Statistical entropies of scalar and spinor fields in Vaidya–de Sitter space-time computed by the thin-layer method

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The brick-wall method based on thermal equilibrium at a large scale cannot be applied to cases out of equilibrium, such as nonstationary space-time with two horizons, for example, Vaidya–de Sitter space-time. We improve the brick-wall method and propose a thin-layer method. The entropies of scalar and spinor fields in Vaidya–de Sitter space-time are calculated by the thin-layer method. The condition of local equilibrium near the two horizons is used as a working postulate and is maintained for a black hole which evaporates slowly enough and whose mass is far greater than the Planck scale. There are two horizons in Vaidya–de Sitter space-time. We think that the total entropy is mainly attributed to the two layers near the two horizons. The entropy of a scalar field in Vaidya–de Sitter space-time is a linear sum of the area of the black hole horizon and that of the cosmological horizon. Thinking of Dirac equations in the Newman-Penrose formalism, there are four components of the wave functions F_1, F_2, G_1, and G_2. The total entropy is summed up from the entropies corresponding to the four components. On the same condition of the scalar field, the resulting entropy is 7/2 times that of the scalar field, and is also a linear sum of the area of the black hole horizon and that of the cosmological horizon. The difference from the stationary black hole is that the result relies on time-dependent cutoffs.

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I. THIN-LAYER METHOD AND ENTROPY OF SCALAR FIELD IN VAIDYA–de SITTER SPACE-TIME

Recently many people have paid much attention to the statistical origin of black hole entropy. From the path approach to the quantization of the gravitational field, Hawking and Horowitz [1] computed the action to obtain the black hole entropy $S=(\chi/8)A$, where χ is the topological invariant quantity (Euler-Poincaré characteristics) and A is the area of the horizon. However, the statistical origin of the Bekenstein-Hawking entropy of black hole has remained unclear until now [2]. To find the statistical origin of black hole entropy, by employing the brick-wall method, 't Hooft [3] first studied the entropy of a scalar field in a Schwarzschild black hole. After this, the method was applied to a scalar field in various stationary black hole backgrounds [4–9]. In the brick-wall method, because the number of energy levels a particle can occupy in the vicinity of the black hole is divergent at the horizon, the wave function is assumed to be radially nonzero only within the range of $r_H+h\leqslant r\leqslant L$, where r_H is the radius of the horizon, h is "ultraviolet cutoff" near the horizon and L is "infrared cutoff." The results [4,7] usually include three parts: the first term is a quantum correction to the Bekenstein-Hawking entropy which is proportional to the area of the horizon; the second term is logarithmically divergent; the third term is a typical contribution from the vacuum surrounding the system at large distances.

Although the brick-wall method is popular, we point out that it has several defects. (1) In order to obtain the result that the black hole entropy is proportional to the area of the horizon, a theory "neglecting the logarithmically divergent term [4,7] and the third term [3,4,7]" is adopted. We think this approach is not natural. (2) In the brick-wall method, it is supposed that there exists a thermal equilibrium between the external field and the hole in a large spatial region. This method cannot be applied to a nonequilibrium system, such as a system of non-stationary space-time with two horizons, for example, Vaidya–de Sitter space-time. In a system of nonstationary space-time with two horizons, we have two problems: (a) the thermal equilibrium between the external field and the hole is unstable, so there exists no thermal equilibrium in the large spatial region and statistical physics laws are invalid there; and (b) since the two horizons have different temperatures, there exists no thermal equilibrium over the entire space-time, and statistical physics laws are also invalid there.

In order to overcome these defects, we shall improve the brick-wall method and propose a thin-layer method. In the

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brick-wall method, the entropy of the black hole is mainly attributed to the degrees of the freedom of the field of the thin layer ($r_H+h\leqslant r\leqslant r_H+h+\delta$) covering the surface of the horizon, where δ is a positive physical infinitesimal quantity. We are inspired by this. In the thin-layer method, the wave function is assumed to be radially nonzero only within the range of $r_H+h\leqslant r\leqslant r_H+h+\delta$. Up to now, to our knowledge, no papers were devoted to the entropy of a spinor field in a nonstationary space-time with two horizons, such as Vaidya–de Sitter space-time. This approach encounters two main difficulties. One is that the calculation is very complex. The other is that statistical physics laws seem to be invalid because there exists no thermal equilibrium in the large region. However, in our opinion, the global thermal equilibrium is not needed. It is well known that Hawking radiation is derived from the vacuum fluctuation near the horizon. The Bekenstein-Hawking entropy should be associated with the field in this small region near the horizon, where local thermal equilibrium exists and statistical laws are still valid. The notion of local equilibrium will be used as a working postulate. Here we suppose that the physical quantities of the thermodynamic properties of the exciting field, such as pressure and temperature in the vicinity of the horizon, vary slightly. On the one hand, the horizon is spherically symmetric and the radial length of the region near the horizon must be small enough on a macroscopic scale so that the physical quantities in the region can be approximately treated as constants, and approximate equilibrium in the small region is obtained. On the other hand, the region must be large enough on a microscopic scale so that the statistical mechanics in the field near the horizon is valid, and the thermodynamic variables can be defined through a partition function. Obviously, the validity of local equilibrium is crucial to the discussion. According to Kreuzer [10], in order for local equilibrium to be maintained, it is necessary that the fluctuation remains within bounds. For instance, if the pressure of radiation p has a fluctuation δp, we must demand that

