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INCLUSION PAIRS SATISFYING ESHELBY'S UNIFORMITY PROPERTY*

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Abstract. Eshelby conjectured that if for a given uniform loading the field inside an inclusion is uniform, then the inclusion must be an ellipse or an ellipsoid. This conjecture has been proved to be true in two and three dimensions provided that the inclusion is simply connected. In this paper we provide an alternative proof of Cherepanov's result that an inclusion with two components can be constructed inside which the field is uniform for any given uniform loading for two-dimensional conductivity or for antiplane elasticity. For planar elasticity, we show that the field inside the inclusion pair is uniform for certain loadings and *not* for others. We also show that the polarization tensor associated with the inclusion pair lies on the lower Hashin–Shtrikman bound, and hence the conjecture of Pólya and Szegö is not true among nonsimply connected inclusions. As a consequence, we construct a simply connected inclusion, which is nothing close to an ellipse, but in which the field is almost uniform.

Key words. Eshelby's conjecture, Pólya–Szegö conjecture, uniformity property, inclusions with multiple components, polarization tensor, Weierstrass zeta function

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1. Introduction.

Consider a conducting or elastic inclusion subject to a uniform applied field. For certain shapes of inclusions the field inside the inclusion is also uniform, and if this is the case, we say the inclusion has *Eshelby's uniformity property.* Eshelby showed in [9] that ellipses and ellipsoids have the uniformity property and conjectured in [10] that these are the only inclusions with the uniformity property. See also [15]. This conjecture of Eshelby has been proved to be true within the class of simply connected inclusions by Sendeckyj [28] for planar elasticity and by Ru and Schiavone [26] for antiplane elasticity or, equivalently, for two-dimensional conductivity. Recently, a completely different proof of the Eshelby conjecture in two dimensions based on the hodographic transformation was given by Kang and Milton [18]. In the same paper, Eshelby's conjecture in three dimensions was resolved as well. They showed that if a simply connected inclusion with Lipschitz boundary has the uniformity property, then the inclusion must take the shape of an ellipse or an ellipsoid. Independently, Liu (private communication) also established this. As a consequence of Eshelby's conjecture, the conjecture of Pólya and Szegö [25], which asserts that the domain whose polarization tensor has the minimal trace is a disk or a ball, is also proved [17].

Finding a structure inside which the field is uniform is important in the study of composite materials since such a property is required in order to reduce the internal stress of the structure [31]. In fact, it was proved by Grabovsky and Kohn [12] that

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ellipses are the low volume fraction limit of the periodic Vigdergauz microstructure [29, 30], which contains a single inclusion per unit cell. The Vigdergauz microstructure is known to have minimal internal stress among periodic composites. There are also periodic geometries, based on the construction of Hashin [14], that contain a countable number of disks in the unit cell, having Eshelby's uniformity property as follows from section 4 of [6].

In this paper we continue our investigation on the shape of inclusions with the uniformity property. The primary concern of this paper is the construction of inclusions (structures) with two components having smooth boundaries which satisfy Eshelby's uniformity property. This was first solved by Cherepanov [8], and here we provide an alternative proof of Cherepanov's results and give explicit numerical computations of the inclusion shapes.

Another closely related question considered here is whether Eshelby's conjecture is true in a "practical" sense: If the field inside the inclusion is very close to being uniform in some sense, does it follow that the inclusion is very close to an ellipse? (By close to an ellipse we specifically mean that the symmetric difference of the inclusion and an ellipse has small measure.) It is a question of stability.

We construct, in a mathematically rigorous way, inclusions with two components inside which the field is uniform. Figure 2.1 in section 2 shows typical shapes of the inclusion pair. The field inside the inclusion is uniform for any uniform loading in the case of antiplane elasticity, as will be proved in section 3. In the case of linear elasticity, the field is uniform for certain loadings and *not* for other loadings. Using these inclusions, we also answer the question of stability. If we connect two components of the inclusion by a thin bridge, as in Figure 5.1, the field does not change much while the bridged inclusion is simply connected, but far from the shape of an ellipse. In order to construct the structures in this paper, we use the Weierstrass zeta function and the Schwarz–Christoffel formula to solve the free boundary problem. The method of construction in this paper is similar to that of Vigdergauz [29, 30] and Grabovsky and Kohn [12], where the Weierstrass $\mathcal{P}$-function is used to construct the Vigdergauz microstructure.

Eshelby's uniformity property is closely related to the conjecture of Pólya and Szegö on the polarization tensor. In [17] Kang and Milton showed that the polarization tensor satisfies the lower Hashin–Shtrikman bound; then the field inside the inclusion must be uniform, and thus the inclusion is an ellipse provided that it is simply connected. See section 4 for the Hashin–Shtrikman bounds on the polarization tensor. The Pólya–Szegö conjecture follows as an immediate consequence of it. It turns out that the polarization tensor associated with the structure constructed in this paper satisfies the lower Hashin–Shtrikman bound. Therefore, the Pólya–Szegö conjecture does not hold among nonsimply connected inclusions. In the same way as above we are also able to show that stability for the Pólya–Szegö conjecture fails to hold among simply connected inclusions: the bridged inclusion is nothing close to a disk, but the trace of its polarization tensor is very close to being minimal.

This paper is organized as follows: In section 2, we construct inclusions with two components using the Weierstrass zeta function and the Schwarz–Christoffel formula. In section 3, we show that these inclusions enjoy the uniformity property for antiplane elasticity. Section 4 shows that the polarization tensor of the inclusions satisfies the lower Hashin–Shtrikman bound, and hence the Pólya–Szegö conjecture fails to be true among nonsimply connected inclusions. In section 5, we discuss the instability of the uniformity property by connecting the inclusion pair by a thin bridge. In section 6,

we analyze the planar elasticity case. We prove that the field inside the inclusion is uniform for certain types of loadings and then show, by numerical computations, that the field is not uniform for some other types of loadings.

2. Construction of the inclusions. This paper is concerned with a structure consisting of two components, each with a smooth (specifically, Lipschitz) boundary, which satisfy Eshelby's uniformity property in antiplane elasticity or for two-dimensional conductivity. More precisely, we construct an inclusion with two components, B_1 and B_2, such that the solution u to the problem

$$\begin{cases} \nabla \cdot \big(1 + (k-1)\chi(B_1 \cup B_2)\big)\nabla u = 0 & \text{in } \mathbb{R}^2, \\ u(x,y) - a \cdot (x,y) = O(r^{-1}) & \text{as } r \to \infty \end{cases} \tag{2.1}$$

is such that ∇u is constant in $B_1 \cup B_2$. Here $\chi(B_1 \cup B_2)$ is the indicator function of $B_1 \cup B_2$, a is a constant vector representing the direction of the uniform loading, and $r = \sqrt{x^2 + y^2}$. The conductivity coefficient $1 + (k-1)\chi(B_1 \cup B_2)$ in (2.1) indicates that the conductivity of the inclusion $B_1 \cup B_2$ is $k \neq 1$ while that of the background $\mathbb{R}^2 \setminus \overline{B_1 \cup B_2}$ is 1. It is worth mentioning that since ∇u is constant (and not 0) in $B_1 \cup B_2$, ∂B_1 and ∂B_2 are analytic due to a regularity result of Alessandrini and Isakov [2, Corollary 2.2].

In order to construct such inclusions B_1 and B_2, we will construct a holomorphic function f in $\mathbb{C} \setminus \overline{B_1 \cup B_2}$ satisfying

$$f(z) = \Re(cz) + q_j, \quad z \in \partial B_j, \tag{2.2}$$

for some complex constants c and q_j, $j = 1, 2$, and

$$f(z) = \alpha z + O(1) \quad \text{as } |z| \to \infty \tag{2.3}$$

for some complex number α. Here and afterward, we identify z with $x + iy$. Let us first briefly see why it is enough to construct such a function.

