Janus and multifaced supersymmetric theories. II

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We explore the physics of supersymmetric Janus gauge theories in four dimensions with spatial dependent coupling constants e^2 and θ. For the 8 supersymmetric case, we study the vacuum and Bogomol'nyi-Prasad-Sommerfield spectrum, and the physics of a sharp interface where the couple constants jump. We also find less supersymmetric cases either due to additional expressions in the Lagrangian or to the fact that coupling constants depend on additional spatial coordinates.

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I. INTRODUCTION AND CONCLUSION

There has been some interest in the Janus-type field theory where the coupling constants are dependent on space [1]. What is exciting about this subject is that some of supersymmetry of the original theory can be preserved even when the coupling constants are spatially dependent once some corrections are made on the Lagrangian and the supersymmetric transformation. Such field theories appear naturally in the context of anti-de Sitter geometry [2]. Recently, a Janus-type field theory has been discovered in the intersecting $D3$-(p, q) 5 brane [3,4]. The field theory on $D3$ branes can have a spatially dependent complex coupling $\tau = \theta/2\pi + 4\pi i/e^2$, preserving half of original 16 supersymmetries.

In this work we explore further these theories and generalize them, extending our previous work [5]. We first study the vacuum and Bogomol'nyi-Prasad-Sommerfield monopoles (BPS) configurations. We also study the wave and dyonic physics near a sharp interface, which acts like an axionic domain wall. We also find additional supersymmetry breaking Janus theories.

The original Janus solution in Ref. [2] is a 1-parameter family of dilatonic deformations of AdS_5 space without supersymmetry. The Janus solution is made of two Minkowski spaces joined along an interface so that the dilaton field interpolates two asymptotic values at two spaces. The conformal field theory dual field theory is suggested to be the deformation of the Yang-Mills theory where the coupling constant changes from one region to another region [2].

Further works revealed that one can have supersymmetric Janus geometries with the various supersymmetries and internal symmetries [6–10]. Starting from the 16 supersymmetric Yang-Mills theory, the various deformations of 2, 4, 8 supersymmetries have been found [1]. Especially, the 16 supersymmetric Janus geometries have been found [9,10], where both dilaton and axion fields vary along a spatial direction. Also, other aspects of the Janus solutions have been discussed in Refs. [11–15].

In our previous work [5], we investigated in detail the vacuum and BPS structure of the supersymmetric Janus theory for the case where only e^2 depends on the spatial coordinates and found that there can be a new type of classical vacua, which are characterized by the Nahm equation when there are planes where the coupling constant e^2 vanishes. In a later work [3], such vacua were shown to arise naturally when $D3$ branes intersect with $D5$ branes. In addition, we have found all supersymmetric Janus field theories where the coupling constant e^2 depends on other spatial coordinates.

In this work we repeat a similar analysis for the case where τ depends on spatial coordinates. In Refs. [3,4], the $SL(2, Z)$ transformation and the brane picture were an important tool to explore Janus-type field theories. Especially the vacuum structure was explored in the detail. However, one can still ask whether there is nontrivial classical vacua besides the usual Coulomb vacua in our case. Our analysis shows that if such vacua exist, they would break the supersymmetry further to only two. However, the generalization of the Nahm equation is too complicated at this moment. The BPS objects are dyonic objects, and their characterization is equally or more complicated than the previous case. In the presence of the θ term, dyons would carry additional electric charge due to Witten effect [16].

One interesting simplification is a sharp interface where the coupling constant τ jumps from one constant value to another in a very small region. As the θ angle jumps, such interface can be interpreted partially as an axionic domain wall [17,18]. Electromagnetic wave reflected or transmitted through such a wall would have rotated polarization. We calculate the reflection and transmission coefficient. We fully investigate the $1/2$ BPS dyonic object near the wall, ignoring the non-Abelian core.

A full classification of four-dimensional Janus gauge theory with partially conserved supersymmetry with spatial dependent coupling $e^2(x, y, z)$ has been done in our previous paper [5]. We think that the same classification works also for the present case with $\tau(x, y, z)$, and have worked out all cases in detail.

One can introduce the interface degrees of freedom on a sharp interface without further breaking of supersymmetry as in Ref. [3]. Our result suggests that one could introduce a more general class of interface Lagrangian to our more general Lagrangian with more parameters and less supersymmetries. It would be interesting to explore these Lagrangians and their properties.

