically the black hole wave function in terms of the confluent hypergeometric function. The quantum model resembles in many respects a quantum Friedmann-Robertson-Walker (FRW) cosmological model minimally coupled to a massless scalar field. The standard interpretation of quantum mechanics and quantum cosmol-

*Electronic address: dsbak@mach.uos.ac.kr
†Electronic address: sangkim@knusun1.kunsan.ac.kr
‡Electronic address: skkim@theory.ewha.ac.kr
§Electronic address: kssoh@phya.snu.ac.kr
‖Electronic address: jhyee@phya.yonsei.ac.kr

ogy enables us to calculate explicitly the rate for black hole formation. We put forth a quantum criterion on black hole formation, whose semiclassical approximation agrees with the classical one. Furthermore, by using the characteristic function defined by $W=\hbar\,\mathrm{Im}\ln\psi$ we are able to compute the quantum effects to the classical apparent horizon and to the critical exponent.

The organization of this paper is as follows. In Sec. II, using the ADM formulation we derive the Wheeler-DeWitt equation for black hole formation by a self-similarly collapsing scalar field. The Wheeler-DeWitt equation has an SU(2) group structure and the wave functions are found in terms of the confluent hypergeometric function. In Sec. III, we find the wave function for black hole formation, which consists of both the incoming and outgoing components. We calculate the incident and transmitted fluxes, in terms of which the rate for black hole formation is found. It is found that the rate for black hole formation gives the correct semiclassical limits in both the supercritical and subcritical cases. In Sec. IV we use the new method for recovering the classical solution from our wave function introduced by us recently in Ref. [22]. The key idea is to modify the Wheeler-DeWitt equation to have a nonvanishing energy E rather than the usual vanishing value. Then we define the Hamilton-Jacobi characteristic function as $W=\hbar\,\mathrm{Im}\ln\psi$, from which we get the Hamilton-Jacobi evolution of classical solution. Finally, we let the energy E vanish. Following this procedure we regain the semiclassical limit of our quantum solutions and find the position of the apparent horizon. We have used the steepest descent method to determine the position of the apparent horizon and found good agreement with numerical evaluation. Even in the critical case ($c_0=1$) there appears no qualitative change from the classical results. In the last section, we discuss the salient points of our work.

II. CANONICAL QUANTIZATION

The model to be studied in this paper is the spherical-symmetric geometry minimally coupled to a massless scalar field. Then the Hilbert-Einstein action

$$S=\frac{1}{16\pi}\int_M d^4x\sqrt{-g}\left[\frac{1}{G}{}^{(4)}R-2(\nabla\phi)^2\right]+\frac{1}{8\pi G}\int_{\partial M}d^3x\,K\sqrt{h}, \tag{1}$$

where G is the Newton constant, reduces to the $(1+1)$ action

$$S_{\mathrm{sph}}=\frac{1}{4}\int d^2x\sqrt{-\gamma}\times r^2\left[\frac{1}{G}\left\{{}^{(2)}R(\gamma)+\frac{2}{r^2}[(\nabla r)^2+1]\right\}-2(\nabla\phi)^2\right], \tag{2}$$

where γ_{ab} is the metric in the remaining two-dimensional manifold. The $(1+1)$ metric for the scalar field collapse has the form

$${}^{(2)}ds^2=-2e^{2\sigma}du\,dv, \tag{3}$$

where u and v are null coordinates, and σ is a function of u and v. To find the continuously self-similar solutions, $r=-u\sqrt{z}y=\sqrt{-uv}y$, $\phi(z)$ and $\sigma(z)$, which depend only on $z=-v/u=e^{-2\tau}$, we introduce a diagonal gauge

$$u=-\omega e^{-\tau},\quad v=\omega e^{\tau}, \tag{4}$$

in terms of which the $(1+1)$ metric takes the form

$${}^{(2)}ds^2=e^{2\sigma}(-2\omega^2d\tau^2+2\,d\omega^2). \tag{5}$$

The classical solutions with vanishing σ are found in Refs. [16,17], and the corresponding spacetime has the geometry $\mathcal{M}=R^{1,1}\times S^2$.