Suppose that there are such simply connected inclusions B_1 and B_2, and let u be the solution to (2.1). Let $u^e := u|_{\mathbb{R}^2 \setminus \overline{B_1 \cup B_2}}$ and $u^i := u|_{B_1 \cup B_2}$. Then there exist holomorphic functions U^e in $\mathbb{C} \setminus \overline{B_1 \cup B_2}$ and U^i in $B_1 \cup B_2$ such that $\Re U^e = u^e$ and $\Re U^i = u^i$. To see the existence of U^e, it suffices to note that $\int_C \frac{\partial u^e}{\partial \nu} ds = 0$ for any closed piecewise C^1-curve C in $\mathbb{C} \setminus \overline{B_1 \cup B_2}$, which can be easily verified using Green's theorem. By (2.1), the solution u satisfies the transmission conditions along the interface ∂B_1 and ∂B_2:

$$u|_+ = u|_- \quad \text{and} \quad \frac{\partial u}{\partial \nu}\Big|_+ = k \frac{\partial u}{\partial \nu}\Big|_- \quad \text{on } \partial B_j,\ j = 1, 2, \tag{2.4}$$

where the subscripts $+$ and $-$ denote the limits from outside and inside ∂B_j, respectively. It then follows from the Cauchy–Riemann equation that

$$\frac{k+1}{2} U^i - \frac{k-1}{2} \overline{U^i} = U^e + i\lambda_j \quad \text{on } \partial B_j,\ j = 1, 2, \tag{2.5}$$

for some real constant λ_j. See [16]. Since u^i is linear in each B_j, so is U^i, say,

$$U^i(z) = b_j z + d_j, \quad z \in B_j,\ j = 1, 2.$$

The constancy of ∇u in $B_1 \cup B_2$ implies $b_1 = b_2 (= b)$. Then (2.5) takes the form

$$\frac{k+1}{2}(bz + d_j) - \frac{k-1}{2}(\overline{bz} + \overline{d}_j) = U^e(z) + i\lambda_j \quad \text{on } \partial B_j,\ j = 1, 2. \tag{2.6}$$

If we put

$$f(z) = U^e(z) - kbz, \tag{2.7}$$

then f is holomorphic in $\mathbb{C} \setminus \overline{B_1 \cup B_2}$ and satisfies (2.2) and (2.3). By reversing the previous arguments we see that if B_1 and B_2 admit a holomorphic function satisfying (2.2) and (2.3), then $B_1 \cup B_2$ has the uniformity property.

For the rest of this section we deal with the problem of constructing two inclusions B_1 and B_2 which admit a function f holomorphic in $\mathbb{C} \setminus \overline{B_1 \cup B_2}$ satisfying (2.2) and (2.3). It turns out that this problem was solved by Cherepanov [8] in a more general setting. Cherepanov showed that there are inclusions with an arbitrary number of components which admit a holomorphic function (outside the inclusion) satisfying (2.2) and (2.3), and then constructed such inclusions with single and double components. The construction of this paper is different from that of [8], and it is more elementary using the explicit formula of the Weierstrass zeta function and the Schwarz–Christoffel formula.

Suppose that f is a holomorphic function in $\mathbb{C} \setminus \overline{B_1 \cup B_2}$ satisfying (2.2) and (2.3). Since such an f maps $\mathbb{C} \setminus \overline{B_1 \cup B_2}$ onto the complex plane with two slits, it is natural to construct an appropriate holomorphic function G on the complex plane with two slits and then define f as the hodographic transform (or the inverse) of G. The use of hodographic transforms is a well-known technique for solving free boundary problems.

Let $0 < a < b$ be two fixed real constants and consider the complex plane with two slits $[-b, -a]$ and $[a, b]$. We first construct a holomorphic function F so that its real parts are constant on each slit while its imaginary part vanishes on the other parts of the real axis. Once we construct such a function F, then the desired function G will be defined as $G(z) = F(z) + \alpha z$ for some real constant α, as we shall see later. For the construction of F, we make use of the Weierstrass zeta function and the Schwarz–Christoffel formula.

For given positive real numbers c and d, let $t_1 = 2c$ and $t_2 = i2d$. Then the Weierstrass zeta function $\zeta(w)$ is defined by

$$\zeta(w) := \frac{1}{w} + \sum_{t \neq 0} \left(\frac{1}{w - t} + \frac{1}{t} + \frac{w}{t^2} \right), \tag{2.8}$$

where the sum is over all $t = n_1 t_1 + n_2 t_2$ with integers n_1 and n_2 not both zero. The function ζ has the periodicity properties

$$\zeta(w + t_1) = \zeta(w) + \eta_1, \quad \zeta(w + t_2) = \zeta(w) + i\eta_2, \tag{2.9}$$

where η_1 and η_2 are constants satisfying

$$d\eta_1 - c\eta_2 = \pi. \tag{2.10}$$

See [1]. For each $t = n_1 t_1 + n_2 t_2$, its conjugate $\bar{t} = n_1 t_1 - n_2 t_2$ is on the same lattice of points as t lies on. Thus one can easily see that

$$\overline{\zeta(\bar{w})} = \zeta(w), \tag{2.11}$$

and hence $\zeta(w)$ is real when w is real. We also have

$$\overline{\zeta(-\bar{w})} = -\zeta(w), \tag{2.12}$$

from which it follows that $\zeta(w)$ is purely imaginary when w is purely imaginary. It then follows from (2.9) that η_1 and η_2 are real.

Note that by (2.11)

$$\overline{\zeta(\bar{w} - 2id)} = \overline{\zeta(\overline{w + t_2})} = \zeta(w) + i\eta_2, \tag{2.13}$$

and hence

$$\zeta(w) - \overline{\zeta(\bar{w} - 2id)} = -i\eta_2. \tag{2.14}$$

Thus we deduce that if $w = u - id$ with u real, then

$$\Im\zeta(u - id) = -\frac{\eta_2}{2}. \tag{2.15}$$

Similarly, using the first identity in (2.9) and (2.12), one can see that if $w = -c + iv$ with v real, then

$$\Re\zeta(-c + iv) = -\frac{\eta_1}{2}, \tag{2.16}$$

and if $w = c + iv$ with v real, then

$$\Re\zeta(c + iv) = \frac{\eta_1}{2}. \tag{2.17}$$

We will also need the following lemma.

LEMMA 2.1. *When $d > c$ the following inequality holds:*

$$\Im\zeta(\pm c + iv) \geq \frac{\eta_2}{2d}v, \quad -d < v < 0. \tag{2.18}$$

Proof. Note first that $\Im\zeta(-c + iv) = \Im\zeta(c + iv)$ because of the first identity in (2.9). By scaling we may assume that $2c = 1$. Put $2d = \tau$ to shorten notation, and note that $\tau > 1$. Let

$$h(v) := \Im\zeta(c + iv) - \frac{\eta_2}{2d}v, \quad -\frac{\tau}{2} < v < 0.$$

Since $h(0) = h(-\frac{\tau}{2}) = 0$, it suffices to show that h is concave in $(-\frac{\tau}{2}, 0)$. Observe that

$$h''(v) = -2\Im \sum_{n_1,n_2} \frac{1}{(\frac{1}{2} + i(v - n_2\tau) - n_1)^3}.$$

From the well-known identity (see [1])

$$\sum_{m=-\infty}^{\infty} \frac{1}{(z - m)^2} = \frac{\pi^2}{\sin^2 \pi z},$$

we have

$$\sum_{m=-\infty}^{\infty} \frac{1}{(z - m)^3} = \frac{\pi^3 \cos \pi z}{\sin^3 \pi z}.$$