This paper is presented as follows: In Sec. II, we review the 8 supersymmetric Janus Yang-Mills theories in four dimensions. In Sec. III, we study the vacuum structure of this theory. We raise the possibility of vacua preserving only two supersymmetries. In Sec. IV, we consider 1/2 BPS field configurations. In Sec. V, we focus on the sharp interface for the coupling constant. The image charges for the magnetic monopoles and electric charges are found. The wave propagation and reflection at the interface is studied. In Sec. VI, less supersymmetric Janus Yang-Mills theories whose coupling constants may have additional space-time dependence are explored briefly.

(While this work is written, a paper [19] has appeared where there is some overlap. We feel some of points raised here seem new.)

II. A BRIEF REVIEW OF 8 SUPERSYMMETRIC JANUS LAGRANGIAN

We start with the ten-dimensional supersymmetric Yang-Mills Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4e^2}\,\mathrm{Tr}(F^{MN}F_{MN} + 2i\bar{\Psi}\Gamma^M D_M\Psi), \qquad (2.1)$$

where $M, N = 0, 1, 2\ldots, 9$ and $\mu,\ \nu = 0,\ 1,\ 2,\ 3$. We use the ten-dimensional notation for convenience. The gamma matrices Γ^M are in the Majorana representation, and the gaugino field Ψ is Majorana and Weyl. The spatial signature is $(-+++\ldots+)$. The Lagrangian is invariant under the supersymmetric transformation

$$\delta_0 A_M = i\bar{\Psi}\Gamma_M\epsilon, \qquad \delta_0\Psi = \tfrac{1}{2}\Gamma^{MN}\epsilon F_{MN}, \qquad (2.2)$$

where the supersymmetry (SUSY) parameter ϵ is Majorana and satisfies the Weyl condition

$$\Gamma^{012\cdots 9}\epsilon = \epsilon. \qquad (2.3)$$

As we consider $1+3$ dimensional space-time $x^\mu = x^0, x^1, x^2, x^3$, the remaining spatial gradient $\partial_M = 0$ with $M = 4, 5\cdots 9$ and the gauge field A_M become scalar fields ϕ_M with $M = 4, 5\cdots 9$. In four-dimensional space-time, one can have an additional term in the Lagrangian

$$\mathcal{L}_\theta = -\frac{\theta}{32\pi^2}\,\mathrm{Tr}\tilde{F}^{\mu\nu}F_{\mu\nu} = \frac{1}{8\pi^2}W^\mu\partial_\mu\theta, \qquad (2.4)$$

where the dual field strength is $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}/2$ with $\epsilon^{0123} = 1$ and W_μ is the Chern-Simons term, $W^\mu = \epsilon^{\mu\nu\nu\rho}\,\mathrm{Tr}(A_\nu\partial_\rho A_\sigma/2 - iA_\mu A_\nu A_\rho/3)$. As $\mathcal{L}_\theta$ is a total derivative, the supersymmetry of the original Lagrangian $\mathcal{L}_0$ would be intact.

We are interested in the case where the coupling constants e^2, θ depend on space-time coordinates. The original Lagrangian $\mathcal{L}_0 + \mathcal{L}_\theta$ is no longer invariant under the original supersymmetric transformation *modulo* a total space-time derivative. Fortunately, one can maintain some of supersymmetries if one modifies the supersymmetric transformation of the gaugino field by $\delta_1\Psi$ and also the Lagrangian by additional terms, which depend on the derivatives of the coupling constant. For this work, we need to take the space-time dependent supersymmetric parameter $\epsilon(x)$. The Lagrangian $\mathcal{L}_0 + \mathcal{L}_\theta$ transforms under the supersymmetric transformation δ_0 of Eq. (2.2) nontrivially as follows:

$$\begin{aligned}\delta_0(\mathcal{L}_0 + \mathcal{L}_\theta) = &-\partial_\mu\left(\frac{1}{4e^2}\right)\mathrm{Tr}(F_{MN}i\bar{\Psi}\Gamma^{MN}\Gamma^\mu\epsilon)\\ &-\frac{1}{2e^2}\,\mathrm{Tr}(F_{MN}i\bar{\Psi}\Gamma^\mu\Gamma^{MN}\partial_\mu\epsilon)\\ &+\left(\frac{\partial_\mu\theta}{16\pi^2}\right)\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(F_{\nu\rho}i\bar{\Psi}\Gamma_\sigma\epsilon). \end{aligned}\qquad (2.5)$$

The additional transformation of the original Lagrangian due to $\delta_1\Psi$ would be

$$\begin{aligned}\delta_1\mathcal{L}_0 = &-\partial_\mu\left(\frac{1}{2e^2}\right)i\,\mathrm{Tr}(\Psi\Gamma^\mu\delta_1\Psi)\\ &-\frac{1}{e^2}\,\mathrm{Tr}(i\bar{\Psi}\Gamma^M D_M\delta_1\Psi).\end{aligned}\qquad (2.6)$$

