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Excited states NLIE for sine-Gordon model in a strip with Dirichlet boundary conditions

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Abstract

We investigate various excited states of sine-Gordon model on a strip with Dirichlet boundary conditions on both boundaries using a non-linear integral equation (NLIE) approach.

1. Introduction

Consider the quantum field theory of a self-interacting scalar boson $\phi(x,t)$ on a $(1+1)$-dimensional strip, infinite in time direction and finite in space with bulk action

$$\mathcal{A}_{\mathrm{DSG}} = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}t \int_0^L \mathrm{d}x \left[(\partial_t\phi)^2 - (\partial_x\phi)^2 + \frac{m_0^2}{\beta^2}\cos\beta\phi \right], \tag{1}$$

and with the Dirichlet boundary conditions $\phi(0,t) \equiv \phi_- + \frac{2\pi}{\beta}m_-$ and $\phi(L,t) = \phi_+ + \frac{2\pi}{\beta}m_+$, $m_\pm \in \mathbb{Z}$. We shall refer to this integrable theory as the Dirichlet sine-Gordon (DSG) model. It has several important applications ranging from condensed matter physics to string theory.

The well-known bulk particle spectrum of sine-Gordon, composed of solitons and antisolitons with topological charge 1 and -1, respectively, and, only in the *attractive* regime $0 < \beta \leqslant \sqrt{4\pi}$, bound states of solitons known as breathers, is complemented by the rich structure of boundary bound states described in [1]. The expressions for the bulk S-matrices can be found in [2], those for the fundamental soliton or antisoliton reflection matrices are given in [3] and the general boundary bound state excited reflection matrices can be found in [1].

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An important feature of DSG is the conservation of the topological charge

$$Q \equiv \frac{\beta}{2\pi}\left[\int_0^L dx \frac{\partial}{\partial x}\phi(x,t) - \phi_+ + \phi_-\right] = m_+ - m_- \in \mathbb{Z}. \tag{2}$$

The model enjoys the discrete symmetry of the field $\phi \to \phi + \frac{2\pi}{\beta}m$ and simultaneously $\phi_\pm \to \phi_\pm + \frac{2\pi}{\beta}m$ ($m \in \mathbb{Z}$). The charge conjugation symmetry $\phi \to -\phi$ sending solitons into antisolitons is also guaranteed provided $\phi_\pm \to -\phi_\pm$ simultaneously. It sends $Q \to -Q$, so one can restrict attention to the study of positive Q and then act with this transformation to obtain states of negative Q. The periodicity allows to restrict the boundary parameters to the range $0 \leqslant \phi_\pm < \frac{2\pi}{\beta}$.[1]

A problem of great interest is to connect the scattering theory approach just mentioned to the somewhat complementary description of perturbed conformal field theory data. In this Letter we attack this problem from the point of view of the nonlinear integral equation (NLIE) approach which was developed for vacuum scaling functions in [4–6] and extended later to the excited states [7–11]. In this framework, exact scaling functions of finite size effects provide a way to investigate the flows from ultraviolet (UV) to infrared (IR) scales in integrable quantum field theory, hence building a bridge between the perturbed CFT description and the factorized scattering one. The NLIE for the vacuum Casimir energy was already deduced, along a similar line to those we shall illustrate here, some years ago in [12]. Our interest here is to establish a general NLIE valid for all states in the finite size spectrum. This allows to identify "particle" states in the scattering description with "conformal" states in the perturbed CFT description.

To make this approach viable, one should start from an exact solution of a lattice regularization of the model. This is normally provided, in NLIE approach, by some 2D light-cone vertex model or equivalently by some inhomogeneous 1D spin chain. By suitable scaling limit the continuum renormalized theory can be reached. The Bethe ansatz equations that solve the lattice model are turned in the continuum limit into a NLIE that has to be considered as the basic renormalized tool to calculate the eigenvalues of the integral of motions of the theory on any state in finite volume.