$$\frac{\delta p}{p}\ll 1 \tag{1}$$

for thermal radiation; $p\sim T^4$, where T is the temperature. On the other hand, for a black hole with mass M, $T\sim 1/M$, inequality (1) requires that

$$\frac{\delta p}{p}\sim\frac{4\,\delta T}{T}\sim\frac{\delta M}{M}\ll 1. \tag{2}$$

We see that inequalities (1) and (2) are easily satisfied by general black holes, since their evaporation is slow enough ($\dot{r}_H\ll 1$) so that $\delta M/M\ll 1$, except for those cases with Planck mass.

The simplest nonstationary space-time with two horizons is described by a Vaidya–de Sitter metric whose line element is [11]

$$ds^2=\left[1-\frac{2M(v)}{r}-\frac{1}{3}\Lambda r^2\right]dv^2-2\,dv\,dr -r^2(d\theta^2+\sin^2\theta\,d\varphi^2), \tag{3}$$

where mass M depends on the advanced Eddington-Finkelstein time v, and Λ is the cosmological constant.

From the semiclassical Einstein gravitational equation

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=-8\pi\langle 0|T_{\mu\nu}|0\rangle_{ren}, \tag{4}$$

the renormalized matter stress-energy tensor of vacuum fluctuation is computed as follows:

$$\langle 0|T_{\mu\nu}|0\rangle_{ren}=-\frac{1}{4\pi r^2}\frac{dM(v)}{dv}\delta_\mu^0\delta_\nu^0. \tag{5}$$

In our paper, the signature is -2, and natural units ($G=c=1$) are adopted. The metric can be used to describe an evaporating black hole when dM/dv satisfies the "Stefan-Boltzmann" law. The position of event horizon r_H is determined by [12]

$$r_H^3-\frac{3(1-2\dot{r}_H)}{\Lambda}r_H+\frac{6M(v)}{\Lambda}=0, \tag{6}$$

which is cubic equation of one variable r_H, where $\dot{r}_H=dr_H/dv$. If $1-2\dot{r}_H>0$, Eq. (6) has three different roots r_-, r_+, and r_C [see Eq. (A2) in the Appendix]. r_- is a negative root, and thus it has no physical meaning; r_+ is a smaller positive root corresponding to black hole horizon; and r_C is a larger positive root corresponding to cosmological horizon. The radiation temperatures at the two horizons are determined by [12]

$$T_H=T_+=\frac{1}{\beta_+}=\frac{1}{2\pi(1-2\dot{r}_+)r_+^2}\left[M(v)-\frac{1}{3}\Lambda r_+^3\right],$$

$$T_H=T_C=\frac{1}{\beta_C}=\frac{1}{2\pi(1-2\dot{r}_C)r_C^2}\left[M(v)-\frac{1}{3}\Lambda r_C^3\right], \tag{7}$$

where $\dot{r}_+=dr_+/dv$ and $\dot{r}_C=dr_C/dv$. The variable temperature is meaningful only in the vicinity of the horizon. The Vaidya–de Sitter space-time has two horizons with two different temperatures, and there exists no thermal equilibrium in the entire space-time. When we compute the free energy, we take only the thin layer near the black hole horizon or the cosmological horizon in Vaidya–de Sitter space-time. Although there exists no thermal equilibrium in the entire space-time, approximate thermal equilibrium exists in the two layers and statistical physics laws are valid there. The subscript H indicates subscript $+$ or subscript C. The following discussion and results are valid for the black hole horizon or the cosmological horizon.

In curved space-time, the wave equation of massless scalar field reads

$$\frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi]=0. \tag{8}$$

With $\Phi=G(v,r)Y(\theta,\varphi)$, the radial equation is obtained as

$$\left(1-\frac{2M(v)}{r}-\frac{1}{3}\Lambda r^2\right)\frac{\partial^2 G}{\partial r^2}+2\frac{\partial^2 G}{\partial v\,\partial r}+\frac{2}{r}\frac{\partial G}{\partial v}+\frac{2}{r}\left(1-\frac{M(v)}{r}-\frac{2}{3}\Lambda r^2\right)\frac{\partial G}{\partial r}-\frac{l(l+1)}{r^2}G=0, \tag{9}$$

where $l=0,1,2,\ldots$. In general, the exact solution of Eq. (9) does not have the form of $\exp(-i\omega v)G(r)$ in the nonstationary space-time. However, It is enough to know the asymptotic behavior of the equation near the horizon, because we only investigate the field in the vicinity of the horizon. Therefore, a reasonable solution is still supposed by the WKB approximations:

$$G_+(v,r)=\exp[-i\omega v+iS_+(r,v)],\qquad G_C(v,r)=\exp[-i\omega v+iS_C(r,v)]. \tag{10}$$