Therefore, we get

$$\begin{aligned}
h''(v) &= -2\pi^3 \Im \sum_{n_2=-\infty}^{\infty} \frac{\cos\pi\left(\frac{1}{2}+i(v-n_2\tau)\right)}{\sin^3\pi\left(\frac{1}{2}+i(v-n_2\tau)\right)} \\
&= 2\pi^3 \sum_{n=-\infty}^{\infty} \frac{\sinh\pi(v-n\tau)}{\cosh^3\pi(v-n\tau)} \\
&= 2\pi^3 \frac{\sinh\pi v}{\cosh^3\pi v} \\
&\quad + 2\pi^3 \sum_{n=1}^{\infty} \frac{\sinh\pi(v-n\tau)\cosh^3\pi(v+n\tau)+\sinh\pi(v+n\tau)\cosh^3\pi(v-n\tau)}{\cosh^3\pi(v-n\tau)\cosh^3\pi(v+n\tau)}.
\end{aligned}$$

Straightforward but tedious computation yields

$$\begin{aligned}
&\sinh\pi(v-n\tau)\cosh^3\pi(v+n\tau)+\sinh\pi(v+n\tau)\cosh^3\pi(v-n\tau) \\
&= \frac{1}{2}\sinh 2\pi v\left[2+\cosh 2\pi v\cosh 2\pi n\tau-\cosh^2 2\pi n\tau\right],
\end{aligned}$$

and hence

$$h''(v) = 8\pi^3\sinh\pi v\left[\frac{1}{4\cosh^3\pi v}+\sum_{n=1}^{\infty}\frac{2+\cosh 2\pi v\cosh 2\pi n\tau-\cosh^2 2\pi n\tau}{\left(\cosh 2\pi v+\cosh 2\pi n\tau\right)^3}\right].$$

Since $v<0$, it is now enough to show that the quantity inside the bracket, which we call $I(v)$, is positive. Indeed, we have

$$I(v) > \frac{1}{4\cosh^3\pi v}-\sum_{n=1}^{\infty}\frac{1}{\cosh 2\pi v+\cosh 2\pi n\tau}.$$

Since $-\frac{\tau}{2}<v<0$ and $\tau>1$, we now have

$$\begin{aligned}
I(v) &> \frac{1}{4\cosh^3\frac{\pi\tau}{2}}-2\sum_{n=1}^{\infty}e^{-2\pi n\tau} \\
&= \frac{1}{4\cosh^3\frac{\pi\tau}{2}}-\frac{2}{1-e^{-2\pi\tau}}e^{-2\pi\tau} \\
&= 2\left[\left(e^{-\frac{1}{6}\pi\tau}+e^{-\frac{7}{6}\pi\tau}\right)^{-3}-\frac{1}{1-e^{-2\pi\tau}}\right]e^{-2\pi\tau} \\
&> 2\left[\left(e^{-\frac{1}{6}\pi}+e^{-\frac{7}{6}\pi}\right)^{-3}-\frac{1}{1-e^{-2\pi}}\right]e^{-2\pi\tau} > 0.
\end{aligned}$$

This completes the proof. We remark that the inequality is proved not only when $d>c$, but also when τ is such that the second to last line in the above chain of inequalities is positive. ∎

For a positive real number β, define h by

$$h(w) := \beta\left(\zeta(w-id)-\frac{\eta_2}{2d}w+i\frac{\eta_2}{2}\right). \tag{2.19}$$

Then h is a meromorphic function with poles at $2n_1c+2in_2d+id$ and satisfies

$$\begin{cases}
\Re h(-c+iv) = -\beta c_0, \\
\Re h(c+iv) = \beta c_0, \\
\Im h(u) = 0, \\
\Im h(u+id) = 0
\end{cases} \tag{2.20}$$

for u and v real, where

$$c_0 = \frac{\eta_1}{2} - \frac{c\eta_2}{2d} = \frac{\pi}{2d} \tag{2.21}$$

because of (2.10). Since $\beta > 0$, we also have from (2.18)

$$\Im h(\pm c + iy) > 0, \quad 0 < y < d, \quad \text{when } d > c. \tag{2.22}$$

Since $\zeta(w) = \frac{1}{w} + O(1)$ as $w \to 0$, we have

$$h(w) = \frac{\beta}{w - id} + O(1) \quad \text{as } w \to id. \tag{2.23}$$

Restricting our attention to the rectangle $R = \{z = x + iy \mid -c < x < c,\ 0 < y < d\}$, we now construct a conformal mapping from the upper half of the complex plane onto R. To this end, it is natural to use the Schwarz–Christoffel formula.

For $b > a > 0$, let

$$\begin{aligned} g(z) :&= (z^2 - a^2)^{-1/2}(z^2 - b^2)^{-1/2} \\ &= (z+b)^{-1/2}(z+a)^{-1/2}(z-a)^{-1/2}(z-b)^{-1/2}, \end{aligned}$$

and define for z in the upper half plane

$$w = \Phi(z) := -\int_0^z g(\xi)d\xi. \tag{2.24}$$

The mapping Φ maps the upper half plane onto the rectangle $R = \{z = x+iy \mid -c < x < c,\ 0 < y < d\}$, where

$$c = \int_0^a \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} \quad \text{and} \quad d = \int_a^b \frac{dx}{\sqrt{(x^2 - a^2)(b^2 - x^2)}}\ . \tag{2.25}$$

Note that the intervals $[-b, -a]$ and $[a, b]$ on the real axis get mapped onto the vertical sides $\{-c + iy \mid 0 \le y \le d\}$ and $\{c + iy \mid 0 \le y \le d\}$ of R, $[-a, a]$ onto the bottom of R, and $(-\infty, -b) \cup (b, \infty)$ into the top of R. The point ∞ is mapped to $w = id$, and

$$\Phi(z) = id + O\left(\frac{1}{|z|}\right) \quad \text{as } |z| \to \infty. \tag{2.26}$$

To see this, we have

$$\begin{aligned} \Phi(z) &= -\int_0^z g(\xi)d\xi = -\int_0^\infty g(\xi)d\xi + \int_z^\infty g(\xi)d\xi \\ &= id + \int_z^\infty g(\xi)d\xi = id + \int_z^\infty O(|\xi|^{-2})d\xi \\ &= id + O(|z|^{-1}) \end{aligned}$$

as $|z| \to \infty$.

We now define F in the upper half of $\mathbb{C}$ by

$$F(z) := (h \circ \Phi)(z), \tag{2.27}$$

where Φ is defined by (2.24). It then follows from (2.20) that

$$\begin{cases} \Re F(x+i0) = -\beta c_0, & x \in (-b,-a), \\ \Re F(x+i0) = \beta c_0, & x \in (a,b), \\ \Im F(x+i0) = 0, & x \in (-\infty,-b) \cup (b,\infty) \cup (-a,a), \end{cases} \tag{2.28}$$

and from (2.22) that

$$\Im F(x+i0) \geq 0, \quad x \in (-b,-a) \cup (a,b). \tag{2.29}$$

It also follows from (2.23) and (2.26) that

$$F(z) = \beta z + O(1) \quad \text{as } |z| \to \infty. \tag{2.30}$$

Because of (2.28), F has an obvious extension as a holomorphic function in $\mathbb{C} \setminus ([-b,-a] \cup [a,b])$ satisfying

$$\overline{F(\bar{z})} = F(z). \tag{2.31}$$

For a positive real number α, define G by

$$G(z) := F(z) + \alpha z, \tag{2.32}$$

and then define curves C_j^+, $j = 1, 2$, by

$$C_1^+ := \left\{ \lim_{y \to 0+} G(x+iy) \mid -b \leq x \leq -a \right\}, \tag{2.33}$$

$$C_2^+ := \left\{ \lim_{y \to 0+} G(x+iy) \mid a \leq x \leq b \right\}. \tag{2.34}$$

Observe from (2.29) that, at least when $d > c$ and $\beta > 0$, the curves C_j^+ (except the endpoints) lie in the upper half plane and their endpoints lie on the real axis. In fact, the endpoints of C_1^+ are

$$G(-b) = -\beta \frac{\pi}{2d} - \alpha b \quad \text{and} \quad G(-a) = -\beta \frac{\pi}{2d} - \alpha a, \tag{2.35}$$

and those of C_2^+ are

$$G(a) = \beta \frac{\pi}{2d} + \alpha a \quad \text{and} \quad G(b) = \beta \frac{\pi}{2d} + \alpha b. \tag{2.36}$$

The positivity of α is necessary to ensure that $G(b) > G(a)$. We now define C_j^- to be the reflection of C_j^+ about the real axis, i.e.,

$$C_j^- := \{\overline{z} \mid z \in C_j^+\}, \quad j = 1, 2. \tag{2.37}$$

Assuming $d > c$, we then define the domain B_j to be the domain whose boundary is $C_j^\pm$ for $j = 1, 2$. These domains are determined by the choice of the four parameters a, b, α, and β. However if we replace a, b by $k_1 a, k_1 b$, then the corresponding inclusions are just rescaled by the factor k_1. The reason is as follows.