Let us focus on the case where the coupling constants e^2, θ depend only on the $x^3 = z$ coordinate. It has been shown recently in Ref. [3] that the half of the original supersymmetry could be maintained if the spatial dependence of two coupling constants is constrained so that

$$\frac{1}{e^2} = D\sin 2\psi, \qquad \theta = \theta_0 + 8\pi^2 D\cos 2\psi, \qquad (2.7)$$

with the space-time dependence arising only from $\psi(z)$, which can be an arbitrary function. Note that in the limit $D\to\infty$, $\theta_0\to\mp\infty$ and $\psi(z)\to 0, \pi/2$ with the combinations $D\psi(z)$ and $\theta_0 \pm 8\pi^2 D$ kept finite, the space-time dependence appears only in the fine structure constant $4\pi/e^2$. Notice also that a constant shift of θ by 2π does not change physics. The complex coupling constant becomes

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} = \tau_0 + 4\pi D e^{2i\psi}, \qquad (2.8)$$

where $\tau_0 = \theta_0/(2\pi)$.

With the coupling constants given by Eq. (2.7), eight of the original 16 supersymmetries can be preserved [3]. The condition on the supersymmetric parameter ϵ compatible with the Weyl condition (2.3) is

$$\epsilon(z) = e^{-(\psi(z)/2)\Gamma^{0123}}\epsilon_0, \qquad (2.9)$$

with a constant spinor ϵ_0 such that

$$\Gamma^{3456}\epsilon_0 = \epsilon_0. \tag{2.10}$$

This condition also breaks the global $SO(6)$ symmetry, which rotates 4, 5, 6, 7, 8, 9 indices to $SO(3) \times SO(3)$, each of which rotates 4, 5, 6 and 7, 8, 9 indices, respectively. As $\Gamma^{012\ldots9}\epsilon = \epsilon$, we get $\Gamma^{3456}e^{\psi\Gamma^{0123}}\epsilon = \epsilon$. The condition on ϵ_0 is identical to the case with the constant θ.

As the $SO(6)$ symmetry is broken to $SO(3) \times SO(3)$, we split six scalar fields to two sets each of which are made of three scalar fields. We denote

$$\begin{aligned} X_a &= (X_1, X_2, X_3) = (\phi_4, \phi_5, \phi_6), \\ Y_a &= (Y_1, Y_2, Y_3) = (\phi_7, \phi_8, \phi_9). \end{aligned} \tag{2.11}$$

We will also interchangeably use $(X_1, X_2, X_3) = (X_4, X_5, X_6)$ and $(Y_1, Y_2, Y_3) = (Y_7, Y_8, Y_9)$. The indices for Γ^a follow the indices for the scalar field whenever they are contracted. To cancel some of the terms in the zeroth order variation of the original Lagrangian (2.5), one needs to add a correction to the SUSY transformation of the gaugino field and also corrections to the original Lagrangian. The correction to the original SUSY transformation (2.2) is

$$\begin{aligned} \delta_1 A_M &= 0, \\ \delta_1 \Psi &= \psi'\Gamma^3((\Gamma\cdot X)\cot\psi - (\Gamma\cdot Y)\tan\psi)\epsilon, \end{aligned} \tag{2.12}$$

where the prime means d/dz. The correction to the original Lagrangian is made of two parts. The first correction, which depends on the first order in the derivative of the couple constant, is given as

$$\begin{aligned} \mathcal{L}_1 = {} & \frac{\psi'}{4e^2}\mathrm{Tr} i\bar\Psi\Big(-\Gamma^{012} + \frac{1}{\sin\psi}\Gamma^{456} - \frac{1}{\cos\psi}\Gamma^{789}\Big)\Psi \\ & + \frac{2\psi'}{e^2}\mathrm{Tr}\Big(-\frac{i}{\sin\psi}X_1[X_2, X_3] + \frac{i}{\cos\psi}Y_1[Y_2, Y_3]\Big). \end{aligned} \tag{2.13}$$

The second correction is quadratic in the derivatives of ψ so that

$$\begin{aligned} \mathcal{L}_2 = {} & -\frac{1}{2e^2}\mathrm{Tr}[(\psi'^2 - (\psi'\cot\psi)')X_aX_a \\ & + (\psi'^2 + (\psi'\tan\psi)')Y_aY_a]. \end{aligned} \tag{2.14}$$