Substituting Eq. (10) into Eq. (9), we obtain the radial momenta

$$k_+=\frac{\partial S_+}{\partial r}=\frac{\omega\pm\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3]l(l+1)/r^2}}{1-2M(v)/r-\Lambda r^2/3},\qquad k_C=\frac{\partial S}{\partial r}=\frac{\omega\pm\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3]l(l+1)/r^2}}{1-2M(v)/r-\Lambda r^2/3}, \tag{11}$$

where the plus signs represent the outgoing waves (k_+^+) and (k_C^+), and the minus signs represent the incoming waves (k_+^-) and (k_C^-). According to the semiclassical quantization rule, the radial wave numbers are quantized as

$$2n_1\pi=\int_{r_++h_+}^{r_++h_++\delta_+}k_+^+dr+\int_{r_++h_+}^{r_++h_++\delta_+}k_+^-dr=2\int_{r_++h_+}^{r_++h_++\delta_+}\frac{\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3]l(l+1)/r^2}}{1-2M(v)/r-\Lambda r^2/3}dr,$$

$$2n_2\pi=\int_{r_C-h_C-\delta_C}^{r_C-h_C}k_C^+dr+\int_{r_C-h_C-\delta_C}^{r_C-h_C}k_C^-dr=2\int_{r_C-h_C-\delta_C}^{r_C-h_C}\frac{\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3]l(l+1)/r^2}}{1-2M(v)/r-\Lambda r^2/3}dr, \tag{12}$$

where h_+ and h_C are "ultraviolet cutoffs" near the black hole horizon and the cosmological horizon, respectively, and δ_+ and δ_C are positive physical infinitesimal quantities. The total numbers of wave solutions with energy not exceeding ω are given by

$$g_+(\omega)=\int n_1(2l+1)dl=\frac{1}{\pi}\int\int_{r_++h_+}^{r_++h_++\delta_+}\frac{\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3]l(l+1)/r^2}}{1-2M(v)/r-\Lambda r^2/3}dr(2l+1)dl,$$

$$g_C(\omega)=\int n_2(2l+1)dl=\frac{1}{\pi}\int\int_{r_C-h_C-\delta_C}^{r_C-h_C}\frac{\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3]l(l+1)/r^2}}{1-2M(v)/r-\Lambda r^2/3}dr(2l+1)dl, \tag{13}$$

where the l integration goes over those values of l for which the argument of the square root is positive. Then the main contributions of integral are given by [see Eqs. (A6), (A7), and (A8) in the Appendix]

$$g_+(\omega)=\frac{2}{3\pi}\int_{r_++h_+}^{r_++h_++\delta_+}\frac{\omega^3 r^2}{[1-2M(v)/r-\Lambda r^2/3]^2}dr$$
$$\simeq\frac{18\omega^3 r_+^4}{3\pi\Lambda^2(r_+-r'_-)^2(r_+-r'_C)^2h'_+},\tag{14}$$
$$g_C(\omega)=\frac{2}{3\pi}\int_{r_C-h_C-\delta_C}^{r_C-h_C}\frac{\omega^3 r^2}{[1-2M(v)/r-\Lambda r^2/3]^2}dr$$
$$\simeq\frac{18\omega^3 r_C^4}{3\pi\Lambda^2(r_C-r'_-)^2(r_C-r'_+)^2h'_C},$$

where

$$h'_+=\frac{(r_++h_+-r'_+)(r_++h_++\delta_+-r'_+)}{\delta_+},$$
$$h'_C=\frac{(r_C-h_C-r'_C)(r_C-h_C-\delta_C-r'_C)}{\delta_C},\tag{15}$$

and r'_-, r'_+, and r'_C are shown by Eq. (A7) in the Appendix.