The homogeneous antiferromagnetic XXZ spin-1/2 model in a chain of N sites with lattice spacing a, coupled to parallel magnetic fields h_+ and h_- at the left and right boundaries, respectively, has Hamiltonian

$$\mathcal{H}(\gamma, h_+, h_-) = -J\sum_{n=1}^{N-1}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \cos\gamma\,\sigma_n^z\sigma_{n+1}^z\right) + h_-\sigma_1^z + h_+\sigma_N^z. \tag{3}$$

Here σ_n^α, $\alpha = x, y, z$, are Pauli matrices and $0 \leqslant \gamma < \pi$. Where convenient, we shall also use the equivalent parameter $p = \frac{\pi}{\gamma} - 1$, whose range is $0 < p < \infty$. The Hamiltonian (3), as well as its inhomogeneous generalizations, can be constructed in a double row transfer matrix framework. For details see [13].

The bare continuum limit $N \to \infty$, $a \to 0$ while $Na = L$ remains fixed, is known to give, in the homogeneous case (3) a massless free boson $\phi(x)$ compactified on a circle of radius $R = 1/\sqrt{2(\pi - \gamma)}$ [14]. Among the many possible deformations of the Hamiltonian (3) leading to sine-Gordon in the bare continuum limit, we choose the one introducing alternating inhomogeneities $\vartheta_n = (-1)^n\Theta$ in the sites of the chain. This choice has been known for sometime [15,16] to give a correct construction of sine-Gordon model in the bulk in cylindrical geometry, when the appropriate scaling limit is chosen, with periodic or twisted boundary conditions. It is then natural to expect that the same construction in the presence of boundary magnetic fields $h_\pm$ can also provide an effective tool to define the renormalized DSG theory. It is worthwhile to recall that, as the homogeneous XXZ chain is equivalent to a 6-vertex model on a square lattice, this modified XXZ chain is also equivalent to a 6-vertex model, but—as a

[1] Notice that, unlike in the single boundary case of [1,3,19], in the present two boundaries model we cannot further restrict both boundary parameters to $0 \leqslant \phi_\pm < \frac{\pi}{\beta}$.

consequence of the introduced inhomogeneities—defined on a lattice rotated by 45, i.e., on what can be thought as a Minkowski space discretized along the light-cone directions. This is why this construction is often referred as *light cone lattice* construction of the sine-Gordon model [15]. For $\Theta \to \infty$, $N \to \infty$ and $a \to 0$ while $L = Na$ is fixed, contact can be made, along lines similar to those in [17], with the Lagrangean formulation of DSG model, Eq. (1). The XXZ anisotropy γ is related to the SG coupling β by $\beta^2 = 8(\pi - \gamma) = \frac{8\pi p}{p+1}$. Details of this bare continuum limit are out of the scope of the present Letter.

In this preliminary Letter we put our accent on the presentation of the NLIE got from this construction and analyze only a few bulk excited states with nonexcited boundaries, in order to show consistency with expected results. The careful treatment of the full situation with boundary bound states would involve more delicate issues of analytic continuation in the boundary parameters $\phi_\pm$ that we choose to postpone to a more extensive forthcoming publication [31] for the reasons explained at the end of Section 2.

2. Bethe ansatz and NLIE

The Bethe ansatz equations for the boundary XXZ chain (3) have been written by Alcaraz et al. [18] and Sklyanin [13] some years ago, using an algebraic approach. It is straightforward to generalize them with the introduction of the alternating inhomogeneities [19].

Eigenvalues of the double row transfer matrix can be constructed in terms of sets of distinct numbers $\vartheta_1, \ldots, \vartheta_M$ called *roots.* They are in number of M ($M \leqslant N$) and must satisfy the Bethe ansatz equations

$$\left[s_{1/2}(\vartheta_j + \Theta)s_{1/2}(\vartheta_j - \Theta)\right]^N s_{H_+/2}(\vartheta_j)s_{H_-/2}(\vartheta_j) = \prod_{k=1,k\neq j}^{M} s_1(\vartheta_j - \vartheta_k)s_1(\vartheta_j + \vartheta_k), \tag{4}$$

where

$$s_\nu(x) = \frac{\sinh\frac{\gamma}{\pi}(x + i\nu\pi)}{\sinh\frac{\gamma}{\pi}(x - i\nu\pi)},$$

and $H_\pm$ is defined such that $h_\pm = \sin\gamma \cot\frac{\gamma}{2}(H_\pm + 1)$, and we choose as fundamental region $-p-1 < H_\pm < p+1$. Notice that the boundary terms in the Bethe equations disappear when $H_\pm = 0$, i.e., $h_\pm = 1 + \cos\gamma \equiv h_c$. In such case, as it was shown in [20,21], the system becomes $SL_q(2)$-invariant.