The total Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\theta + \mathcal{L}_1 + \mathcal{L}_2$ is invariant under the corrected SUSY transformation

$$\begin{aligned} \delta A_M &= (\delta_0 + \delta_1)A_M = i\bar\Psi\Gamma_M\epsilon, \\ \delta\Psi &= (\delta_0 + \delta_1)\Psi \\ &= \tfrac{1}{2}F_{MN}\Gamma^{MN}\epsilon + \psi'\Gamma^3(\cot\psi(\Gamma\cdot X) - \tan\psi(\Gamma\cdot Y))\epsilon. \end{aligned} \tag{2.15}$$

Redefine the scalar fields so that

$$\tilde X_a = X_a \sin\psi, \qquad \tilde Y_a = Y_a\cos\psi. \tag{2.16}$$

The correction to the supersymmetric transformation of the gaugino field can be absorbed as

$$\begin{aligned} \delta\Psi = {} & \Big(\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{\sin\psi}\Gamma^\mu D_\mu\tilde X\cdot\Gamma \\ & + \frac{1}{\cos\psi}\Gamma^\mu D_\mu \tilde Y\cdot\Gamma + \cdots\Big)\epsilon. \end{aligned} \tag{2.17}$$

The $\mathcal{L}_2$ can be absorbed into the scalar kinetic energy

$$\begin{aligned} \mathcal{L}_0 + \mathcal{L}_2 = {} & \ldots - \frac{1}{2e^2}\Big(\frac{1}{\sin^2\psi}(D_\mu\tilde X^a)^2 + \frac{1}{\cos^2\psi}(D_\mu\tilde Y^a)^2\Big) \\ & + \ldots. \end{aligned} \tag{2.18}$$

The whole Lagrangian $\mathcal{L}$ becomes

$$\begin{aligned} \mathcal{L} = {} & -\frac{1}{4e^2}\mathrm{Tr}\Big(F^{\mu\nu}F_{\mu\nu} + \frac{e^2\theta}{8\pi^2}\tilde F^{\mu\nu}F_{\mu\nu} + \frac{2}{\sin^2\psi}D^\mu\tilde X_a D^\mu\tilde X^a + \frac{2}{\cos^2\psi}D_\mu\tilde Y^a D^\mu\tilde Y^a\Big) + \frac{1}{4e^2}\mathrm{Tr}\Big(\frac{1}{\sin^4\psi}[\tilde X^a, \tilde X^b]^2 \\ & + \frac{1}{\cos^4\psi}[\tilde Y^a, \tilde Y^b]^2 + \frac{8}{\sin^2 2\psi}[\tilde X^a, \tilde Y^b]^2\Big) - \frac{1}{2e^2}\mathrm{Tr}\Big(i\bar\Psi\Gamma^\mu D_\mu\Psi + \frac{1}{\sin\psi}\bar\Psi\Gamma^a[\tilde X^a, \Psi] + \frac{1}{\cos\psi}\bar\Psi\Gamma^p[\tilde Y^p, \Psi]\Big) \\ & + \frac{\psi'}{4e^2}\mathrm{Tr} i\bar\Psi\Big(-\Gamma^{012} + \frac{1}{\sin\psi}\Gamma^{456} - \frac{1}{\cos\psi}\Gamma^{789}\Big)\Psi + \frac{2\psi'}{e^2}\mathrm{Tr}\Big(-\frac{i}{\sin^4\psi}\tilde X_1[\tilde X_2, \tilde X_3] + \frac{i}{\cos^4\psi}\tilde Y_1[\tilde Y_2, \tilde Y_3]\Big). \end{aligned} \tag{2.19}$$

The combined SUSY transformation (2.15) becomes

$$\delta A_\mu = i\bar{\Psi}\Gamma_\mu \epsilon, \qquad \delta\tilde{X}_a = \frac{1}{\sin\psi}\bar{\Psi}\Gamma_a\epsilon, \qquad \delta\tilde{Y}_p = \frac{1}{\cos\psi}\bar{\Psi}\Gamma_p\epsilon$$

$$\delta\Psi = \left(\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{\sin\psi}D_\mu\tilde{X}_a\Gamma^{\mu a} + \frac{1}{\cos\psi}D_\mu\tilde{Y}_p\Gamma^{\mu p} - \frac{i}{\cos\psi\sin\psi}[\tilde{X}_a, \tilde{Y}_p]\Gamma^{ap} - \frac{i}{2\sin^2\psi}[\tilde{X}_a, \tilde{X}_b]\Gamma^{ab} - \frac{i}{2\cos^2\psi}[\tilde{Y}_p, \tilde{Y}_q]\Gamma^{pq}\right)\epsilon. \tag{2.20}$$

We can choose the gauge group to be any simple Lie group G.