According to statistical laws, the free energy reads [see Eq. (A9) in the Appendix]

$$F\simeq F_++F_C$$
$$=\frac{1}{\beta_+}\int dg_+(\omega)\ln(1-e^{-\beta_+\omega})$$
$$+\frac{1}{\beta_C}\int dg_C(\omega)\ln(1-e^{-\beta_C\omega})$$
$$\simeq-\frac{2\pi^3 r_+^4}{5\Lambda^2(r_+-r_-)^2(r_+-r_C)^2\beta_+^4h'_+}$$
$$-\frac{2\pi^3 r_C^4}{5\Lambda^2(r_C-r_-)^2(r_C-r_+)^2\beta_C^4h'_C},\tag{16}$$

since the contribution to the free energy F from the field existing between the two horizons or from the cosmological horizon to L (infrared cutoff [3,4,7]) can be neglected as compared with F_+ or F_C. The entropy is given by [see Eq. (A10) in the Appendix]

$$S\simeq\beta_+^2\frac{\partial F_+}{\partial\beta_+}+\beta_C^2\frac{\partial F_C}{\partial\beta_C}$$
$$=\frac{1}{90\beta_+(1-2\dot r_+)^2h'_+}\frac{A_+}{4}+\frac{1}{90\beta_C(1-2\dot r_C)^2h'_C}\frac{A_C}{4},\tag{17}$$

where $A_+=4\pi r_+^2$ and $A_C=4\pi r_C^2$. When we redefine the cutoffs

$$\eta_+=90\beta_+(1-2\dot r_+)^2h'_+,$$
$$\eta_C=90\beta_C(1-2\dot r_C)^2h'_C,\tag{18}$$

the entropy of scalar field in Vaidya–de Sitter space-time is given by the form

$$S\simeq\frac{A_+}{4\eta_+}+\frac{A_C}{4\eta_C}.\tag{19}$$

The difference from the stationary case is that time-dependent cutoffs η_+ and η_C have been introduced. One can easily see that the result returns to the situation of a Schwarzschild black hole when $\dot r_+=\dot r_C=0$ and $A_C=0$. In the case of a nonstationary black hole, time-dependent cutoffs seem to be more natural than time-independent cutoffs. It is shown that the entropy of a scalar field in Vaidya–de Sitter space-time is a linear sum of the area of the black hole horizon and that of the cosmological horizon.

II. ENTROPY OF SPINOR FIELD IN VAIDYA–de SITTER SPACE-TIME COMPUTED BY THE THIN-LAYER METHOD

Choose the coordinates $x^\mu=(v,r,\theta,\varphi)$. From Eq. (3), the null tetrad is established as [13]

$$l^\mu=[0,1,0,0],$$
$$n^\mu=-\frac{1}{2r^2}[2r^2,\Delta,0,0],\tag{20}$$
$$m^\mu=\frac{1}{\sqrt{2}r}[0,0,1,i\cos ec(\theta)],$$

where

$$\Delta=r^2-2M(v)r-\frac{1}{3}\Lambda r^4.\tag{21}$$

The spin coefficients are

$$\rho=-\frac{1}{r},\quad\gamma=\frac{M(v)}{2r^2}-\frac{\Lambda r}{6},\quad\mu=-\frac{\Delta}{2r^3},$$
$$\alpha=-\frac{\cot\theta}{2\sqrt{2}r}=-\beta,\tag{22}$$
$$\kappa=\pi=\varepsilon=\lambda=\sigma=\nu=\tau=0,$$

whereas the only nonvanishing component of the Weyl tensor is given by

$$\Psi_2=-\frac{M(v)}{r^3}.\tag{23}$$

Equations (22) and (23) tell us that the Vaidya–de Sitter metric is of Petrov type D.

In curved space-time, the Dirac equations of the spinor dynamics in the Newman-Penrose formalism are as follows [14]:

$$(D+\varepsilon-\rho)F_1+(\overline{\delta'}+\pi-\alpha)F_2-i\frac{\mu_0}{\sqrt{2}}G_1=0,$$

$$(\Delta'+\mu-\gamma)F_2+(\delta'+\beta-\tau)F_1-i\frac{\mu_0}{\sqrt{2}}G_2=0,$$

$$(D+\bar{\varepsilon}-\bar{\rho})G_2-(\delta'+\bar{\pi}-\bar{\alpha})G_1-i\frac{\mu_0}{\sqrt{2}}F_2=0, \tag{24}$$

$$(\Delta'+\bar{\mu}-\bar{\gamma})G_1-(\overline{\delta'}+\bar{\beta}-\bar{\tau})G_2-i\frac{\mu_0}{\sqrt{2}}F_1=0,$$

where μ_0 is the mass of spinor, F_1, F_2, G_1, and G_2 are the four components of the wave function, and

$$D=l^\mu\partial_\mu,\quad \Delta'=n^\mu\partial_\mu,\quad \delta'=m^\mu\partial_\mu. \tag{25}$$

Using Eqs. (3), (20)–(22), and (25), and making the transformations

$$[F_1,F_2,G_1,G_2]=[r^{-1}\cdot{}_{-1/2}R_l(v,r)\cdot{}_{-1/2}Y_l^m(\theta,\varphi),r^{-2}\cdot{}_{+1/2}R_l(v,r)\cdot{}_{+1/2}Y_l^m(\theta,\varphi),$$
$$r^{-2}\cdot{}_{+1/2}R_l(v,r)\cdot{}_{-1/2}Y_l^m(\theta,\varphi),r^{-1}\cdot{}_{-1/2}R_l(v,r)\cdot{}_{+1/2}Y_l^m(\theta,\varphi)], \tag{26}$$

we find that Eqs. (24) are separable into the forms [13]