In order to canonically quantize the system we adopt the ADM formulation and introduce a lapse function $N(\tau)$ to write the $(1+1)$ metric as

$$ds^2_{(2)}=-2N^2(\tau)\omega^2d\tau^2+2\,d\omega^2. \tag{6}$$

After taking into account the boundary terms, we obtain the action[1]

$$S_{\mathrm{sph}}=\int d\left(\frac{m_P^2}{\hbar}\frac{\omega^2}{2}\right)d\tau\left[-\frac{1}{2N}\dot{y}^2+\frac{\hbar y^2}{2m_P^2N}\dot{\phi}^2-N\left\{\frac{1}{2}y^2-1-2\left(\omega\frac{\partial}{\partial\omega}y\right)^2+\frac{2\hbar}{m_P^2}\left(\omega\frac{\partial}{\partial\omega}\phi\right)^2\right\}\right], \tag{7}$$

where overdots denote derivative with respect to τ. Note that $K\equiv(m_P^2/\hbar)(\omega^2/2)=\hbar(\omega^2/2l_P^2)$ has the dimension of action $\hbar$ and is proportional to the area factor measured in the Planck length l_P, and plays the role of a cutoff parameter of the model. Hence $K/\hbar$ is a dimensionless parameter. The canonical momenta for y and ϕ are given by

$$\pi_y=-\frac{K}{N}\dot{y},\quad \pi_\phi=\frac{\hbar Ky^2}{m_P^2N}\dot{\phi}. \tag{8}$$

Then Eq. (7) can be rewritten in the ADM formulation as

$$S_{\mathrm{sph}}=\int dx[\pi_y\dot{y}+\pi_\phi\dot{\phi}-N\mathcal{H}], \tag{9}$$

where

[1]In this paper we use a specific unit system $c=1$ keeping explicitly the Planck constant $\hbar$ and the gravitational constant G, in which $m_P=\sqrt{\hbar/G}$, $l_P=\sqrt{\hbar G}$, and $m_P=\hbar/l_P$. The variables ω and r have the dimension of l_P and the timelike variables τ and y become dimensionless.

$$\mathcal{H}=-\frac{1}{2K}\pi_y^2+\frac{m_P^2}{2\hbar K y^2}\pi_\phi^2-K\left\{1-\frac{1}{2}y^2+2\left(\omega\frac{\partial}{\partial\omega}y\right)^2-\frac{2\hbar}{m_P^2}\left(\omega\frac{\partial}{\partial\omega}\phi\right)^2\right\}, \tag{10}$$

is the Hamiltonian. The N acts as a Lagrange multiplier, so one gets the Hamiltonian constraint

$$\mathcal{H}=0. \tag{11}$$

According to the Dirac quantization method, the constraint (11) becomes a quantum constraint on the wave function

$$\hat{\mathcal{H}}(y,\pi_y;\phi,\pi_\phi)|\Psi\rangle=0. \tag{12}$$

We further impose the constraints from the self-similarity

$$\frac{\partial\hat{\phi}}{\partial\omega}|\Psi\rangle=0=\frac{\partial\hat{y}}{\partial\omega}|\Psi\rangle. \tag{13}$$

Finally, we obtain the Wheeler-DeWitt equation for the quantum black hole formation

$$\left[\frac{\hbar^2}{2K}\frac{\partial^2}{\partial y^2}-\frac{\hbar m_P^2}{2Ky^2}\frac{\partial^2}{\partial\phi^2}-K\left(1-\frac{y^2}{2}\right)\right]\Psi(y,\phi)=0, \tag{14}$$

which is the same equation that was used to calculate the tunneling rate for black hole formation in the subcritical case [20].

The wave function can be factorized into the scalar and gravitational field parts

$$\Psi(y,\phi)=\exp\left(\pm i\frac{Kc_0}{\hbar^{1/2}m_P}\phi\right)\psi(y). \tag{15}$$

Here the wave function for the scalar field was chosen to yield the classical momentum $\pi_\phi=\hbar K y^2\dot{\phi}/m_P^2 N=\pm Kc_0$, where c_0 is a dimensionless parameter. Then Eq. (14) reduces to the gravitational field equation

$$\left[\frac{\hbar^2}{2K}\frac{\partial^2}{\partial y^2}-K\left(1-\frac{y^2}{2}-\frac{c_0^2}{2y^2}\right)\right]\psi(y)=0. \tag{16}$$

It is worthy to note that Eq. (16), as one-dimensional quantum system, describes an inverted Calogero model with energy K and has the group structure SU(2) with the basis [23]

$$\hat{L}_-=\frac{\hat{\pi}_y^2}{2}-\frac{K^2c_0^2}{2\hat{y}^2},\quad \hat{L}_0=\frac{\hat{\pi}_y\hat{y}+\hat{y}\hat{\pi}_y}{2},\quad \hat{L}_+=\frac{\hat{y}^2}{2}. \tag{17}$$