III. THE VACUUM STRUCTURE

The energy density from the above Lagrangian (2.19) is not positive definite. To consider the vacuum, we put $A_\mu = 0$ and $\Psi = 0$. The scalar fields can depend only on $x^3 = z$ coordinates. The nontrivial part of the energy density is

$$\mathcal{E} = \frac{1}{2e^2}\,\mathrm{Tr}\left(\frac{\tilde{X}_a'^2}{\sin^2\psi} + \frac{\tilde{Y}_a'^2}{\cos^2\psi} - \frac{[\tilde{X}_a, \tilde{X}_b]^2}{2\sin^4\psi} - \frac{[\tilde{Y}_a, \tilde{Y}_b]^2}{2\cos^4\psi} - \frac{[\tilde{X}_a, \tilde{Y}_b]^2}{\cos^2\psi\sin^2\psi}\right) + \frac{2i\psi'}{e^2}\,\mathrm{Tr}\left(\frac{1}{\sin^4\psi}\tilde{X}_1[\tilde{X}_2, \tilde{X}_3] - \frac{1}{\cos^4\psi}\tilde{Y}_1[\tilde{Y}_2, \tilde{Y}_3]\right). \tag{3.1}$$

We rewrite the above energy density as

$$\mathcal{E} = \frac{1}{2e^2}\,\mathrm{Tr}\left(\frac{\tilde{X}_a'}{\sin\psi} + \frac{i}{2}\epsilon^{abc}([X_b, X_c] - [Y_b, Y_c])\cos\psi - i\epsilon^{abc}[X_b, Y_c]\sin\psi\right)^2 + \frac{1}{2e^2}\,\mathrm{Tr}\left(\frac{\tilde{Y}_a'}{\cos\psi} - \frac{i}{2}\epsilon^{abc}([X_b, X_c] - [Y_b, Y_c])\sin\psi - i\epsilon^{abc}[X_b, Y_c]\cos\psi\right)^2 + \frac{1}{2e^2}\left(\sum_a i[X_a, Y_a]\right)^2 + \mathcal{E}_b. \tag{3.2}$$

The boundary term is

$$\mathcal{E}_b = -2iD\,\mathrm{Tr}(\tilde{X}_1[\tilde{X}_2, \tilde{X}_3]\cot^2\psi + \tilde{Y}_1[\tilde{Y}_2, \tilde{Y}_3]\tan^2\psi)' + iD\epsilon^{abc}\,\mathrm{Tr}(\tilde{X}_a[\tilde{Y}_b, \tilde{Y}_c] + \tilde{Y}_a[\tilde{X}_b, \tilde{X}_c])'. \tag{3.3}$$

Assuming the boundary contributions vanish, the energy would be non-negative and the classical vacuum would satisfy the vacuum equations

$$\frac{\tilde{X}_a'}{\sin\psi} + \frac{i}{2}\epsilon^{abc}([X_b, X_c] - [Y_b, Y_c])\cos\psi - i\epsilon^{abc}[X_b, Y_c]\sin\psi = 0,$$
$$\frac{\tilde{Y}_a'}{\cos\psi} - \frac{i}{2}\epsilon^{abc}([X_b, X_c] - [Y_b, Y_c])\sin\psi - i\epsilon^{abc}[X_b, Y_c]\cos\psi = 0. \tag{3.4}$$

Of course the obvious vacuum configurations satisfying the above equations are the Abelian Coulomb vacua

$$\tilde{X}_a' = 0, \qquad \tilde{Y}_a' = 0, \qquad [X_a, X_b] = [Y_a, Y_b] = [X_a, Y_b] = 0. \tag{3.5}$$

Thus, $\tilde{X}_a$, $\tilde{Y}_a$ are constant and can be diagonalized. As $\delta\Psi = 0$ of (2.20), these Abelian vacua are fully supersymmetric.

The interesting question is whether there exist any nontrivial solution for the vacuum Eq. (3.4). For a constant θ case, this vacuum equation turns out to be the Nahm equation for the magnetic monopoles, and nontrivial vacuum are allowed when e^2 vanishes on some planes. This has nice interpretation as $D3$ branes intersecting with $D5$ branes. In the present case, it is not clear at all whether nontrivial, or non-Abelian, solutions exist. If they do, one may wonder the number of supersymmetries preserved. Let us consider the generic case. The SUSY transformation for the vacuum configuration would be

$$\delta\Psi = \left(\frac{1}{\sin\psi}\tilde{X}_a'\Gamma^{3a} + \frac{1}{\cos\psi}\tilde{Y}_p'\Gamma^{3p} - \frac{i}{2}[X_a, X_b]\Gamma^{ab} - \frac{i}{2}[Y_p, Y_q]\Gamma^{pq} - i[X_a, Y_p]\Gamma^{ap}\right)\epsilon. \tag{3.6}$$

It vanishes for the vacua satisfying the vacuum Eq. (3.4) only if the SUSY parameter ϵ_0 satisfies the additional conditions

$$\Gamma^{3489}\epsilon_0 = \Gamma^{3597}\epsilon_0 = \Gamma^{3678}\epsilon_0 = -\epsilon_0. \tag{3.7}$$

As only two of the above conditions are independent, any generic non-Abelian vacuum, if exist, would break the number of supersymmetries to two.