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}-\frac{1}{\sin^2\theta}\left(p^2\cos^2\theta-i2p\cos\theta\frac{\partial}{\partial\varphi}-\frac{\partial^2}{\partial\varphi^2}\right)-\frac{1}{2}+j^2\right]_p Y_l^m(\theta,\varphi)=0, \tag{27}$$

$$\Bigg\{A\frac{\partial^2}{\partial r^2}+2\frac{\partial^2}{\partial v\,\partial r}+2\left(p+\frac{1}{2}\right)\frac{2ip\mu_0}{j-2ip\mu_0 r}\frac{\partial}{\partial v}+\left[(p+1)A'+\left(\frac{2ip\mu_0}{j-2ip\mu_0 r}-\frac{2p}{r}\right)A\right]\frac{\partial}{\partial r}+\left(p+\frac{1}{2}\right)\frac{2ip\mu_0}{j-2ip\mu_0 r}\left(pA'-\frac{A}{r}\right)+\frac{1}{6}(2p+1)(p+1)r\left(\frac{A}{r}\right)''-\left(\mu_0^2+\frac{j^2}{r^2}\right)\Bigg\}_p R_l(v,r)=0, \tag{28}$$

where

$$A=1-\frac{2M(v)}{r}-\frac{1}{3}\Lambda r^2 \tag{29}$$

and the prime denotes the derivative with respect to r. Equation (27) shows that ${}_pY_l^m(\theta,\varphi)$ is the spin-weighted spherical harmonics, and the separation constant j satisfies

$$j=l+\frac{1}{2}. \tag{30}$$

Here l and m are integers satisfying the inequalities $-l\leqslant m\leqslant l$ and $p=\pm\frac{1}{2}$. In this paper only radial equation (28) needs to be used. For simplicity, we set $\mu_0=0$, because the calculation of entropy usually uses the small-mass limit.

Considering that the wave function of spinor field is composed of four components, we will calculate the entropy corresponding to every component respectively. The total entropy should be the sum of the four entropies. According to this idea, we calculate the entropy corresponding to F_1 first.

Putting $p=-\frac{1}{2}$ and $\mu_0=0$ into Eq. (28), and using Eqs. (29) and (30), we find that Eq. (28) becomes

$$\left[\left(1-\frac{2M(v)}{r}-\frac{1}{3}\Lambda r^2\right)\frac{\partial^2}{\partial r^2}+2\frac{\partial^2}{\partial v\,\partial r}+\left(\frac{1}{r}-\frac{M(v)}{r^2}-\frac{2}{3}\Lambda r\right)\frac{\partial}{\partial r}-\frac{\left(l+\frac{1}{2}\right)^2}{r^2}\right]_{-1/2}R_l(v,r)=0. \tag{31}$$

As in the above method to compute the entropy of a scalar field in Vaidya–de Sitter space-time, reasonable solutions of Eq. (31) are still supposed by the WKB approximation:

$${}_{-1/2}R_{l+}(v,r)=\exp[-i\omega v+iS_{1+}(r,v)],$$
$${}_{-1/2}R_{lC}(v,r)=\exp[-i\omega v+iS_{1C}(r,v)]. \tag{32}$$

Substituting Eq. (32) into Eq. (31), we obtain the radial momenta

$$k_{1+}=\frac{\partial S_{1+}}{\partial r}=\frac{\omega\pm\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3](l+1/2)^2/r^2}}{1-2M(v)/r-\Lambda r^2/3},$$
$$k_{1C}=\frac{\partial S_{1C}}{\partial r}=\frac{\omega\pm\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3](l+1/2)^2/r^2}}{1-2M(v)/r-\Lambda r^2/3}. \tag{33}$$

The plus signs represent the outgoing waves (k_{1+}^+) and (k_{1C}^+), and the minus signs represent the incoming waves (k_{1+}^-) and (k_{1C}^-). According to the semiclassical quantization rule, the radial wave numbers are quantized as

$$2n_1\pi=\int_{r_++h_+}^{r_++h_++\delta_+}k_{1+}^+\,dr+\int_{r_++h_+}^{r_++h_++\delta_+}k_{1+}^-dr$$
$$=2\int_{r_++h_+}^{r_++h_++\delta_+}\frac{\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3](l+1/2)^2/r^2}}{1-2M(v)/r-\Lambda r^2/3}dr,$$
$$2n_2\pi=\int_{r_C-h_C-\delta_C}^{r_C-h_C}k_{1C}^+\,dr+\int_{r_C-h_C-\delta_C}^{r_C-h_C}k_{1C}^-\,dr$$
$$=2\int_{r_C-h_C-\delta_C}^{r_C-h_C}\frac{\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3](l+1/2)^2/r^2}}{1-2M(v)/r-\Lambda r^2/3}dr. \tag{34}$$