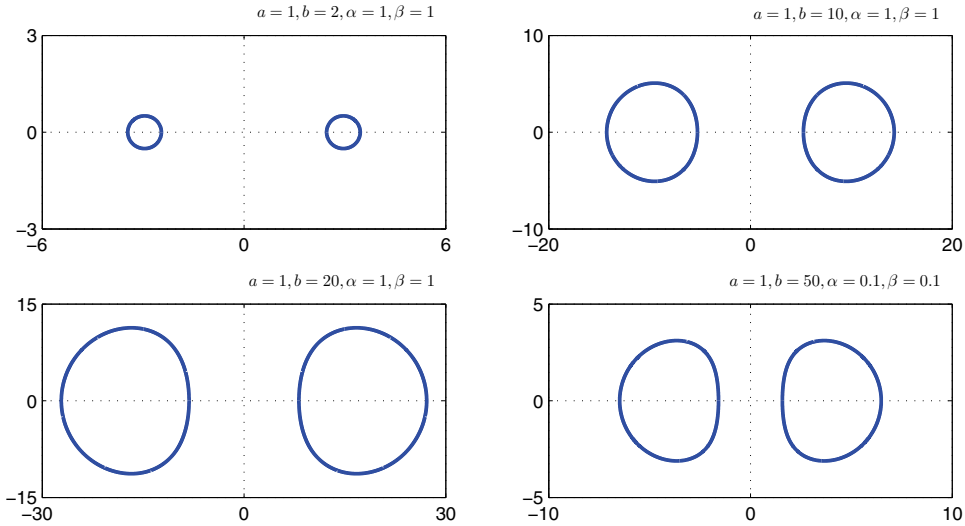

FIG. 2.1. *The typical shapes of inclusions. The scales for the figures are different. In the bottom right figure $\alpha = \beta = 0.1$, and the figure for $\alpha = \beta = 1$ is a similar one magnified ten times. In all figures, $c < d$.*

Let h_1, Φ_1, F_1, and G_1 be the functions defined by (2.19), (2.24), (2.27), (2.32), corresponding to $k_1 a, k_1 b$. Let h_0, etc., be those functions corresponding to a, b. Then we can see the following relations easily:

$$h_1(w) = k_1 h_0(k_1 w),$$
$$\Phi_1(z) = \frac{1}{k_1}\Phi_0\left(\frac{z}{k_1}\right).$$

Therefore, we have

$$G_1(z) = k_1 G_0\left(\frac{z}{k_1}\right).$$

This relation shows that the image of $[k_1 a, k_2 b]$ under G_1 is $k_1 C_2^+$, where C_2^+ is the image of $[a, b]$ under G_0 as given in (2.34).

If we replace α, β by $k_2\alpha, k_2\beta$, then the corresponding inclusions are just rescaled by k_2. This is more obvious. Thus without loss of generality one can choose $\alpha = \beta = 1$. If we just replace α by $k_3\alpha$, one can check that (2.28) implies that the boundary of each inclusion undergoes a linear stretching in the x-direction by a factor of k_3 (which is not in proportion to the change in the distance $2G(a)$ separating the inclusion pair). Thus, among all variations of the four parameters, changing only the ratio a/b leads to a nontrivial change in the inclusion shape. Figure 2.1 shows the shapes of B_1 and B_2, which are obtained numerically for various ratios a/b. Figure 2.2 shows a shape when $c > d$.

The following proposition shows that the inclusion constructed above enjoys the desired property.

PROPOSITION 2.2. *Let $B = B_1 \cup B_2$ be the inclusion constructed above. Then*

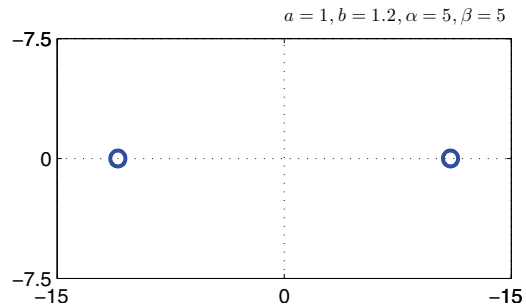

FIG. 2.2. *A shape when $c < d$. In this figure, $c = 1.7227$ and $d = 1.4310$.*

there is f holomorphic in $\mathbb{C} \setminus \overline{B}$ satisfying

$$f(z) = z + O(|z|^{-1}) \quad \text{as } |z| \to \infty \tag{2.38}$$

and

$$f(z) = px + q_1 \quad \text{for } z = x + iy \in \partial B_1, \tag{2.39}$$

$$f(z) = px + q_2 \quad \text{for } z = x + iy \in \partial B_2, \tag{2.40}$$

for some real constant p and complex constants q_1 and q_2.

Proof. One can see from (2.33) and (2.34) that G is a homeomorphism from $[-b, -a] \cup [a, b]$ onto $C_1^+ \cup C_2^+$. One can also see that G is monotonically increasing on $(-\infty, -b]$, $[-a, a]$, and $[b, +\infty)$. Thus G is a homeomorphism from $\partial\Pi^+$ onto $\partial(\Pi^+ \setminus \overline{B_1 \cup B_2})$, where Π^+ is the complex upper half plane. Let φ and ψ be the conformal mappings from the unit disc Δ onto Π^+ and $\Pi^+ \setminus \overline{B_1 \cup B_2}$, respectively. Then $\psi^{-1} \circ G \circ \varphi : \Delta \to \Delta$ is holomorphic and a homeomorphism on $\partial\Delta$. Thus by Rado's theorem [27, p. 4], $\psi^{-1} \circ G \circ \varphi : \Delta \to \Delta$ is conformal, and hence univalent. Therefore, $G : \Pi^+ \to \Pi^+ \setminus \overline{B_1 \cup B_2}$ is univalent. Since $G(\bar{z}) = \overline{G(z)}$ and C_j^+ lies on the upper half plane, we conclude that G is univalent from $\mathbb{C} \setminus ([-b, -a] \cup [a, b])$ onto $\mathbb{C} \setminus \overline{B_1 \cup B_2}$. We emphasize that in order for G to be univalent, the upper part of ∂B_j, C_j^+ should lie on the upper half plane, as we proved before under the assumption that $d > c$. When $c < d$, the mapping G can sometimes be univalent and thus lead to other inclusion shapes, but we do not explore this possibility here.