When θ is constant, it is known that there may be nontrivial vacua when the $e^2(z)$ vanishes at some points [5]. When $D3$ branes are connecting $D5$ branes, the dilation field vanishes at the location of $D5$ branes and so the coupling constant $e^2(z)$ in the theory on $D3$ branes varies, while vanishing at the $D5$ locations. The vacua of the theory characterize $D3$ branes. After T-dual transformation to $D1$-$D3$, the vacuum structure characterizes how $D1$ branes end on $D3$ branes. As $D1$ branes ending on $D3$ branes appear as magnetic monopoles, we know that the Nahm equation characterizes $D1$ branes ending on $D3$ branes. The S-dual version of the above vacuum configuration, which appears as the boundary field theory, has been studied by Gaiotto and Witten [4].

IV. BPS EQUATIONS

Similar to the constant θ case, we can introduce two possible 1/2 BPS conditions:

$$\Gamma^{1234}\epsilon_0 = \epsilon_0, \qquad \Gamma^{70}\epsilon_0 = \epsilon_0. \tag{4.1}$$

The first one was for magnetic monopoles and the second one was for the charged massive particles in the Coulomb phase. With nontrivial θ, both conditions become the 1/2 BPS condition for dyons. If we impose both conditions, we would get 1/4 BPS configurations. If there is nontrivial non-Abelian vacuum of only two supersymmetries (3.7), the above BPS condition would be incompatible, implying that there would be no BPS dyons in such a non-Abelian vacuum.

With the first condition of (4.1) in the Coulomb vacuum, one can read the 1/2 BPS equations for dyons from the supersymmetric transformation to be $Y_a = 0$. The remaining 1/2 BPS equations are

$$\begin{aligned} E_i - D_i\tilde{X}_1 &= 0,\\ (B_1 + iB_2) - \cot\psi(D_1 + iD_2)\tilde{X}_1 &= 0,\\ B_3 - \cot\psi D_3\tilde{X}_1 - \frac{i}{\sin^2\psi}[\tilde{X}_2, \tilde{X}_3] &= 0,\\ (D_1 + iD_2)(\tilde{X}_2 + i\tilde{X}_3) &= 0,\\ D_3(\tilde{X}_2 + i\tilde{X}_3) - \cot\psi[\tilde{X}_1, \tilde{X}_2 + i\tilde{X}_3] &= 0, \end{aligned} \tag{4.2}$$

where the electric and magnetic components of the field strength are

$$E_i = F_{i0}, \qquad B_i = \tfrac{1}{2}\epsilon_{ijk}F_{jk}. \tag{4.3}$$

These equations are consistent with the Gauss law

$$D_i\left(\frac{1}{e^2}E_i + \frac{\theta}{8\pi^2}B_i\right) + \frac{i}{e^2}([X_a, D_0X_a] + [Y_a, D_0Y_a]) = 0. \tag{4.4}$$

One could impose a further constraint $\tilde{X}_2 = \tilde{X}_3 = 0$, and then the above BPS equations become

$$B_i - \cot\psi D_i\tilde{X}_1 = 0, \qquad E_i - D_i\tilde{X}_1 = 0. \tag{4.5}$$

These equations can also be obtained from the energy functional, which can be reshuffled to