The total number of wave solutions with energy not exceeding ω is given by

$$g_{1+}(\omega)=\int n_1(2l+1)dl$$
$$=\frac{1}{\pi}\int\int_{r_++h_+}^{r_++h_++\delta_+}\frac{\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3](l+1/2)^2/r^2}}{1-2M(v)/r-\Lambda r^2/3}dr(2l+1)dl,$$
$$g_{1C}(\omega)=\int n_2(2l+1)dl$$
$$=\frac{1}{\pi}\int\int_{r_C-h_C-\delta_C}^{r_C-h_C}\frac{\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3](l+1/2)^2/r^2}}{1-2M(v)/r-\Lambda r^2/3}dr(2l+1)dl, \tag{35}$$

where the l integration goes over those values of l for which the argument of the square root is positive. Then the main contributions of the integral are given by [see Eqs. (14) and (15)]

$$g_{1+}(\omega)=\frac{2}{3\pi}\int_{r_++h_+}^{r_++h_++\delta_+}\frac{\omega^3r^2}{[1-2M(v)/r-\Lambda r^2/3]^2}dr$$
$$=g_+(\omega),$$
$$g_{1C}(\omega)=\frac{2}{3\pi}\int_{r_C-h_C-\delta_C}^{r_C-h_C}\frac{\omega^3r^2}{[1-2M(v)/r-\Lambda r^2/3]^2}dr$$
$$=g_C(\omega). \tag{36}$$

According to statistical physics laws, the free energy reads [see Eq. (A9) in the Appendix]

$$F_1\simeq F_{1+}+F_{1C}$$
$$=-\frac{1}{\beta_+}\int dg_{1+}(\omega)\ln(1+e^{-\beta_+\omega})$$
$$-\frac{1}{\beta_C}\int dg_{1C}(\omega)\ln(1+e^{-\beta_C\omega})$$
$$\simeq-\frac{7\pi^3r_+^4}{20\Lambda^2(r_+-r_-)^2(r_+-r_C)^2\beta_+^4h'_+}$$
$$-\frac{7\pi^3r_C^4}{20\Lambda^2(r_C-r_-)^2(r_C-r_+)^2\beta_C^4h'_C}, \tag{37}$$

since the contribution to the free energy F_1 from the field existing between the two horizons or from the cosmological horizon to L (the infrared cutoff [3,4,7]) can be neglected as compared with F_{1+} or F_{1C}. The entropy is given by [see Eq. (A10) in the Appendix]

$$S_1 \simeq \beta_+^2 \frac{\partial F_{1+}}{\partial \beta_+} + \beta_C^2 \frac{\partial F_{1C}}{\partial \beta_C} = \frac{7}{8} \frac{1}{90\beta_+(1-2\dot{r}_+)^2 h'_+} \frac{A_+}{4} + \frac{7}{8} \frac{1}{90\beta_C(1-2\dot{r}_C)^2 h'_C} \frac{A_C}{4}, \tag{38}$$

where $A_+ = 4\pi r_+^2$ and $A_C = 4\pi r_C^2$.

Next we shall calculate the entropy corresponding to F_2. Putting $p=\frac{1}{2}$ and $\mu_0=0$ into Eq. (28) and using Eqs. (29) and (30), we find that Eq. (28) becomes

$$\left[\left(1-\frac{2M(v)}{r}-\frac{1}{3}\Lambda r^2\right)\frac{\partial^2}{\partial r^2}+2\frac{\partial^2}{\partial v\,\partial r} + \left(-\frac{1}{r}+\frac{5M(v)}{r^2}-\frac{2}{3}\Lambda r\right)\frac{\partial}{\partial r}+\frac{1}{r^2} - \frac{6M(v)}{r^3}-\frac{\left(l+\frac{1}{2}\right)^2}{r^2}\right]{}_{1/2}R_l(v,r)=0. \tag{39}$$

As in the above method to compute the entropy corresponding to F_1, reasonable solutions of Eq. (39) are still supposed by the WKB approximation:

$$\begin{aligned} {}_{1/2}R_{l+}(v,r) &= \exp[-i\omega v + iS_{2+}(r,v)], \\ {}_{1/2}R_{lC}(v,r) &= \exp[-i\omega v + iS_{2C}(r,v)]. \end{aligned} \tag{40}$$