Due to the group structure we are able to find the solutions to Eq. (16) in terms of the confluent hypergeometric function [24]:

$$\psi_I(y)=D_I\left[\exp\left(-\frac{i}{2}\frac{K}{\hbar}y^2\right)\right]\left(\frac{K}{\hbar}y^2\right)^{\mu_-}M\left(a_-,b_-,i\frac{K}{\hbar}y^2\right), \tag{18}$$

where

$$\mu_-=\frac{1}{4}-\frac{i}{2\hbar}Q,$$
$$a_-=\frac{1}{2}-\frac{i}{2\hbar}(Q+K),$$
$$b_-=1-\frac{i}{\hbar}Q, \tag{19}$$

with

$$Q=\left(K^2c_0^2-\frac{\hbar^2}{4}\right)^{1/2}. \tag{20}$$

The other independent solution is given by

$$\psi_{II}(y)=D_{II}\left[\exp\left(-\frac{i}{2}\frac{K}{\hbar}y^2\right)\right]\left(\frac{K}{\hbar}y^2\right)^{\mu_-}U\left(a_-,b_-,i\frac{K}{\hbar}y^2\right). \tag{21}$$

However, the wave function (21) is a linear combination of ψ_I and another solution

$$\psi_{III}(y)=D_{III}\left[\exp\left(-\frac{i}{2}\frac{K}{\hbar}y^2\right)\right]\times\left(\frac{K}{\hbar}y^2\right)^{\mu_+}M\left(a_+,b_+,i\frac{K}{\hbar}y^2\right), \tag{22}$$

where

$$\mu_+=\frac{1}{4}+\frac{i}{2\hbar}Q,$$
$$a_+=\frac{1}{2}+\frac{i}{2\hbar}(Q-K),$$
$$b_+=1+\frac{i}{\hbar}Q. \tag{23}$$

In the above equations D's are constants, and μ's, a's, and b's are dimensionless parameters. Hence Eqs. (18), (21), and (22) are dimensionless functions.

III. WAVE FUNCTION FOR BLACK HOLE FORMATION

In the classical context, the massless scalar field imploding self-similarly from spatial infinity with momentum above the critical value collapses to form a black hole without leaving any remnant and with momentum below the critical value it reflects back to spatial infinity [17]. However, in the quantum context, we may follow the analogy to the scattering problem of quantum mechanics. We prescribe the *boundary condition* for the wave functions for black hole formation such that they should be incident from spatial infinity and some part of them be reflected by the potential barrier back to spatial infinity but the remaining part be transmitted to-

ward the black hole singularity inside their apparent horizons.

We now wish to calculate the quantum-mechanical rate for black hole formation. It is not difficult to see that Eq. (18) has the asymptotic form [24] at spatial infinity

$$\psi_{\rm BH}=\psi_I(y)\cong\tilde{D}_I\left[\frac{\Gamma(b_-)}{\Gamma(a_+^*)}e^{i\pi a_-}\left(i\frac{K}{\hbar}y^2\right)^{\mu_- - a_-}e^{-(i/2)(K/\hbar)y^2}+\frac{\Gamma(b_-)}{\Gamma(a_-)}\left(i\frac{K}{\hbar}y^2\right)^{\mu_- - a_+^*}e^{(i/2)(K/\hbar)y^2}\right], \tag{24}$$

where $\tilde{D}_I=D_I(-i)^{\mu_-}$. To show that Eq. (18) satisfies indeed the boundary condition for the black hole formation, we note that the first term describes the incoming component and the second term describes the outgoing (reflected) component at spatial infinity, $(y\gg 1)$. Near $y=0$, Eq. (18) has the form [24]

$$\psi_{\rm BH}\cong D_I\left(\frac{K}{\hbar}y^2\right)^{1/4}\exp\left[-\frac{i}{2}\left(\frac{K}{\hbar}y^2+\frac{Q}{\hbar}\ln\left(\frac{K}{\hbar}y^2\right)\right)\right], \tag{25}$$

so it has only the incoming flux toward the black hole singularity as will be shown later.