$$\begin{aligned} \mathcal{H} = &\int d^3x \frac{1}{2e^2}\,\mathrm{Tr}\Big[(E_i - D_i\tilde{X}_1)^2 + (B_1 - \cot\psi D_1\tilde{X}_1)^2 + (B_2 - \cot\psi D_2\tilde{X}_1)^2 + \left(B_3 - \cot\psi D_3\tilde{X}_1 - \frac{i}{\sin^2\psi}[\tilde{X}_2, \tilde{X}_3]\right)^2\\ &+ \frac{1}{\sin^4\psi}|D_1(\tilde{X}_2 + i\tilde{X}_3) + iD_2(\tilde{X}_2 + i\tilde{X}_3)|^2 + \left(\frac{i}{\sin^2\psi}[\tilde{X}_2, \tilde{X}_3] - \cot\psi D_3\tilde{X}_2 + D_0\tilde{X}_3\right)^2\\ &+ \left(\frac{i}{\sin^2\psi}[\tilde{X}_2, \tilde{X}_3] + \cot\psi D_3\tilde{X}_3 + D_0\tilde{X}_2\right)^2\Big] + \int \partial_i\,\mathrm{Tr}\frac{1}{e^2}\tilde{X}_1\left(E_i - \tan\psi B_i + \delta_{i3}\frac{i}{\sin^2\psi}[\tilde{X}_2, \tilde{X}_3]\right)\\ &- \frac{2D}{\sin^2\psi}\,\mathrm{Tr}\tilde{X}_1[\tilde{X}_2, \tilde{X}_3]\Big|_{z=-\infty}^{z=\infty} + 2D\int \partial_i\,\mathrm{Tr}B_i\tilde{X}_1 + (Y\text{-dependent terms}), \end{aligned} \tag{4.6}$$

where we have used the Gauss law in completing the squares. Note that there are three boundary terms. Among these, the first term vanishes on imposing the BPS equations and the second term is zero for Coulomb vacua. Therefore, for the half-BPS configurations the energy is proportional to the magnetic charge

$$\mathcal{H} = 2D\int d^3x\partial_i\,\mathrm{Tr}B_i\tilde{X}_1. \tag{4.7}$$

Thus, the 1/2 BPS configuration with nonzero energy would be those with $X_2 = X_3 = 0$ and satisfies Eq. (4.5).

The 1/2-BPS equations for the supersymmetric condition $\Gamma^{70}\epsilon_0 = \epsilon_0$ can be obtained in a similar manner. The resulting equations and the energy are exactly the same as Eqs. (4.2) and (4.7) with $\tilde{X}_a$ and ψ replaced by $\tilde{Y}_a$ and $\pi/2 - \psi$, respectively. After imposing both conditions, we get the 1/4 BPS equations, which are complicated and mixed version of the above 1/2 BPS equations. One novel aspect of 1/4 BPS equations is that the electric and magnetic fields are not parallel to each other.

Since the solution of the BPS equation is a dyonic object in the presence of θ term, we would like to briefly discuss the Witten effect [16] in this case. Assume that the vacuum is given by

$$\mathrm{Tr}\,\phi\phi = \tilde{v}^2, \tag{4.8}$$

where $\phi = \tilde{X}_1$ or $\phi = \tilde{Y}_1$ depending of the supersymmetric condition. The Noether charge n generating the gauge transformation around the direction ϕ is

$$n = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i^a)} \delta A_i^a = \int d^3x \,\mathrm{Tr}\left(\frac{1}{e^2} E_i + \frac{\theta}{8\pi^2} B_i\right) \frac{1}{\tilde{v}} D_i \phi. \quad (4.9)$$

This is quantized as an integer. Using Eqs. (2.7) and (4.5), we find

$$n = \left(\frac{\theta_0}{8\pi^2} \pm D\right) \int d^3x \partial_i \,\mathrm{Tr}(B_i \phi / \tilde{v}) = \left(\frac{\theta_0}{8\pi^2} \pm D\right) Q_M = (\tau_0 \pm 4\pi D) n_m, \quad (4.10)$$

where the upper (lower) sign is for $\phi = \tilde{X}_1 (\tilde{Y}_1)$. Q_M is the magnetic charge quantized as $Q_M = 4\pi n_m$ with n_m being an integer. Therefore, for 1/2-BPS solutions to survive quantum mechanically, the parameters in the coupling should be quantized. Note, however, that there is no simple relation between the electric charge and the magnetic charge if the couplings are not constants. For example, with Eq. (4.5),

$$Q_E = \int d^3x \partial_i \,\mathrm{Tr}\left(\frac{1}{e^2 \tilde{v}} E_i \tilde{X}_1\right) = 2D \int d^3x \partial_i \,\mathrm{Tr}\left(\frac{\sin^2\psi}{\tilde{v}} B_i \tilde{X}_1\right). \quad (4.11)$$

The electric charge would have been a sum of Noether charge and that of the Witten effect if θ is constant.