Substituting Eq. (40) into Eq. (39), we obtain the radial momenta

$$\begin{aligned} k_{2+} &= \frac{\partial S_{2+}}{\partial r} = \frac{\omega \pm \sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3][-1/r^2+6M(v)/r^3+(l+1/2)^2/r^2]}}{1-2M(v)/r-\Lambda r^2/3}, \\ k_{2C} &= \frac{\partial S_{2C}}{\partial r} = \frac{\omega \pm \sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3][-1/r^2+6M(v)/r^3+(l+1/2)^2/r^2]}}{1-2M(v)/r-\Lambda r^2/3}, \end{aligned} \tag{41}$$

where the plus signs represent the outgoing waves (k_{2+}^+) and (k_{2C}^+), and the minus signs represent the incoming waves (k_{2+}^-) and (k_{2C}^-). According to the semiclassical quantization rule, the radial wave numbers are quantized as

$$\begin{aligned} 2n_1\pi &= \int_{r_++h_+}^{r_++h_++\delta_+} k_{2+}^+\,dr + \int_{r_++h_+}^{r_++h_++\delta_+} k_{2+}^-\,dr \\ &= 2\int_{r_++h_+}^{r_++h_++\delta_+} \frac{\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3][-1/r^2+6M(v)/r^3+(l+1/2)^2/r^2]}}{1-2M(v)/r-\Lambda r^2/3}dr, \\ 2n_2\pi &= \int_{r_C-h_C-\delta_C}^{r_C-h_C} k_{2C}^+\,dr + \int_{r_C-h_C-\delta_C}^{r_C-h_C} k_{2C}^-\,dr \\ &= 2\int_{r_C-h_C-\delta_C}^{r_C-h_C} \frac{\sqrt{\omega^2-[1-2M(v)/r-\Lambda r^2/3][-1/r^2+6M(v)/r^3+(l+1/2)^2/r^2]}}{1-2M(v)/r-\Lambda r^2/3}dr. \end{aligned} \tag{42}$$

The total numbers of wave solutions with energy not exceeding ω are given by

$$\begin{aligned} g_{2+}(\omega) &= \int n_1(2l+1)dl \\ &= \frac{1}{\pi}\int\int_{r_++h_+}^{r_++h_++\delta_+} \frac{\sqrt{\omega^2-(1-2M(v)/r-\Lambda r^2/3)[-1/r^2+6M(v)/r^3+(l+1/2)^2/r^2]}}{1-2M(v)/r-\Lambda r^2/3}dr(2l+1)dl, \\ g_{2C}(\omega) &= \int n_2(2l+1)dl = \frac{1}{\pi}\int\int_{r_C-h_C-\delta_C}^{r_C-h_C} \frac{\sqrt{\omega^2-(1-2M(v)/r-\Lambda r^2/3)[-1/r^2+6M(v)/r^3+(l+1/2)^2/r^2]}}{1-2M(v)/r-\Lambda r^2/3}dr(2l \\ &\quad +1)dl, \end{aligned} \tag{43}$$

where the l integration goes over those values of l for which the argument of the square root is positive. Then the main contributions of the integral are given by [see Eqs. (14) and (15)]

$$g_{2+}(\omega)=\frac{2}{3\pi}\int_{r_+ + h_+}^{r_+ + h_+ + \delta_+}\frac{\omega^3 r^2}{[1-2M(v)/r-\Lambda r^2/3]^2}dr = g_+(\omega), \tag{44}$$

$$g_{2C}(\omega)=\frac{2}{3\pi}\int_{r_C - h_C - \delta_C}^{r_C - h_C}\frac{\omega^3 r^2}{[1-2M(v)/r-\Lambda r^2/3]^2}dr = g_C(\omega).$$

According to statistical physics laws, the free energy reads [see Eq. (A9) in the Appendix]

$$\begin{aligned} F_2 &\simeq F_{2+}+F_{2C} \\ &= -\frac{1}{\beta_+}\int dg_{2+}(\omega)\ln(1+e^{-\beta_+\omega}) \\ &\quad -\frac{1}{\beta_C}\int dg_{2C}(\omega)\ln(1+e^{-\beta_C\omega}) \\ &\simeq -\frac{7\pi^3 r_+^4}{20\Lambda^2(r_+-r_-)^2(r_+-r_C)^2\beta_+^4 h_+'} \\ &\quad -\frac{7\pi^3 r_C^4}{20\Lambda^2(r_C-r_-)^2(r_C-r_+)^2\beta_C^4 h_C'}, \end{aligned} \tag{45}$$

since the contribution to the free energy F_2 from the field existing between the two horizons or from the cosmological horizon to L (the infrared cutoff [3,4,7]) can be neglected as compared with F_{2+} or F_{2C}. The entropy is given by [see Eq. (A10) in the Appendix]

$$\begin{aligned} S_2 &\simeq \beta_+^2\frac{\partial F_{2+}}{\partial\beta_+}+\beta_C^2\frac{\partial F_{2C}}{\partial\beta_C} \\ &= \frac{7}{8}\frac{1}{90\beta_+(1-2\dot r_+)^2 h_+'}\frac{A_+}{4}+\frac{7}{8}\frac{1}{90\beta_C(1-2\dot r_C)^2 h_C'}\frac{A_C}{4} \\ &= S_1, \end{aligned} \tag{46}$$