We calculate the incoming (incident) flux at spatial infinity

$$j_{\rm in}={\rm Im}\left[\psi^*(y)i\frac{\hat{\pi}_y}{K}\psi(y)\right]=-|\tilde{D}_I|^2\frac{Q\cosh\frac{\pi}{2\hbar}(Q-K)}{(\hbar K)^{1/2}\sinh\frac{\pi}{\hbar}Q}e^{\pi/2\hbar(2Q+K)}, \tag{26}$$

where we used the relations $|\Gamma(1+ix)|^2=\pi x/\sinh\pi x$ and $|\Gamma(\frac{1}{2}+ix)|^2=\pi/\cosh\pi x$ [24]. On the other hand, the outgoing (reflected) flux at spatial infinity is similarly found:

$$j_{\rm ref}=|\tilde{D}_I|^2\frac{Q\cosh\frac{\pi}{2\hbar}(Q+K)}{(\hbar K)^{1/2}\sinh\frac{\pi}{\hbar}Q}e^{(\pi/2\hbar)K}. \tag{27}$$

The rate for black hole formation is the ratio of the transmitted flux to the incident flux. From the flux conservation applied to Eqs. (26) and (27) we obtain the (transmission) rate for black hole formation

$$\frac{j_{\rm tran}}{j_{\rm in}}=1-\frac{j_{\rm ref}}{j_{\rm in}}=1-\frac{\cosh\frac{\pi}{2\hbar}(Q+K)}{\cosh\frac{\pi}{2\hbar}(Q-K)}e^{-(\pi/\hbar)}Q. \tag{28}$$

Equation (28) determines the probability for black hole formation for all values of c_0.

In order to understand the classical limit of this solution, let us consider the large-K limit. K always enters Eqs. (18), (24), and (28) in the combination of $K/\hbar$, so the large-K limit is equivalent to the small-$\hbar$ limit, which is the semiclassical limit. From the mass of a black hole formed in the supercritical case [17],

$$M_{AH}=\frac{m_P^2}{\hbar}\left(\frac{1}{2}r_{AH}\right)=\frac{m_P^2}{\hbar}\left(\frac{\omega_c}{2}y_{AH}\right), \tag{29}$$

we can express $K/\hbar$ as

$$\frac{K}{\hbar}=\frac{m_P^2}{\hbar^2}\left(\frac{\omega_c^2}{2}\right)=\frac{4}{c_0^2}\left(\frac{M_{AH}}{m_P}\right)^2, \tag{30}$$

where ω_c denotes the cutoff of the ω variable. In the supercritical case the influx from spatial infinity falls into the black hole singularity and the black hole mass increases infinitely for an infinite duration. Hence the large-K limit is the large black hole mass limit. In this limit the rate for black hole formation should be calculated separately for $c_0>1$ and $c_0<1$. For $c_0>1$ Eq. (28) becomes, asymptotically,

$$\frac{j_{\rm tran}}{j_{\rm in}}=1-e^{-(\pi/\hbar)K(c_0-1)}. \tag{31}$$

Therefore, the rate for black hole formation becomes unity for $c_0>1$, which implies that a black hole is formed from the collapsing scalar field, as in the classical case [17].

In the case of $c_0<1$, we carefully rewrite Eq. (28) as

$$\frac{j_{\rm tran}}{j_{\rm in}}=\frac{e^{-\pi/2\hbar(K+Q)}\sinh\frac{\pi}{\hbar}Q}{\cosh\frac{\pi}{2\hbar}(K-Q)}. \tag{32}$$

The asymptotic value of Eq. (32),

$$\frac{j_{\rm tran}}{j_{\rm in}}=e^{-(\pi/\hbar)K(1-c_0)}, \tag{33}$$

is the tunneling rate for quantum black hole formation in the subcritical case [20]. The classical result is obtained in the very large limit of K, in which the scalar field with $c_0>1$ evolves supercritically to form a black hole and with $c_0<1$, it bounces back without forming a black hole.

A few comments are in order. First, the flux conservation is valid at spatial infinity and the singularity. To show the conservation we compute the transmission flux from spatial infinity toward the singularity

$$j_{\rm tran}=j_{\rm in}-j_{\rm ref}=|\tilde{D}_I|^2\frac{Qe^{(\pi/2\hbar)Q}}{(\hbar K)^{1/2}}, \tag{34}$$

which coincides with the flux near $y=0$ obtained by a direct computation using Eq. (25), as expected. Second, since Eq. (14) is linear and c_0 is just a separation parameter, one may construct a more general solution for black hole formation by superposing the solutions (24) with different c_0:

$$\Psi^F_{\rm BH}(y,\phi)=\int dc_0F(c_0)\psi_{\rm BH}(y)\exp\left(\pm i\frac{Kc_0}{\hbar^{1/2}m_P}\phi\right), \tag{35}$$

where $F(c_0)$ is an arbitrary weighting factor. By recalling that (μ_-a_-) from Eq. (19) and $(\mu_-a_+^*)$ from Eq. (23) are independent of c_0, one still has the factorized asymptotic form for Eq. (35):

$$\Psi^F_{\rm BH}(y,\phi)\cong F_{(-)}(\phi)\left(i\frac{K}{\hbar}y^2\right)^{\mu_--a_-}e^{-(i/2)(K/\hbar)y^2}+F_{(+)}(\phi)\left(i\frac{K}{\hbar}y^2\right)^{\mu_--a_+^*}e^{(i/2)(K/\hbar)y^2}. \tag{36}$$

Here, $F_{(\pm)}(\phi)$, which are obtained by integrating over c_0, may represent wave packets of plane waves, $e^{\pm i(Kc_0/\hbar^{1/2}m_P)\phi}$, in Eq. (15).