Since G is univalent, G^{-1} is holomorphic in $\mathbb{C} \setminus \overline{B_1 \cup B_2}$ and satisfies

$$G^{-1}(z) = \frac{1}{\alpha + \beta} z + O(1) \quad \text{as } |z| \to \infty, \tag{2.41}$$

and

$$G^{-1}(z) = \frac{1}{\alpha} x + \frac{\beta d}{2\pi\alpha} \quad \text{for } z = x + iy \in \partial B_1, \tag{2.42}$$

$$G^{-1}(z) = \frac{1}{\alpha} x - \frac{\beta d}{2\pi\alpha} \quad \text{for } z = x + iy \in \partial B_2. \tag{2.43}$$

Let

$$f(z) := (\alpha + \beta)[G^{-1}(z) - \gamma], \tag{2.44}$$

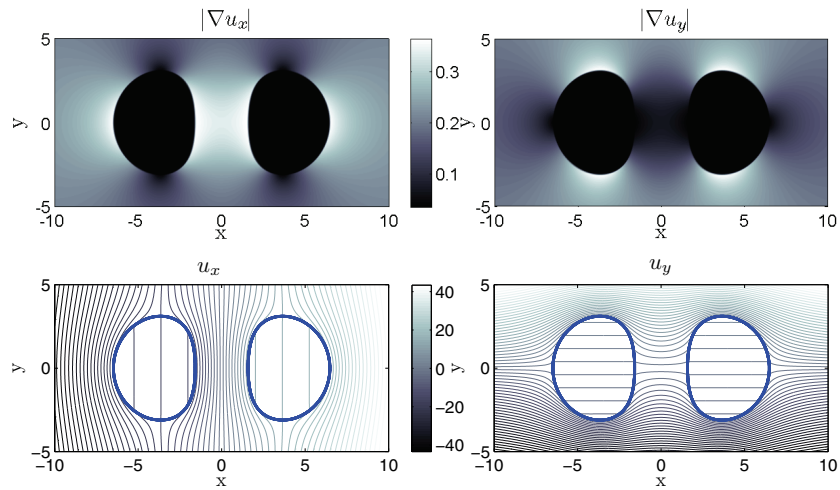

FIG. 3.1. *u_x and u_y are solutions corresponding to the $\mathbf{e}_1$ and $\mathbf{e}_2$ fields, respectively. The top figures are $|\nabla u_x|$ and $|\nabla u_y|$, and the bottom figures are equipotential lines of u_x and u_y.*

where γ is chosen so that f satisfies (2.38). By putting

$$p = \frac{\alpha+\beta}{\alpha}, \quad q_1 = (\alpha+\beta)\left[\frac{\beta d}{2\pi\alpha} - \gamma\right], \quad q_2 = (\alpha+\beta)\left[-\frac{\beta d}{2\pi\alpha} - \gamma\right], \tag{2.45}$$

we have (2.39) and (2.40). This completes the proof. ∎

We note that the most important property of f is that

$$\overline{-\frac{p}{2}z + f(z)} = \frac{p}{2}z + \overline{q_j} \quad \text{on } \partial B_j, \tag{2.46}$$

so that the function on the left-hand side of this equation, which is antiholomorphic outside the inclusion, can be extended inside B_j as a linear holomorphic function.

3. The uniformity property for antiplane elasticity. We now show that the inclusions B_1 and B_2 have the uniformity property for antiplane elasticity (or for two-dimensional conductivity): For any uniform loading the field inside the inclusions is uniform. Before proving this, it may be helpful to the reader to refer to Figure 3.1, which clearly exhibits the uniformity property. This figure was obtained by solving (2.1) numerically using the boundary integral method.

We now prove the following theorem, which is a precise statement of the uniformity property for two-dimensional conductivity.

THEOREM 3.1. *Let $B = B_1 \cup B_2$ be the inclusion constructed in section 2 with $\alpha > 0$ and $\beta > 0$. Let $k \neq 1$. For each nonzero constant vector a, let u be the solution to (2.1). Then ∇u is constant in B.*

Proof. Define U^e and U^i by

$$U^e(z) := \left(k + \frac{1-k}{p}\right)^{-1}\left(kz + \frac{1-k}{p}f(z)\right), \quad z \in \mathbb{C} \setminus \overline{B_1 \cup B_2}, \tag{3.1}$$

$$U^i(z) := \left(k + \frac{1-k}{p}\right)^{-1}(z + c_j), \quad z \in B_j,\ j = 1, 2, \tag{3.2}$$

where f is defined by (2.44). Choosing the complex constants c_j for $j = 1, 2$ properly, one can easily see that U^e and U^i satisfy (2.5) and $U^e(z) = z + O(|z|^{-1})$ as $|z| \to \infty$. Let $u := \Re U^e$ in $\mathbb{R}^2 \setminus \overline{B_1 \cup B_2}$ and $u := \Re U^i$ in $B_1 \cup B_2$. Then u satisfies (2.4), and hence u is the solution to (2.1) with $a = (1, 0)$. Note that we have

$$\nabla u = \frac{\alpha + \beta}{\alpha + k\beta}\mathbf{e}_1 \quad \text{in } B_j,\ j = 1, 2, \tag{3.3}$$

where $\mathbf{e}_1 = (1, 0)$. Thus for the uniform loading $\mathbf{e}_1 = (1, 0)$, the field inside B_1 and B_2 is given by (3.3).

One can show that the field inside the inclusion due to the loading $\mathbf{e}_2 = (0, 1)$ is also uniform using Keller's duality argument [19]. In fact, for a given $k \neq 1$, let $k_0 = 1/k$ and let u_0 be the solution to (2.1) with $a = \mathbf{e}_1$ and k replaced with k_0. Let v^e be the harmonic conjugate of u_0 in $\mathbb{C} \setminus \overline{B_1 \cup B_2}$ so that

$$v^e(x, y) - y = O(r^{-1}) \quad \text{as } r \to \infty. \tag{3.4}$$

The existence of such a harmonic conjugate is proved in [5]. Let v^i be the harmonic conjugate of u_0 in $B_1 \cup B_2$. Define w by

$$w(x, y) = \begin{cases} v^e(x, y), & (x, y) \in \mathbb{R}^2 \setminus \overline{B_1 \cup B_2}, \\ k_0 v^i(x, y) + C, & (x, y) \in B_1 \cup B_2, \end{cases} \tag{3.5}$$

where the constant C is chosen so that w is continuous across ∂B_j, $j = 1, 2$. Then using the Cauchy–Riemann equations one can show (see [5]) that w is the solution to (2.1) with $a = (0, 1)$. We also have from (3.3) and the Cauchy–Riemann equation that

$$\nabla w = \frac{\alpha + \beta}{k\alpha + \beta}\mathbf{e}_2 \quad \text{in } B_j,\ j = 1, 2. \tag{3.6}$$

This completes the proof. ∎

So far we have shown that the inclusions $B := B_1 \cup B_2$ have the uniformity property for the antiplane elasticity model: Given the applied field $\mathbf{e}_1$, the field inside B is uniform and given by $\frac{\alpha+\beta}{\alpha+k\beta}\mathbf{e}_1$, and for the applied field $\mathbf{e}_2$, the field inside B is uniform and given by $\frac{\alpha+\beta}{k\alpha+\beta}\mathbf{e}_2$.

4. Polarization tensors: Polyá–Szegö conjecture. In this section we compute the polarization tensor associated with $B = B_1 \cup B_2$ and show that the Polyá–Szegö conjecture fails to be true among inclusions with multiple components. To explain the polarization tensor associated with the inclusion B consisting of m components $B_1, \ldots, B_m$, we consider the following problem: For a vector $\xi \in \mathbb{R}^2$,

$$\begin{cases} \nabla \cdot \big(1 + (k-1)\chi(B)\big)\nabla u = 0 & \text{in } \mathbb{R}^2, \\ u(x, y) - \xi \cdot (x, y) = O(r^{-1}) & \text{as } r \to \infty. \end{cases} \tag{4.1}$$

The solution u to (4.1) admits the asymptotic expansion

$$u(x, y) = \xi \cdot (x, y) + \frac{1}{2\pi}\xi \cdot M\frac{(x, y)^T}{r^2} + O(r^{-2}) \quad \text{as } r \to \infty \tag{4.2}$$

for some 2×2 matrix M. This matrix $M = M(B)$ is the polarization tensor associated with B. It should be noted that the polarization tensor associated with the inclusion

consisting of multiple components $B_1, \ldots, B_m$ is not the sum or a combination of the polarization tensors of individual inclusions. It incorporates the interactions among components.