The antiferromagnetic vacuum turns out to be a maximal set $M = N/2$ of real roots and it exists for N even only. In the region $0 \leqslant \gamma < \pi$ of interest for us, and for small enough boundary magnetic fields, this is the true ground state of the theory. For N odd instead the states with lowest possible total spin have $M = \frac{N-1}{2}$ roots and one hole. However, to deal correctly with the continuum limit one has to consider *both* N even and odd sectors, like it was shown in the periodic case in [9–11]. The symmetry of (4) $\{\vartheta_j\} \to \{-\vartheta_j\}$, evident from the Bethe equations, implies that only roots with positive real part are independent parameters characterizing a Bethe state. The value $\vartheta_j = 0$ is a solution of (4) for any N and M. However, the corresponding Bethe state would vanish, so one has always to subtract this unwanted root, i.e., to create a hole at $\vartheta = 0$.

The domain of root distribution can be considered as a semistrip $\mathbb{U}_+$ of the complex ϑ plane:

$$\mathbb{U}_+ = \left\{\vartheta \in \mathbb{C} \,\middle|\, \operatorname{Re}\vartheta > 0,\ -\frac{\pi^2}{2\gamma} < \operatorname{Im}\vartheta \leqslant \frac{\pi^2}{2\gamma} \quad \text{or} \quad \operatorname{Re}\vartheta = 0,\ 0 < \operatorname{Im}\vartheta < \frac{\pi^2}{2\gamma}\right\}.$$

This also excludes another unwanted root at $i\frac{\pi^2}{2\gamma}$ and considers only half of the imaginary axis, as it should for symmetry. However, for computational purposes, it is often better to double this strip by mirroring all the roots

$$\mathbb{U} = \left\{\vartheta \in \mathbb{C} \,\middle|\, \operatorname{Re}\vartheta \in \mathbb{R},\ -\frac{\pi^2}{2\gamma} < \operatorname{Im}\vartheta \leqslant \frac{\pi^2}{2\gamma}\right\}.$$

To each root ϑ_j associate its mirror root $\vartheta_{-j} \equiv -\vartheta_j$. Define the function

$$\varphi_\nu(\vartheta) \equiv \pi + i \log s_\nu(\vartheta), \tag{5}$$

with the oddity condition $\varphi_\nu(-u) = -\varphi_\nu(u)$ fixing the fundamental branch of the logarithm. It is periodic in the imaginary direction, with period $i\frac{\pi^2}{\gamma}$, and real on the real axis. We choose as fundamental periodicity the strip $\vartheta \in \mathbb{U}$. Singularities of this function are distributed along the imaginary axis:

$$\mathrm{Re}\,\vartheta = 0, \qquad \mathrm{Im}\,\vartheta = \pm\pi\big(k(p+1) - \nu\big), \quad k \in \mathbb{Z},$$

so that the fundamental analyticity strip is limited to $|\mathrm{Im}\,\vartheta| < \pi \min(\nu, p+1-\nu)$.

In terms of the function (5) the logarithm of the Bethe equations (4) can be expressed as

$$N\big[\varphi_{1/2}(\vartheta_j + \Theta) + \varphi_{1/2}(\vartheta_j - \Theta)\big] + \varphi_{H_+/2}(\vartheta_j) + \varphi_{H_-/2}(\vartheta_j) + \varphi_1(\vartheta_j) + \varphi_1(2\vartheta_j) - \sum_{k=1}^{2M} \varphi_1(\vartheta_j - \vartheta_k) = 2\pi I_j, \quad I_j \in \mathbb{Z}.$$