Finally, we explain the reason why the other wave functions for Eq. (14) are irrelevant for black hole formation. The wave function (22) has incident components from both spatial infinity and the black hole singularity to the potential barrier, so it does not satisfy the boundary condition for black hole formation. Another wave function is obtained by taking a linear combination of Eqs. (18) and (22) which describes the outgoing component alone at spatial infinity,

$$\psi_{IV}(y)=\psi_I-\frac{\tilde{D}_I}{\tilde{D}_{III}}e^{i\pi(a_-a_+)}\frac{\Gamma(b_-)\Gamma(a_-^*)}{\Gamma(b_+)\Gamma(a_+^*)}\psi_{III}\cong\left[1-e^{i\pi(a_--a_+)}\frac{\Gamma(a_-)\Gamma(a_-^*)}{\Gamma(a_+)\Gamma(a_+^*)}\right]\times\tilde{D}_I\frac{\Gamma(b_-)}{\Gamma(a_-)}\left(i\frac{K}{\hbar}y^2\right)^{\mu_--a_+^*}e^{(i/2)(K/\hbar)y^2}. \tag{37}$$

Since the net flux is the difference between those of the incoming and outgoing components, it is easy to see that the one-parameter (θ) family of wave functions,

$$\psi_\theta(y)=\psi_I+\frac{e^{i\theta}-1}{1-e^{i\pi(a_--a_+)}\dfrac{\Gamma(a_-)\Gamma(a_-^*)}{\Gamma(a_+)\Gamma(a_+^*)}}\psi_{IV} \tag{38}$$

has the same rate as Eq. (28). However, the branches ψ_I and ψ_{IV} of the wave function (38) have the incident components both from spatial infinity and from the black hole singularity to the potential barrier, respectively. We also note that the wave function ψ_{IV}^*, the complex conjugate of Eq. (37), is also a solution to the linear real equation (14) and is purely incident from spatial infinity. However, near $y=0$, ψ_{IV}^* consists of two branches of wave functions: ψ_I^* having an outgoing flux and ψ_{III}^* having an incoming flux. Thus ψ_{IV}^* describes the incident waves from both spatial infinity and the black hole singularity to the potential barrier. Hence, these wave functions, ψ_{IV}^* and $\psi_\theta(\theta\neq 0)$, do not satisfy in a strict sense the boundary condition for black hole formation.

IV. SEMICLASSICAL LIMIT AND APPARENT HORIZON

In order to regain the semiclassical picture from our quantum wave function we use our recently proposed method [22]. For this we first consider the Wheeler-DeWitt equation with nonvanishing energy parameter E given by

$$\left[\frac{\hbar^2}{2K}\frac{\partial^2}{\partial y^2}-\frac{\hbar m_P^2}{2Ky^2}\frac{\partial^2}{\partial\phi^2}-K\left(1-\frac{y^2}{2}\right)\right]\Psi_E(y,\phi)=E\Psi_E(y,\phi), \tag{39}$$

where E has the dimension of $\hbar$. The solution corresponding to black hole formation is

$$\psi_E(y)=D\left[\exp\left(-\frac{i}{2}\frac{K}{\hbar}y^2\right)\right]\left(\frac{K}{\hbar}y^2\right)^{\mu_-}M\left(a_E,b_-,i\frac{K}{\hbar}y^2\right), \tag{40}$$

where μ_- and b_- are the same as Eq. (19), while

$$a_E=\frac{1}{2}-\frac{i}{2\hbar}(Q+K+E). \tag{41}$$

Note that the parameter E appears indirectly only through a_E. The Hamilton-Jacobi characteristic function $W_E(y)$ is given as

$$W_E(y)=\hbar\,\mathrm{Im}\ln\psi_E(y)=-\frac{1}{2}Ky^2-\frac{1}{2}Q\ln\left(\frac{K}{\hbar}y^2\right)+\hbar\,\mathrm{Im}\,M\left(a_E,b_-,i\frac{K}{\hbar}y^2\right)+\text{const.} \tag{42}$$

From Eq. (42) we recover the evolution equation of the gravitational collapse as

$$\tau+\beta=\frac{\partial}{\partial E}W_E(y)\bigg|_{E=0}$$
$$=\hbar\frac{\partial}{\partial E}\left[\mathrm{Im}\ln M\left(a_E,b_-,i\frac{K}{\hbar}y^2\right)\right]\bigg|_{E=0}, \tag{43}$$

where β is a constant to be determined by an initial condition.