It is known that the eigenvalues of the polarization tensor must be confined within the so-called Hashin–Shtrikman bounds [21, 7] (see also [20, 23]):

$$\mathrm{Tr}(M) \leq (k-1)\left(1+\frac{1}{k}\right)|B| \tag{4.3}$$

and

$$\mathrm{Tr}(M^{-1}) \leq \frac{1+k}{(k-1)|B|}, \tag{4.4}$$

where Tr denotes the trace and $|B|$ is the area of B. If M has minimal trace, then M satisfies (4.4) and M is diagonal. These bounds are known to be optimal in the sense that all the points inside the bound, except the upper bound, are realized as the pair of eigenvalues of the polarization tensor associated with a certain shape—coated ellipses [7] and crosses [3]. The lower bound (4.4) is attained by ellipses. Thus a conjecture, which implies the Polyá–Szegö conjecture, is that if (4.4) holds for an inclusion, then that inclusion must be an ellipse.

Kang and Milton [18] proved this new conjecture affirmatively in two dimensions (and three dimensions) within the class of simply connected inclusions with Lipschitz boundaries. In fact, in [17], they showed that if the polarization tensor $M(B)$ satisfies the lower bound (4.4), then B must have the uniformity property and is therefore an ellipse by Eshelby's conjecture. The Pólya–Szegö conjecture, which asserts that the inclusion whose polarization tensor has the minimal trace is a disk, follows from this.

We now show that the polarization tensor associated with the inclusion constructed in section 2 satisfies (4.4), and hence the Pólya–Szegö conjecture is not true among nonsimply connected inclusions. To do that, let u^1 and u^2 be solutions to (2.1) with $a = \mathbf{e}_1$ and $a = \mathbf{e}_2$, respectively, and put $\mathbf{u} := (u^1, u^2)$. Then, the polarization tensor M is given by

$$M = (k-1)\int_B \nabla \mathbf{u}\, dxdy, \tag{4.5}$$

where $\nabla \mathbf{u}$ is the Jacobian matrix. See [4], for example, for the proof of (4.5). As an immediate consequence of (3.3) and (3.6) we obtain the following corollary.

COROLLARY 4.1. *The polarization tensor associated with the inclusion constructed in section* 2 *is given by*

$$M = (k-1)|B| \begin{pmatrix} \dfrac{\alpha+\beta}{\alpha+k\beta} & 0 \\ 0 & \dfrac{\alpha+\beta}{k\alpha+\beta} \end{pmatrix}. \tag{4.6}$$

Note that this tensor satisfies

$$\mathrm{Tr}(M^{-1}) = \frac{k+1}{(k-1)|B|}, \tag{4.7}$$

which is the lower Hashin–Shtrikman bound. It is quite interesting to observe that the polarization tensor (4.6) is the same as that for the ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2} \leq 1$. In particular, the inclusion in Figure 2.1, which has $\alpha = \beta$, has the same polarization tensor as that of a circular disk.

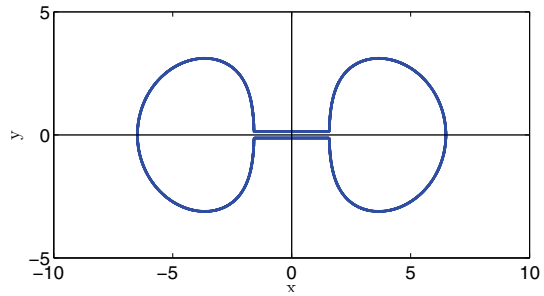

FIG. 5.1. *Bridged inclusion.*

5. Instability of the uniformity property. For a given $\epsilon > 0$, let δ be a positive number such that the rectangle $(-\delta, \delta) \times (\epsilon, \epsilon)$ is contained in the convex hull of B_1 and B_2. Let $B_\epsilon := B \cup ((-\delta, \delta) \times (\epsilon, \epsilon))$. B_ϵ is B_1 and B_2 connected by a thin bridge. Figure 5.1 shows the bridged inclusion.

Let γ and γ_ϵ be the conductivity distributions with inclusions B and B_ϵ, respectively, namely,

$$\gamma = 1 + (k-1)\chi(B), \quad \gamma_\epsilon = 1 + (k-1)\chi(B_\epsilon). \tag{5.1}$$

Let $h(x, y)$ be a harmonic function in $\mathbb{R}^2$, e.g., $h(x, y) = x$ or y. Let u be the solution to

$$\begin{cases} \nabla \cdot \gamma \nabla u = 0 \quad \text{in } \mathbb{R}^2, \\ u(x, y) - h(x, y) = O(r^{-1}) \quad \text{as } r \to \infty \end{cases} \tag{5.2}$$

and u_ϵ be the solution to (5.2) with γ replaced with γ_ϵ. Then a standard regularity theory of elliptic equations shows that

$$\|\nabla(u - u_\epsilon)\|_2 \to 0 \quad \text{as } \epsilon \to 0. \tag{5.3}$$

Here $\|\cdot\|_2$ is the norm of the square integral. In fact, if we put $w = u - u_\epsilon$, then w satisfies

$$\begin{cases} \nabla \cdot \gamma_\epsilon \nabla w = \nabla \cdot (\gamma_\epsilon - \gamma)\nabla u \quad \text{in } \mathbb{R}^2, \\ w(x, y) = O(r^{-1}) \quad \text{as } r \to \infty. \end{cases} \tag{5.4}$$

Thus it follows from a regularity theorem for the elliptic operator $\nabla \cdot \gamma_\epsilon \nabla$ (see [11]) that provided k is strictly positive

$$\|w\|_{H^1(\mathbb{R}^2)} \le C \left(\int_{R_\epsilon} |\nabla u|^2 \right)^{1/2} \tag{5.5}$$

for some constant C independent of ϵ, where $R_\epsilon = B_\epsilon \setminus B$. In particular, we have (5.3).

If $h(x, y) = x$ or y, ∇u is constant in B_1 and B_2, as we have seen in section 3. Therefore, by (5.3), ∇u_ϵ is almost uniform (in the H^1 sense) if ϵ is small. It is obvious that B_ϵ is simply connected but nothing similar to an ellipse. Figure 5.2 shows the absolute value of the gradient of u_ϵ.

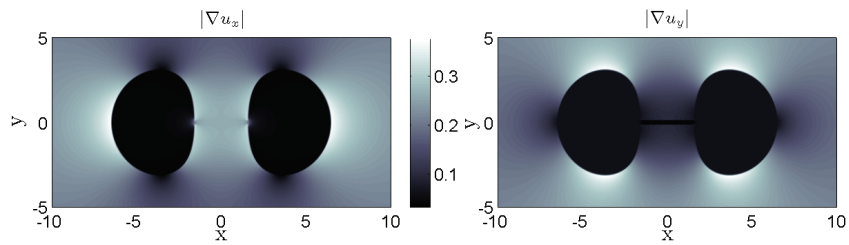

FIG. 5.2. *The graph of the absolute value of the gradient of the solutions corresponding to the bridged inclusion.*

We note that (5.5) implies that

$$\|M(B) - M(B_\epsilon)\| \leq C\epsilon, \tag{5.6}$$

where $M(B)$ is the polarization tensor of B and $M(B_\epsilon)$ is that of B_ϵ. In the case when $\alpha = \beta$, this equation shows that from a practical standpoint the Polyá–Szegö conjecture is false in two dimensions: a simply connected inclusion can have a polarizability tensor arbitrarily close to that of a circular disk yet not resemble a disk at all. We remark that in the extreme cases not treated here, when $k = 0$ or $k = \infty$, the insertion of even an infinitesimal bridge drastically changes the polarization tensor. So it is still an open question whether a void or perfectly conducting region is necessarily close in shape to an ellipse if it is simply connected and almost has the polarizability tensor of an ellipse.

6. Uniformity property: The elasticity case. In this section we consider the uniformity property of the inclusion $B_1 \cup B_2$ for planar elasticity and show that for a certain loading the field inside $B_1 \cup B_2$ is uniform while for other loadings it is not uniform.