where $A_+=4\pi r_+^2$ and $A_C=4\pi r_C^2$. In the same way, we can calculate the entropies corresponding to G_1 and G_2, respectively. By using the thin-layer method, we find that the entropy corresponding to G_1 or G_2 is equal to S_1 or S_2. The total entropy is written as

$$\begin{aligned} S &= 4S_1 \\ &\simeq \frac{7}{2}\left[\frac{1}{90\beta_+(1-2\dot r_+)^2 h_+'}\frac{A_+}{4}+\frac{1}{90\beta_C(1-2\dot r_C)^2 h_C'}\frac{A_C}{4}\right] \\ &= \frac{7}{2}\left(\frac{A_+}{4\eta_+}+\frac{A_C}{4\eta_C}\right), \end{aligned}$$

where the difference from the stationary case is that time-dependent cutoffs η_+ and η_C [see Eq. (18)] have been introduced. It is shown that the entropy of the spinor field in Vaidya–de Sitter space-time is 7/2 times that of scalar field in Vaidya–de Sitter space-time. This result is similar to Refs. [15,16] that are calculated by other method [17].

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APPENDIX

Equation (6) is

$$r_H^3-\frac{3(1-2\dot r_H)}{\Lambda}r_H+\frac{6M(v)}{\Lambda}=0, \tag{A1}$$

which is a cubic equation of one variable r_H. Equation (A1) has three different roots which are

$$\begin{aligned} r_H &= r_- = -2\sqrt{\frac{1-2\dot r_-}{\Lambda}}\cos\Theta_1, \\ r_H &= r_+ = 2\sqrt{\frac{1-2\dot r_+}{\Lambda}}\cos\left(\Theta_2+\frac{\pi}{3}\right), \\ r_H &= r_C = 2\sqrt{\frac{1-2\dot r_C}{\Lambda}}\cos\left(\Theta_3-\frac{\pi}{3}\right), \end{aligned} \tag{A2}$$

where

$$\begin{aligned} \Theta_1 &= \frac{1}{3}\text{Arccos}\left[\frac{3M(v)\Lambda^{1/2}}{(1-2\dot r_-)^{3/2}}\right], \\ \Theta_2 &= \frac{1}{3}\text{Arccos}\left[\frac{3M(v)\Lambda^{1/2}}{(1-2\dot r_+)^{3/2}}\right], \end{aligned} \tag{A3}$$

$$\Theta_3=\frac{1}{3}\text{Arccos}\left[\frac{3M(v)\Lambda^{1/2}}{(1-2\dot{r}_C)^{3/2}}\right],$$

where $\dot{r}_-=dr_-/dv$, $\dot{r}_+=dr_+/dv$, and $\dot{r}_C=dr_C/dv$. r_- is a negative root, so it has no physical meaning; r_+ is a smaller positive root corresponding to a black hole horizon; and r_C is a larger positive root corresponding to a cosmological horizon. For Eq. (A1), the relations between the coefficients and the roots are

$$r_++r_C+r_-=0, \tag{A4}$$

$$r_+r_Cr_-=-\frac{6M(v)}{\Lambda}. \tag{A5}$$

From Eq. (14), we require that

$$r^3-\frac{3}{\Lambda}r+\frac{6M(v)}{\Lambda}=0, \tag{A6}$$

which is a cubic equation of one variable r. Equation (A6) has three different roots,

$$r'_-=-2\frac{1}{\sqrt{\Lambda}}\cos\Theta',$$

$$r'_+=2\frac{1}{\sqrt{\Lambda}}\cos\left(\Theta'+\frac{\pi}{3}\right), \tag{A7}$$

$$r'_C=2\frac{1}{\sqrt{\Lambda}}\cos\left(\Theta'-\frac{\pi}{3}\right),$$

where

$$\Theta'=\frac{1}{3}\text{Arccos}[3M(v)\Lambda^{1/2}]. \tag{A8}$$

If $\dot{r}_-\ll 1, \dot{r}_+\ll 1, \dot{r}_C\ll 1$, from Eqs. (A2) and (A7) we obtain

$$r'_-\simeq r_-, r'_+\simeq r_+, r'_C\simeq r_C. \tag{A9}$$

Using Eqs. (A9), (A4), and (A5), we obtain

$$\begin{aligned}(r_+-r'_-)^2(r_+-r'_C)^2&\simeq(r_+-r_-)^2(r_+-r_C)^2\\&=\frac{1}{r_+^2}[r_+^3-(r_++r_-)r_+^2+r_+r_Cr_-]^2\\&=\frac{1}{r_+^2}\left[2r_+^3-\frac{6M(v)}{\Lambda}\right]^2\\&=\frac{4}{r_+^2}\left[r_+^3-\frac{3M(v)}{\Lambda}\right]^2.\end{aligned} \tag{A10}$$

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