Before turning to the evolution equation (43) in the region where quantum effects are important, we compare it with the classical equation of motion in the regions where classical effects are dominant. The classical solution of the evolution equation

$$\frac{\dot{y}^2}{2}+1-\frac{y^2}{2}-\frac{c_0^2}{2y^2}=0, \tag{44}$$

is given by

$$\tau+\beta=\pm\int dy\frac{1}{\sqrt{y^2+\frac{c_0^2}{y^2}-2}}$$
$$=\pm\frac{1}{2}\ln(y^2-1+\sqrt{y^4-2y^2+c_0^2}). \tag{45}$$

First, at spatial infinity ($y\to\infty$) we have the asymptotic expression

$$\tau+\beta=\pm\frac{1}{2}\ln y^2+\mathrm{const.} \tag{46}$$

We compare this classical evolution with the quantum one. We compute the characteristic function for the incoming and outgoing components, separately, using Eq. (43)

$$\tau+\beta=\hbar\ \mathrm{Im}\left[\frac{-i}{2M}\frac{\partial}{\partial a}M\left(a,b_-,i\frac{K}{\hbar}y^2\right)\bigg|_{a=a_-,y\to\infty}\right]$$
$$=\pm\frac{1}{2}\ln y^2, \tag{47}$$

where the upper (lower) sign comes from the outgoing (incoming) component in Eq. (24). Hence, in the asymptotic region the Hamilton-Jacobi solution and the classical limit give the same evolution of the collapsing process.

Second, near the singularity ($y\to 0$) the allowed motion is toward the singularity $y=0$. So the classical solution is given by the lower sign in Eq. (45) and has the asymptotic form

$$\tau+\beta=-\frac{1}{2}\ln\left[c_0-1+y^2\left(1-\frac{1}{c_0}\right)\right]$$
$$=\frac{y^2}{2c_0}+\mathrm{const.} \tag{48}$$

On the other hand, the wave function (40) depends on E only through the confluent hypergeometric function $M[a_E,b_-,i(Ky^2/\hbar)]$, and the semiclassical limit of the quantum solution is

$$\tau+\beta=\hbar\frac{\partial}{\partial E}\left[\mathrm{Im}\ln M\left(a_E,b_-,i\frac{K}{\hbar}y^2\right)\right]$$
$$=\frac{y^2}{2c_0}. \tag{49}$$

Thus, in the semiclassical limit ($\hbar\to 0$, which is equivalent to $K\to\infty$) the quantum wave function has the correct limit to the Hamilton-Jacobi solution both in the asymptotic region and near the singularity.

In the intermediate region between the asymptotic region and the singularity it is most interesting to see how the apparent horizon and black hole mass are affected by quantum effects. We note that the position of the apparent horizon is given by the trapped surface $(\nabla r)^2=0$, which translates to $\dot{y}^2-y^2=0$. In the classical case the apparent horizon is determined from the equation of motion,

$$\frac{dW_c}{dy}=-\sqrt{K^2\left(y^2+\frac{c_0^2}{y^2}-2\right)}=K\dot{y}=-Ky, \tag{50}$$

which has the solution

$$y_{AH}^2=\frac{c_0^2}{2}. \tag{51}$$

On the other hand, in the quantum case we use the quantum counterpart Eq. (42) to Eq. (50),

$$\frac{dW_E}{dy}\bigg|_{E=0}=K\dot{y}=-Ky. \tag{52}$$

Since $W|_{E=0}=\hbar\ \mathrm{Im}\ln\psi$, Eq. (52) becomes

$$-Ky=\hbar\frac{d}{dy}(\mathrm{Im}\ln\psi)$$
$$=-Ky-\frac{Q}{y}+\hbar\ \mathrm{Im}\left[\frac{d}{dy}\ln M\right], \tag{53}$$

which can be rewritten as

$$\frac{Q}{2K}=\hbar y^2\ \mathrm{Re}\left[\frac{a}{b}\frac{M(a+1,b+1,z)}{M(a,b,z)}\right]. \tag{54}$$