Let $C = (C_{ijkl})$ be the elasticity tensor of the inclusion-matrix composite, namely,

$$C_{ijkl} := \Big(\lambda\,\chi(\mathbb{R}^2 \setminus \overline{B}) + \widetilde{\lambda}\,\chi(B)\Big)\delta_{ij}\delta_{kl} + \Big(\mu\,\chi(\mathbb{R}^2 \setminus \overline{B}) + \widetilde{\mu}\,\chi(B)\Big)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where $B = B_1 \cup B_2$. The elasticity tensor C indicates that the matrix (the background) has Lamé parameters (λ, μ), while the inclusion has parameters $(\widetilde{\lambda}, \widetilde{\mu})$. It is always assumed that

$$\mu > 0, \quad d\lambda + 2\mu > 0, \quad \widetilde{\mu} > 0, \quad \text{and} \quad d\widetilde{\lambda} + 2\widetilde{\mu} > 0$$

for ellipticity. For given constants a_{ij}, $i, j = 1, 2$, consider the following linear elastic problem:

$$\begin{cases} \nabla \cdot \big(C(\nabla \mathbf{u} + \nabla \mathbf{u}^T)\big) = 0 & \text{in } \mathbb{R}^2, \\ \mathbf{u}(x) - \displaystyle\sum_{i,j=1}^{2} a_{ij} x_i \mathbf{e}_j = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases} \tag{6.1}$$

where $\mathbf{e}_j$, $j = 1, \ldots, d$, denotes the standard basis for $\mathbb{R}^2$. The uniform applied loading is determined by the matrix (a_{ij}).

Let us first seek a type of loading which yields a uniform field inside the inclusions. The existence of such a loading is expected due to the link between conductivity

problems and elasticity problems in composites when the field in one phase is uniform [13]. We first invoke the following complex representation of the solution to (6.1) from [24, 4]: Let $\mathbf{u} = (u, v)$ be the solution of (6.1) and let $\mathbf{u}_e := \mathbf{u}|_{\mathbb{C}\setminus\overline{B}}$ and $\mathbf{u}_i := \mathbf{u}|_B$. Then there are unique functions φ_e and ψ_e holomorphic in $\mathbb{C} \setminus \overline{B}$ and φ_i and ψ_i holomorphic in B such that

$$2\mu(u_e + iv_e)(z) = \kappa\varphi_e(z) - z\overline{\varphi_e'(z)} - \overline{\psi_e(z)}, \quad z \in \mathbb{C} \setminus \overline{B}, \tag{6.2}$$

$$2\widetilde{\mu}(u_i + iv_i)(z) = \widetilde{\kappa}\varphi_i(z) - z\overline{\varphi_i'(z)} - \overline{\psi_i(z)}, \quad z \in B, \tag{6.3}$$

where

$$\kappa = \frac{\lambda + 3\mu}{\lambda + \mu}, \qquad \widetilde{\kappa} = \frac{\widetilde{\lambda} + 3\widetilde{\mu}}{\widetilde{\lambda} + \widetilde{\mu}}. \tag{6.4}$$

Moreover, the following hold on ∂B_j, $j = 1, 2$:

$$\frac{1}{2\mu}\left(\kappa\varphi_e(z) - z\overline{\varphi_e'(z)} - \overline{\psi_e(z)}\right) = \frac{1}{2\widetilde{\mu}}\left(\widetilde{\kappa}\varphi_i(z) - z\overline{\varphi_i'(z)} - \overline{\psi_i(z)}\right), \tag{6.5}$$

$$\varphi_e(z) + z\overline{\varphi_e'(z)} + \overline{\psi_e(z)} = \varphi_i(z) + z\overline{\varphi_i'(z)} + \overline{\psi_i(z)} + c, \tag{6.6}$$

where c is a constant. Equation (6.5) expresses continuity of displacement, and (6.6) expresses continuity of traction.

Let f be the function in (2.44) and let

$$\varphi_e(z) = A_e z, \quad \psi_e(z) = C_e\left[-\frac{p}{2}z + f(z)\right], \quad z \in \mathbb{C} \setminus \overline{B}, \tag{6.7}$$

where A_e and C_e are complex and real constants, respectively. As was observed in (2.46), $\overline{\psi}$ on ∂B_j has an extension to B_j as the linear holomorphic function $C_e(\frac{p}{2}z + \overline{q_j})$. Therefore, on ∂B_j, $j = 1, 2$, (6.5) and (6.6) now take the forms

$$\left(\widetilde{\kappa}\varphi_i(z) - z\overline{\varphi_i'(z)} - \overline{\psi_i(z)}\right) = \frac{\widetilde{\mu}}{\mu}\left(\kappa A_e - \overline{A_e} - \frac{C_e p}{2}\right)z + D_j, \tag{6.8}$$

$$\varphi_i(z) + z\overline{\varphi_i'(z)} + \overline{\psi_i(z)} = \left(A_e + \overline{A_e} + \frac{C_e p}{2}\right)z + E_j \tag{6.9}$$

for some constants D_j and E_j. Equations (6.8) and (6.9) force us to take $\varphi_i(z) = A_i z + \text{constant}$ and $\psi_i = \text{constant}$, and the complex number A_i should satisfy

$$\begin{cases} \widetilde{\kappa}A_i - \overline{A_i} = \dfrac{\widetilde{\mu}}{\mu}\left(\kappa A_e - \overline{A_e} - \dfrac{C_e p}{2}\right), \\ A_i + \overline{A_i} = A_e + \overline{A_e} + \dfrac{C_e p}{2}. \end{cases} \tag{6.10}$$

Let $A_e = a_1 + ia_2$. Equation (6.10) has a solution A_i if and only if

$$C_e = \frac{4}{p}\left[1 + \frac{2\widetilde{\mu}}{\mu(\widetilde{\kappa} - 1)}\right]^{-1}\left[\frac{\widetilde{\mu}(\kappa - 1)}{\mu(\widetilde{\kappa} - 1)} - 1\right]a_1, \tag{6.11}$$

and in this case

$$2\mu\begin{pmatrix} u_e \\ v_e \end{pmatrix} = \begin{pmatrix} (\kappa - 1)a_1 - C_e(1 - \frac{p}{2}) & -(\kappa + 1)a_2 \\ (\kappa + 1)a_2 & (\kappa - 1)a_1 + C_e(1 - \frac{p}{2}) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + O(r^{-1}) \tag{6.12}$$

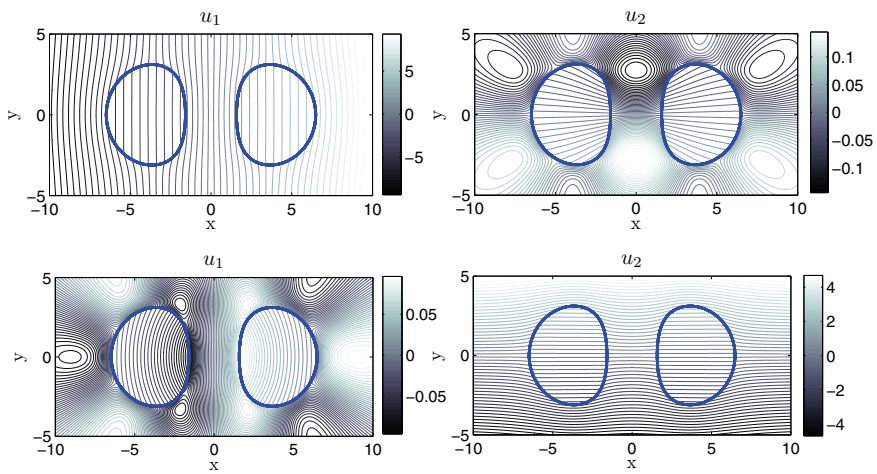

FIG. 6.1. *Equipotential lines of the solution* $\mathbf{u} = (u_1, u_2)$ *for the loading* $(x, 0)$ *(top) and* $(0, y)$ *(bottom).* ∇u_2 *is not uniform for the top, while* ∇u_1 *is not uniform for the bottom.*

as $r \to \infty$. Putting $t = \frac{(\kappa-1)a_1}{2\mu}$ and $s = \frac{(\kappa+1)a_2}{2\mu}$ and simplifying expressions using (2.45) and (6.4), we arrive at the following: If the loading (a_{ij}) is of the form

$$(a_{ij}) = \begin{pmatrix} (1-\theta)t & -s \\ s & (1+\theta)t \end{pmatrix}, \tag{6.13}$$

where t and s are real constants and

$$\theta = \frac{(\alpha-\beta)(\widetilde{\lambda}+\widetilde{\mu}-\lambda-\mu)}{(\alpha+\beta)(\mu+\widetilde{\lambda}+\widetilde{\mu})}, \tag{6.14}$$

then $\nabla\mathbf{u}$ is constant in B where $\mathbf{u}$ is the solution to (6.1). In particular, when $\alpha = \beta$, this corresponds to a hydrostatic loading. We mention that the inclusions constructed in this paper depend on the parameters α and β. In summary, we have proved the following theorem.