Eigenvalues of the transfer matrix can be expressed in terms of its roots. In the following, we shall be mainly interested in the energy spectrum, for which the formula is

$$E = -\frac{1}{2a}\sum_{k=1}^{M}\left(\frac{\mathrm{d}}{\mathrm{d}\Theta}\varphi_{1/2}(\Theta - \vartheta_k) + \frac{\mathrm{d}}{\mathrm{d}\Theta}\varphi_{1/2}(\Theta + \vartheta_k)\right) + \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}\Theta}\varphi_{1/2}(\Theta). \tag{6}$$

Define for $\vartheta \in \mathbb{U}$ the so-called *counting function*

$$Z_N(\vartheta) = N\big[\varphi_{1/2}(\vartheta + \Theta) + \varphi_{1/2}(\vartheta - \Theta)\big] + \varphi_{H_+/2}(\vartheta) + \varphi_{H_-/2}(\vartheta) - \sum_{k=-M}^{M} \varphi_1(\vartheta - \vartheta_k) + \varphi_1(\vartheta) + \varphi_1(2\vartheta) \tag{7}$$

in terms of which the logarithm of the Bethe equations simply becomes the condition

$$Z_N(\vartheta_j) = 2\pi I_j. \tag{8}$$

The last term in (7) takes care of the fact that in the second member of (4) the product does not include factors with $k = j$. The last but one instead explicitly subtracts the unwanted root $\vartheta_0 = 0$. The integers I_j play the role of quantum numbers.

The analytic structure of $Z_N(\vartheta)$ makes it convenient to classify the roots and related objects of the Bethe ansatz (4) as follows

1. *Real roots* ϑ_k, $k = 1, \ldots, M_R$, $\vartheta_k > 0$: they are strictly positive real solutions of (4) and (8);
2. *Holes* ϑ_k, $k = 1, \ldots, N_H$, $\vartheta_k > 0$: strictly positive real solutions of (8) that are not solutions of the Bethe ansatz (4);
3. *Close roots* ϑ_k, $k = 1, \ldots, M_C$: complex solutions with $\mathrm{Re}\,\vartheta_k \geqslant 0$ and imaginary part in the range $0 < |\mathrm{Im}\,\vartheta_k| < \min \pi(1, p)$;
4. *Wide roots* ϑ_k, $k = 1, \ldots, M_W$, $\mathrm{Re}\,\vartheta_k \geqslant 0$: complex conjugate solutions with imaginary part $\pi \min(1, p) < |\mathrm{Im}\, w_k| < \frac{\pi^2}{2\gamma}$.

For further convenience it is useful to introduce the notion of *self-conjugate* roots, i.e., wide roots with $\mathrm{Im}\,\vartheta_k = \frac{\pi^2}{2\gamma}$, whose complex conjugate is the root itself, due to the periodicity of Bethe equations. Their number will be indicated as M_{SC}. Also we shall refer to roots or holes lying on the imaginary axis as *magnetic*.

The function $Z_N(\vartheta)$ for $\vartheta \in \mathbb{R}$ is globally monotonically increasing. However, there may be points where locally $\dot{Z}_N(\vartheta) < 0$. In particular holes or roots s_j such that $\dot{Z}_N(s_j) < 0$ are called *special* holes or roots. If a special object appears, then there must be also two other objects (real roots or holes) with the same quantum number, as imposed by the global increasing monotonicity of the counting function. Moreover, as two roots with the same quantum number are not allowed in Bethe ansatz, at maximum one object of this triple can be a root, the others are forced to be holes. In the following we indicate the number of specials with N_S. A special object should be counted both as special (i.e., in N_S) and as root or hole (i.e., in M_R or N_H) according to its nature.

One may relate the numbers of various types of roots to the 3rd component of total spin of the system $S = \frac{N}{2} - M$. To do that, we express the asymptotics of the function $Z_N(\vartheta)$ on the real axis of ϑ where they can be compared with the counting of real roots and holes. As a result we get the following *counting equation* to be satisfied by any allowed root configuration

$$N_H - 2N_S = 2S + M_C + 2\,\mathrm{step}(p-1)M_W + \mathrm{step}(p-1) + \left\lfloor -\frac{2S}{p+1} - \frac{H}{p+1} \right\rfloor, \tag{9}$$

where $\lfloor x \rfloor$ denotes the integer part of x and $H = \frac{H_+ + H_-}{2}$.