In the semiclassical limit ($K/\hbar\to\infty$) we can evaluate the apparent horizon (y_{AH}^2) using the steepest descent method (see the Appendix for details). We find the apparent horizon up to the first order as

$$y_{AH}^2=\frac{c_0^2}{2}+\frac{\hbar^2}{K^2}\frac{3-\frac{64}{c_0^2}+\frac{80}{c_0^4}}{8}+O\left(\frac{\hbar^3}{K^3}\right), \tag{55}$$

where the first term on the right-hand side is the classical result and the second term is the first-order quantum correction. We checked this with numerical calculation for large $K/\hbar(=40)$. They are

$$y_{AH}^2|_{\text{numerical}}=1.9616(c_0=2);0.5000(c_0=1);\\ 0.2569\left(c_0=\frac{1}{\sqrt{2}}\right),$$

$$y_{AH}^2|_{\text{Eq. (55)}}=1.9994(c_0=2);0.5001(c_0=1);\\ 0.2652\left(c_0=\frac{1}{\sqrt{2}}\right). \tag{56}$$

Even in the critical case $(c_0=1)$ the apparent horizon deviates only slightly from the classical one, and the first-order correction is quite good. As far as the apparent horizon is concerned the critical case seems to show no special behavior compared to the super and subcritical cases. For small $K/\hbar$ there is a large quantum correction, and the approximation (55) is not valid. For example, we calculated numerically when $K/\hbar=1$, and the deviation from the classical value is substantial as

$$y_{AH}^2|_{\text{numerical}}-y_{AH}^2|_{\text{classical}}\\ =-0.412(c_0=2);0.127(c_0=1);0.128\left(c_0=\frac{1}{\sqrt{2}}\right). \tag{57}$$

However, there is no qualitative change in the position of the horizon.

V. DISCUSSION

In this paper we have studied black hole formation by a self-similarly collapsing massless scalar field. The analytic wave functions for black hole formation were found in terms of the confluent hypergeometric function. By evaluating the incoming and outgoing flux at spatial infinity we were able to compute the rate for quantum black hole formation, which agrees with our previous result in the subcritical case. To find other quantum effects we used the characteristic function defined by the imaginary part of the wave function and recovered the evolution of black hole formation in time with quantum effects taken into account. We now compare our results with those from the classical solution, and the wave functions by Tomimatsu, and discuss the physical implications.

First, in the classical case [17] the evolution of the gravitational collapse is governed by Eq. (44) and the solution is given by Eq. (45). In the supercritical case the incoming component, the lower sign of Eq. (45), falls into the singularity and any part of it is not reflected back to spatial infinity. On the other hand, in the quantum case the wave function (18) for black hole formation has both the incoming and outgoing components (24). In particular, in the supercritical case it has a small fraction of outgoing component. In the subcritical case black hole formation is allowed by quantum-mechanical tunneling.

Second, among all the wave functions (18), (21), and (22) for the Wheeler-DeWitt equation, Eq. (18) has the desired semiclassical limit for black hole formation. On the other hand, Eq. (37) represents a wave function with outgoing component alone. Though the physical meaning of this wave function is not clear, it may be interpreted as an amplitude for black hole decay or white hole creation, which requires a further study.

Third, as mentioned in Sec. IV, the apparent horizon in the quantum case deviates from the classical one only slightly. Thus there appears no qualitative change from the classical to the quantum treatment of the apparent horizon.

Finally, we compare our wave functions with those by Tomimatsu who found the wave functions for quantum black hole formation and decay [19]. The difference is that he used a different coordinate system from ours and quantized both the classical constraint equation and the Hamiltonian from the reduced action, whereas we adopted directly the ADM formulation, obtained the Hamiltonian constraint, and quantized the constraint according to the Dirac quantization method. In contrast to his wave functions diverging at the origin in both the supercritical and subcritical cases, our wave functions are always regular at the origin. The regularity of wave function for a black hole seems to suggest that quantum gravity effects may cure some singularity problems in classical gravity.

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APPENDIX: USEFUL FORMULAS FOR THE CONFLUENT HYPERGEOMETRIC FUNCTIONS

The asymptotic form of the confluent hypergeometric function [24] used in this paper is

$$\frac{M(a,b,z)}{\Gamma(b)}=\frac{e^{i\pi a}z^{-a}}{\Gamma(b-a)}\left[1+\frac{e^{z}z^{a-b}}{\Gamma(a)}\right], \tag{A1}$$

where $-\pi/2<\arg z<3\pi/2$.