THEOREM 6.1. *If the* (a_{ij}) *are given by* (6.13) *for some real numbers* s *and* t *where* θ *is defined by* (6.14)*, then the solution* $\mathbf{u}$ *to* (6.1) *has the property that* $\nabla\mathbf{u}$ *is constant in* B.

We do not have a complete characterization of those loadings which yield a uniform strain field inside the inclusion, but numerical computations show that for certain loadings the field is not uniform. Figure 6.1 shows the equipotential lines for the solution $\mathbf{u} = (u_1, u_2)$ for the loadings $(x, 0)$ and $(0, y)$. It is worthwhile to compare the result of this paper with that for the simply connected inclusion in [28, 18]. For a simply connected inclusion, if the field inside the inclusion is uniform for a single loading, then the inclusion is of elliptical shape, and hence the field is uniform for any loading. Here we established that it is not the case for an inclusion with multiple components. It is an open question whether the uniformity of the interior field for all uniform applied loadings forces the inclusion (with possibly multiple components) to be an ellipse or not.

Conclusion. Providing an alternative proof to that of Cherepanov [8], we constructed a family of inclusions with two components which have the uniformity prop-

erty for antiplane elasticity: for any loading the field inside the inclusions is uniform. In the case of planar elasticity the field is uniform for certain types of loadings but not uniform for other loadings. These results show that the conjectures of Eshelby and Pólya–Szegö are not true among nonsimply connected inclusions. By connecting two inclusions by a thin bridge we showed that these conjectures do not hold in a practical sense even for simply connected inclusions: even if the field inside an inclusion is very close to being uniform, the inclusion need not be close to an ellipse.

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REFERENCES

[1] L. V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1978.

[2] G. Alessandrini and V. Isakov, *Analyticity and uniqueness for the inverse conductivity problem*, Rend. Istit. Mat. Univ. Trieste, 28 (1996), pp. 351–369.

[3] H. Ammari, Y. Capdeboscq, H. Kang, E. Kim, and M. Lim, *Attainability by simply connected domains of optimal bounds for polarization tensors*, European J. Appl. Math., 17 (2006), pp. 201–219.

[4] H. Ammari and H. Kang, *Reconstruction of Small Inhomogeneities from Boundary Measurements*, Lecture Notes in Math. 1846, Springer-Verlag, New York, 2004.

[5] H. Ammari, H. Kang, and M. Lim, *Gradient estimates for solutions to the conductivity problem*, Math. Ann., 332 (2005), pp. 277–286.

[6] K. Astala and V. Nesi, *Composites and quasiconformal mappings: New optimal bounds in two dimensions*, Calc. Var. Partial Differential Equations, 18 (2003), pp. 335–355.

[7] Y. Capdeboscq and M. S. Vogelius, *A review of some recent work on impedance imaging for inhomogeneities of low volume fraction*, in Partial Differential Equations and Inverse Problems, Contemp. Math. 362, AMS, Providence, RI, 2004, pp. 69–87.

[8] G. P. Cherepanov, *Inverse problems of the plane theory of elasticity*, Prikl. Mat. Meh., 38 (1974), pp. 963–979 (in Russian).

[9] J. D. Eshelby, *The determination of the elastic field of an ellipsoidal inclusion, and related problems*, Proc. Roy. Soc. London Ser. A, 241 (1957), pp. 376–396.

[10] J. D. Eshelby, *Elastic inclusions and inhomogeneities*, in Progress in Solid Mechanics, Vol. II, North–Holland, Amsterdam, 1961, pp. 87–140.

[11] L. C. Evans, *Partial Differential Equations*, Grad. Stud. Math. 19, AMS, Providence, RI, 1998.

[12] Y. Grabovsky and R. Kohn, *Microstructures minimizing the energy of a two phase elastic composite in two space dimensions.* II. *The Vigdergauz microstructure*, J. Mech. Phys. Solids, 43 (1995), pp. 949–972.

[13] Y. Grabovsky, *Bounds and extremal microstructures for two-component composites: A unified treatment based on the translation method*, Proc. Roy. Soc. London Ser. A, 452 (1996), pp. 919–944.

[14] Z. Hashin, *The elastic moduli of heterogeneous materials*, Trans. ASME Ser. E J. Appl. Mech., 29 (1962), pp. 143–150.

[15] C. O. Horgan, *Anti-plane shear deformations in linear and nonlinear solid mechanics*, SIAM Rev., 37 (1995), pp. 53–81.

[16] V. Isakov and J. Powell, *On the inverse conductivity problem with one measurement*, Inverse Problems, 6 (1990), pp. 311–318.

[17] H. Kang and G. W. Milton, *On Conjectures of Pólya–Szegö and Eshelby*, in Inverse Problems, Multi-scale Analysis and Effective Medium Theory, Contemp. Math. 408, AMS, Providence, RI, 2006, pp. 75–80.

[18] H. Kang and G. W. Milton, *Solutions to the Pólya–Szegö conjecture and the weak Eshelby conjecture*, Arch. Ration. Mech. Anal., 188 (2008), pp. 93–116.

[19] J. B. Keller, *A theorem on the conductivity of a composite medium*, J. Math. Phys., 5 (1964), pp. 548–549.

[20] R. V. Kohn and G. W. Milton, *On bounding the effective conductivity of anisotropic composites*, in Homogenization and Effective Moduli of Materials and Media, J. L. Ericksen, D. Kinderlehrer, R. V. Kohn, and J. L. Lions, eds., IMA Vol. Math. Appl. 1, Springer-Verlag, New York, 1986, pp. 97–125.

[21] R. Lipton, *Inequalities for electric and elastic polarization tensors with applications to random composites*, J. Mech. Phys. Solids, 41 (1993), pp. 809–833.

[22] L. P. Liu, *Solutions to the Eshelby conjectures*, Proc. Roy. Soc. A Math. Phys. Engrg. Sci., 464 (2008), pp. 573–594.

[23] G. W. Milton, *The Theory of Composites*, Cambridge Monogr. Appl. Comput. Math., Cambridge University Press, Cambridge, UK, 2002.

[24] N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, English translation, Noordhoff International Publishing, Leiden, 1977.

[25] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Ann. Math. Stud. 27, Princeton University Press, Princeton, NJ, 1951.

[26] C.-Q. Ru and P. Schiavone, *On the elliptic inclusion in anti-plane shear*, Math. Mech. Solids, 1 (1996), pp. 327–333.

[27] R. Schoen and S. T. Yau, *Lectures on Harmonic Maps*, International Press, Boston, 1997.

[28] G. P. Sendeckyj, *Elastic inclusion problems in plane elastostatics*, Internat. J. Solids Structures, 6 (1970), pp. 1535–1543.

[29] S. B. Vigdergauz, *Effective elastic parameters of a plate with a regular system of equal-strength holes*, Inzhenernyi Zhurnal. Mekhnika Tverdogo Tela, 21 (1986), pp. 165–169.

[30] S. B. Vigdergauz, *Two dimensional grained composites of extreme rigidity*, J. Appl. Mech., 61 (1994), pp. 390–394.

[31] L. T. Wheeler, *Stress minimum forms for elastic solids*, AMR, 45 (1992), pp. 1–12.