Following the standard derivation as illustrated in [8,10,22], we obtain a NLIE for the function $Z_N(\vartheta)$. The continuum limit can be taken by sending $N \to \infty$ and $a \to 0$ in such a way that $L = Na$ remains constant. The only way to get a sensible NLIE satisfied by the limiting counting function $Z(\vartheta) \equiv \lim_{N\to\infty} Z_N(\vartheta)$ is to admit that also Θ rescales as

$$\Theta \sim \log \frac{2N}{\mathcal{M}L},$$

where $\mathcal{M}$ is a mass scale. We often use in the following the dimensionless scale parameter $l = \mathcal{M}L$. As a result of this limit procedure, one can define the NLIE on the continuum

$$Z(\vartheta) = 2l \sinh\vartheta + g\big(\vartheta \big| \{\vartheta_k\}\big) + P(\vartheta|H_+, H_-) - 2i\,\mathrm{Im} \int \mathrm{d}x\, G(\vartheta - x - i\varepsilon) \log\big[1 - (-1)^{M_{\mathrm{SC}}} e^{iZ(x+i\varepsilon)}\big],$$

where the boundary contribution is given by

$$P(\vartheta|H_+, H_-) = 2\pi \int_0^{\vartheta} \mathrm{d}x \left[F(x, H_+) + F(x, H_-) + G(x) + J(x)\right]$$

with

$$G(\vartheta) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} e^{ik\vartheta} \frac{\sinh\frac{\pi}{2}(p-1)k}{2\sinh\frac{\pi}{2}pk \cosh\frac{\pi}{2}k} \quad \text{for } |\mathrm{Im}\,\vartheta| < \pi \min(1, p), \tag{10}$$

$$J(\vartheta) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} e^{ik\vartheta} \frac{\sinh\frac{\pi}{4}(p-1)k \cosh\frac{\pi}{4}(p+1)k}{\sinh\frac{\pi}{2}pk \cosh\frac{\pi}{2}k} \quad \text{for } |\mathrm{Im}\,\vartheta| < \frac{\pi}{2}\min(1, p),$$

$$F(\vartheta, H) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} e^{ik\vartheta}\, \mathrm{sign}(H) \frac{\sinh\frac{\pi}{2}(p+1-|H|)k}{\sinh\frac{\pi}{2}pk \cosh\frac{\pi}{2}k} \quad \text{for } |\mathrm{Im}\,\vartheta| < \frac{\pi}{2}|H|. \tag{11}$$

The source term is given by

$$g\big(\vartheta \big| \{\vartheta_k\}\big) \equiv \sum_k c_k \big[\chi_{(k)}(\vartheta - \vartheta_k) + \chi_{(k)}(\vartheta + \vartheta_k)\big],$$

where

$$\chi(\vartheta) = 2\pi \int_0^{\vartheta} \mathrm{d}x\, G(x), \tag{12}$$

and $\{\vartheta_k\}$ is the set of position of the various objects (holes, close and wide roots, specials) characterizing a certain state. They are characterized by the quantization rule

$$Z_N(\vartheta_j) = 2\pi I_j, \quad I_j \in \mathbb{Z} + \frac{\rho}{2}, \quad \rho = M_{\mathrm{SC}} \bmod 2.$$

The coefficients c_k are given by

$$c_k = \begin{cases} +1 & \text{for holes,} \\ -1 & \text{for all other objects,} \end{cases}$$

and for any function $f(\vartheta)$ we define

$$f_{(k)}(\vartheta) = \begin{cases} f_{\mathrm{II}}(\vartheta) & \text{for wide roots,} \\ f(\vartheta + i\varepsilon) + f(\vartheta - i\varepsilon) & \text{for specials,} \\ f(\vartheta) & \text{for all other objects,} \end{cases}$$

where the *second determination* of $f(\vartheta)$ is defined as

$$f_{\mathrm{II}}(\vartheta) = \begin{cases} f(\vartheta) + f(\vartheta - i\pi\, \mathrm{sign}\, \mathrm{Im}\, \vartheta) & \text{if } p > 1 \\ f(\vartheta) - f(\vartheta - i\pi p\, \mathrm{sign}\, \mathrm{Im}\, \vartheta) & \text{if } p < 1 \end{cases} \quad \text{for } |\mathrm{Im}\, \vartheta| > \pi \min(1, p).$$