To obtain the characteristic function valid at intermediate region we use the integral formula for the confluent hypergeometric function [24]

$$\frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b)}M(a,b,z)=\int_0^1 dt\, e^{zt}t^{a-1}(1-t)^{b-a-1}, \tag{A2}$$

which is valid for $\mathrm{Re}\, b>\mathrm{Re}\, a>0$. In this paper $z=i(K/\hbar)y^2$ and $\mathrm{Re}\, b=1>\mathrm{Re}\, a=\frac{1}{2}>0$. In order to evaluate $M(a,b,z)$ in the large-K limit we evaluate it with the steepest descent method. For this we define

$$\frac{iK}{\hbar}f(t)=tz+(a-1)\ln(t)+(b-a-1)\ln(1-t). \tag{A3}$$

We find the relevant root of $df/dt=0$ as

$$t_-=\frac{c+y^2-(y^4+c^2-2y^2)^{1/2}}{2y^2}, \tag{A4}$$

where $c=(c_0^2-\hbar^2/4K^2)^{1/2}-i(\hbar/K)$. We evaluate the integral

$$\int dt\, \exp\left(i\frac{K}{\hbar}f(t)\right)=\left[\exp\left(i\frac{K}{\hbar}f(t_-)\right)\right]\left(\frac{2\pi\hbar}{-iKf''(t_-)}\right)^{1/2}\times\left[1+O\left(\frac{\hbar}{K}\right)\right]. \tag{A5}$$

After straightforward calculations we obtain

$$y_{AH}^2=\frac{c_0^2}{2}\left[1+\frac{\hbar^2}{K^2}\frac{3-\frac{64}{c_0^2}+\frac{80}{c_0^4}}{4c_0^2}+O\left(\frac{\hbar^2}{K^3}\right)\right], \tag{A6}$$

which is in good agreement with the numerical calculations in the large-K limit as shown at the end of Sec. IV.

[1] D. Christodoulou, Commun. Math. Phys. **105**, 337 (1986); **106**, 587 (1986); **109**, 591 (1987).
[2] D. Goldwirth and T. Piran, Phys. Rev. D **36**, 3575 (1987); R. Gómez, R.A. Isaacson and J. Winnicour, J. Comput. Phys. **98**, 11 (1992).
[3] M.W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
[4] C.R. Evans and J.S. Coleman, Phys. Rev. Lett. **72**, 1782 (1994).
[5] T. Koike and T. Mishima, Phys. Rev. D **51**, 4045 (1995); T. Koike, T. Hara, and S. Adachi, Phys. Rev. Lett. **74**, 5170 (1995).
[6] D. Maison, Phys. Lett. B **366**, 82 (1996).
[7] E.W. Hirschmann and D.M. Eardley, Phys. Rev. D **51**, 4198 (1995); **52**, 5850 (1995).
[8] C. Gundlach, Phys. Rev. D **54**, 7353 (1996).
[9] S. Hod and T. Piran, Phys. Rev. D **55**, R440 (1997); **55**, 3485 (1997).
[10] C. Gundlach, Phys. Rev. D **55**, 6002 (1997).
[11] E.W. Hirschmann and D.M. Eardley, Phys. Rev. D **56**, 4696 (1997).
[12] D.M. Eardley and E.W. Hirschmann, Phys. Rev. D **52**, 5397 (1995).
[13] R.S. Hamade, J.H. Horne, and J.M. Stewart, Class. Quantum Grav. **13**, 2241 (1996).
[14] A.M. Abrahams and C.R. Evans, Phys. Rev. Lett. **70**, 2980 (1993).
[15] M.D. Roberts, Gen. Relativ. Gravit. **21**, 907 (1989).
[16] P.R. Brady, Class. Quantum Grav. **11**, 1255 (1994).
[17] A.V. Frolov, Class. Quantum Grav. **16**, 407 (1999).
[18] Y. Oshiro, K. Nakamura, and A. Tomimatsu, Prog. Theor. Phys. **91**, 1265 (1994).
[19] A. Tomimatsu, Phys. Rev. D **52**, 4540 (1995).
[20] D. Bak, S.P. Kim, S.K. Kim, K.-S. Soh, and J.H. Yee, Phys. Rev. D **60**, 064005 (1999).
[21] M.W. Choptuik, in the *Proceedings of YKIS'99 Workshop on Black Holes and Gravitational Waves*.
[22] D. Bak, S.P. Kim, S.K. Kim, K.-S. Soh, and J.H. Yee, ''Classical Limit and Time in Quantum Cosmology,'' gr-qc/9907031.
[23] F. Calogero, J. Math. Phys. **12**, 419 (1971).
[24] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).