The contribution of special objects comes from the fact that the logarithmic term inside the integral can go off the fundamental branch right when $\dot{Z}_N < 0$. In this case, the contribution of the jump in the logarithm amounts exactly to the source term of a special object. For large l where the driving term dominates, monotonicity excludes the presence of special holes or roots. Therefore, they should not be regarded as objects that can be added at will, but better as artefacts that appear only for relatively small values of l, dictated by the breakdown of analyticity of the equation at certain points. It is also clear from analysis done in [8,10,22,23] that the number $N_H - 2N_s$ remains constant along a flow in l, and equals N_H for l sufficiently large.

For the vacuum state containing real roots only, this equation coincides with the one found some years ago in [12]. Once the equation is solved for $Z(\vartheta + i\varepsilon)$ one can use this result to compute the $Z(\vartheta)$ function at any value in the analyticity strip $|\mathrm{Im}\, \vartheta| < \pi \min(1, p)$, provided the function $P(\vartheta|H_+, H_-)$ is well defined there (see comments below). To extend the function outside this analyticity strip one has to resort to the following modification of the NLIE

$$Z(\vartheta) = 2l \sinh_{\mathrm{II}} \vartheta + g_{\mathrm{II}}\big(\vartheta \big| \{\vartheta_k\}\big) + P_{\mathrm{II}}(\vartheta|H_+, H_-) - 2i\, \mathrm{Im} \int \mathrm{d}x\, G_{\mathrm{II}}(\vartheta - x - i\varepsilon) \log\big[1 - (-1)^{M_{\mathrm{SC}}} e^{iZ(x+i\varepsilon)}\big].$$

Once $Z(\vartheta)$ is known, it can be used to compute the energy. It is composed of bulk and boundary terms whose expression can be found in [12] and a Casimir energy scaling function given by

$$E = \mathcal{M} \sum_k c_k \cosh_{(k)} \vartheta_k - \mathcal{M} \int \frac{\mathrm{d}x}{2\pi} \sinh x\, Q(x).$$

In the IR limit $l = \mathcal{M}L \to \infty$ one can make contact between the NLIE excited states and the scattering theory of an underlying field theory. The integral terms in the NLIE and in the energy formula go as $O(e^{-l})$ and can be discarded. In the case of a single hole placed at ϑ_1, the NLIE becomes

$$Z(\vartheta_1) = 2l \sinh \vartheta_1 + \chi(2\vartheta_1) + P(\vartheta_1|H_+, H_-) = 2l \sinh \vartheta_1 + \mathcal{F}(\vartheta_1, H_+) + \mathcal{F}(\vartheta_1, H_-)$$

with

$$\mathcal{F}(\vartheta, H) = 2\pi \int_0^{\vartheta} dx \left[F(x, H) + \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \frac{\sinh \frac{3\pi}{2} k \sinh \frac{\pi}{2}(p-1)k}{\sinh \frac{\pi}{4} pk \sinh \pi k} \right].$$

Interpreted, along lines of analysis very similar to those of [10,22], as a quantization rule for momentum of a particle of energy $\mathcal{M}\cosh\vartheta_1$, this yields its reflection amplitudes at both boundaries, given by $\mathcal{R}(\vartheta, \xi_\pm) = e^{i\mathcal{F}(\vartheta, H_\pm)}$. Known identities [12] allow to identify such reflection matrix with the Ghoshal–Zamolodchikov soliton–soliton one, upon suitable identification of the parameters $H_\pm$ and $\xi_\pm$. $\xi_\pm$ in turn are connected to the DSG boundary parameters $\phi_\pm$ [3] thus allowing us to relate $H_\pm$ and $\phi_\pm$ as

$$H_\pm = p\left(1 \mp \frac{8}{\beta}\phi_\pm\right).$$

Notice that the periodicity $\phi_\pm \to \phi_\pm + \frac{2\pi}{\beta}$ reflects in the periodicity $H_\pm \to H_\pm \pm 2(p+1)$. By changing sign to both magnetic fields $h_\pm \to -h_\pm$ simultaneously, one can check that also the antisoliton reflection matrix is correctly reproduced.

We also checked, along similar lines, that by considering suitable combinations of holes and non-magnetic complex roots we reproduce the correct bulk S-matrix as well as the expected Ghoshal–Zamolodchikov reflection matrices in some simple multiparticle states.

Our analysis is limited to the case of boundary magnetic fields $|h_\pm| < h_c = 1 + \cos\gamma$ so that there is no way to accommodate any magnetic root. This corresponds to absence of boundary bound states, as observed in [19]. All bulk states then scatter through the unexcited Ghoshal–Zamolodchikov reflection matrix. To go beyond this h_c limitation, one should modify the Fourier representation of F, i.e., consider analytic continuation. The hope is that such analytic continuation should naturally introduce source terms for magnetic roots in $g(\vartheta|\{\vartheta_k\})$ in accordance to Saleur–Skorik analysis of Bethe equations, where they found that vacuum changes by adding the maximal string of magnetic roots. Boundary bound states should then be obtained by removing some of these magnetic roots, thus creating holes on the imaginary axis. This procedure turns out quite cumbersome and delicate, and needs more investigation, so we choose to postpone the treatment of boundary bound states to a future more detailed paper [31]. In the IR limit, however, by dropping the convolution term in NLIE, it is easy to check that one can reproduce the Mattson–Dorey excited reflection matrices.

3. UV limit and conformal theory

In the limit $l \to 0$ we make contact with the UV regime of DSG. The roots and holes may rescale to infinity or stay in a finite region as

$$\vartheta = \vartheta^\pm \pm \log\frac{1}{l} \quad \text{or} \quad \vartheta = \vartheta^0.$$

Therefore, we classify these into three types which will be denoted by indices "±, 0". Accordingly, the NLIE splits into left and right kink equations. Standard manipulations [8,22] lead to the following energy formula

$$E(L) \sim \frac{\pi}{L}\left(\Delta - \frac{1}{24}\right)$$

with

$$\Delta = \frac{p}{p+1}\left[\frac{1}{2}\left(\frac{H_+ + H_-}{2p} - 1 - 2\left(S - S^+\right) - 2\left(K - K_W^+\right)\right) + \frac{p+1}{2p}S\right]^2 + N,$$
$$N = \left(I_H^+ - I_C^+ - I_W^+ - 2I_S^+ + L_W^+ S^+ + 2K - 2K_W^+ - S^+ - 2\left(S^+\right)^2\right) \in \mathbb{Z}, \tag{13}$$

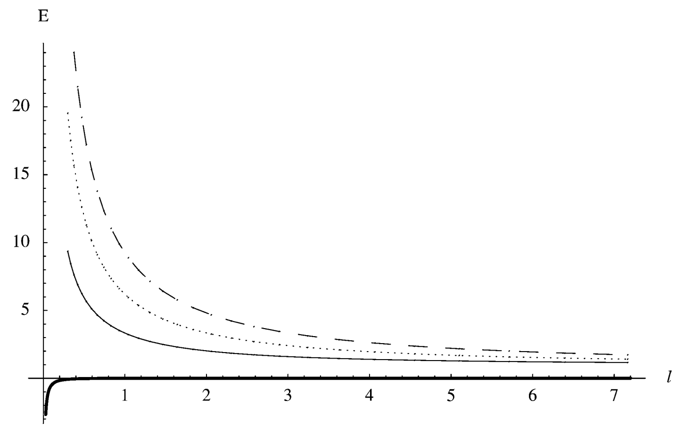

Fig. 1. Casimir energy levels for the vacuum state (thick line) and soliton states with quantum number $I = 1$ (filled line), $I = 2$ (dotted line), $I = 3$ (dotted-dashed line). In this example $p = 2.3$, $H_+ = 2.2$ and $H_- = 2.4$.

where $I_A^+ = \sum_k I_{A,k}^+$ with $A = H, C, W, \ldots$ the various types of roots and holes, and

$$2S^+ = N_H^+ - 2N_S^+ - M_C^+ - 2M_W^+ \operatorname{step}(p-1), \qquad L_W^+ = \operatorname{sign}(p-1)\big(M_W - M_W^+\big).$$

The integers K and K_W^+ are defined through the equations

$$Z_+(-\infty) = 2\operatorname{Im}\log\big[1 - (-1)^{M_{\mathrm{SC}}} e^{iZ_+(-\infty)}\big] + \pi + 2\pi K,$$
$$g_+\big(-\infty\big|\big\{\vartheta_k^+\big\}\big) = 2\chi(\infty)\big(S - 2S^+\big) + 2\pi K_W^+.$$

This result should be compared with $c = 1$ CFT of a boson with Dirichlet conformal boundary conditions which is compactified on a circle of radius $R = \frac{\sqrt{4\pi}}{\beta}$ [24,25]. The Hilbert space is composed of Heisenberg algebra representations $\mathcal{Q}_m$ of $c = 1$ CFT whose primary states $|m\rangle$ are created from the vacuum by vertex operators $:e^{i\kappa\phi}:$ of $U(1)$ charge

$$\kappa = \frac{1}{2}\left(\frac{\phi_+ - \phi_-}{\sqrt{\pi}} + \frac{1}{2} m R\right)$$

and conformal dimensions $\Delta_m = 2\kappa^2$. All other states are created from these ones by applying repeatedly the creation operators

$$\mathcal{Q}_m = \big\{a_{-k_1} \cdots a_{-k_p}|m\rangle, k_1, \ldots, k_p \in \mathbb{Z}_+\big\}.$$

For a generic state $|i\rangle \in \mathcal{Q}_m$ the energy is given by

$$E_i = \frac{\pi}{L}\left(\Delta_m + N_i - \frac{1}{24}\right), \qquad N_i = \sum_{j=1}^{p} k_p \in \mathbb{Z}_+.$$

Comparing this formula with Eq. (13) we see that the winding number m is related to the XXZ spin by $m = 2S$. Notice that, according to this formula, the ground state ($m = 0$) with Dirichlet boundary condition has not conformal dimension 0 as in the periodic CFT: the nontrivial boundaries contribute some energy to the Casimir effect.

We end this section by presenting, only in graphical form (Fig. 1), a simple example of numerical integration of NLIE for the vacuum and few solitonic excited states. A detailed numerical comparison of these and other numerical data with a truncated conformal space approach is planned in [31].

4. Conclusions and perspectives

In this Letter we have obtained the NLIE governing the finite size effects for excited states in sine-Gordon field theory with two boundaries, each with an independent Dirichlet boundary condition as a continuum limit of the Bethe ansatz equations of the alternating inhomogeneous XXZ spin chain. Analysis of the IR and UV limits gives strong evidence that the underlying model is actually the DSG theory. Understanding of states describing the scattering of bulk solitonic particles with Ghoshal–Zamolodchikov unexcited boundaries are under control.

A better understanding of the generation mechanism of boundary bound states for $|h_{\pm}| > h_c$ as analytic continuation of the NLIE due to the singularities of the boundary source terms F, and of the general structure of vacuum in this case, could lead to a full control of the excited boundary situations too, which represents an achievement of crucial importance in the framework of NLIE approach.

Also, other bulk states should be analyzed more carefully, as breather scattering off unexcited or excited boundaries. In this preliminary study we did not perform numerical analysis of NLIE. It should, however, be very valuable in itself and even better if comparable with suitable truncated conformal space approach data.

Finally the understanding of this Dirichlet boundary conditions case should be seen just as a step towards the full investigation of NLIE for general integrable boundary conditions in sine-Gordon model, whose route has been recently opened by the deduction of vacuum low magnetic field NLIE in [26] starting from the Bethe ansatz for non-diagonal boundary conditions proposed and studied in [27–30].

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