V. SHARP INTERFACE

Here, we consider the 1/2-BPS case that the coupling constants $e^2(z)$, $\theta(z)$ change from one value to another at a sharp interface so that

$$(e, \theta, \psi) = \begin{cases} (e_1, \theta_1, \psi_1) & \text{for } z > 0 \\ (e_2, \theta_2, \psi_2) & \text{for } z < 0 \end{cases}. \quad (5.1)$$

As there is no matter source at the interface, we get various continuity conditions from the equations of motion. The following quantities are continuous at the interface $z = 0$:

$$E_i, \quad \left(\frac{1}{e^2} E_3 + \frac{\theta}{8\pi^2} B_3\right), \quad B_3, \quad \left(\frac{1}{e^2} B_i - \frac{\theta}{8\pi^2} E_i\right), \quad \tilde{X}_a, \quad D_i \tilde{X}_a, \quad \cot\psi D_3 \tilde{X}_a, \quad \tilde{Y}_p, \quad D_i \tilde{Y}_p, \quad \tan\psi D_3 \tilde{Y}_p, \quad (i = 1, 2). \quad (5.2)$$

From this boundary condition we see that, if θ is not constant, electric charge is induced on the boundary, which is proportional to the magnetic flux Φ through the boundary

$$Q_E^{\text{induced}} = \int \partial_i \left(\frac{1}{e^2} E_i\right) = \frac{\theta_2 - \theta_1}{8\pi^2} \Phi. \quad (5.3)$$

A. Reflection and transmission of waves

Let us consider now a massless wave propagating toward the interface of the two coupling constants from $z > 0$ region. The fields and their derivatives in (5.2) should be continuous across the interface $z = 0$. A part of the incident wave will be reflected and the rest may get refracted or transmitted. Let us call the electromagnetic field of the incident wave to be $\mathbf{E}$, $\mathbf{B}$, the reflected wave to be $\mathbf{E}''$, $\mathbf{B}''$, and the transmitted wave to be $\mathbf{E}'$, $\mathbf{B}'$. The continuity equations at $z = 0$ are

$$\begin{aligned} &(\mathbf{E} + \mathbf{E}'' - \mathbf{E}') \times \hat{\mathbf{z}} = 0, \\ &(\mathbf{B} + \mathbf{B}'' - \mathbf{B}) \cdot \hat{\mathbf{z}} = 0, \\ &\left(\frac{\mathbf{E} + \mathbf{E}''}{e_1^2} - \frac{\mathbf{E}'}{e_2^2} + \frac{\theta_1}{8\pi^2}(\mathbf{B} + \mathbf{B}'') - \frac{\theta_2}{8\pi^2}\mathbf{B}'\right) \cdot \hat{\mathbf{z}} = 0, \\ &\left(\frac{\mathbf{B} + \mathbf{B}''}{e_1^2} - \frac{\mathbf{B}'}{e_2^2} - \frac{\theta_1}{8\pi^2}(\mathbf{E} + \mathbf{E}'') + \frac{\theta_2}{8\pi^2}\mathbf{E}'\right) \times \hat{\mathbf{z}} = 0. \end{aligned} \quad (5.4)$$

The space-time dependence waves would be $e^{-iwt + i\mathbf{k}\cdot\mathbf{x}}$, $e^{-iwt + i\mathbf{k}''\cdot\mathbf{x}}$, and $e^{-iwt + i\mathbf{k}'\cdot\mathbf{x}}$ for the incident, reflected, and transmitted waves, respectively. The wave equation at each region and the above continuity equations imply that

$$w = |\mathbf{k}| = |\mathbf{k}''| = |\mathbf{k}'|, \qquad \mathbf{k} = \mathbf{k}', \qquad (\mathbf{k} + \mathbf{k}'') \cdot \hat{z} = 0. \quad (5.5)$$

Thus the transmitted wave is not refracted at all. After taking out the space-time dependence, we can express the electric fields of the reflected and transmitted waves in terms of the the incident wave. The amplitudes of the transmitted electric fields is given by

$$\mathbf{E}_0' = \frac{\sin 2\psi_1}{\sin(\psi_1 + \psi_2)} [\cos(\psi_1 - \psi_2)\mathbf{E}_0 - \sin(\psi_1 - \psi_2)\mathbf{B}_0]. \quad (5.6)$$

Note that the polarization direction is rotated. For the reflected electric field, the expression explicitly depends on the incident angle and will not be shown here since it is rather complicated. However, if the incident electric field has the form $\mathbf{E}_0 = E_0(\cos 2\psi_1 \hat{\mathbf{m}} + \sin 2\psi_1 \hat{\mathbf{n}})$, where $\hat{\mathbf{m}}$ is the unit vector normal to the plane formed by $\hat{\mathbf{z}}$ and $\mathbf{k}$ and $\hat{\mathbf{n}} = \hat{\mathbf{k}} \times \hat{\mathbf{m}}$, it can be written in a simple form

$$\mathbf{E}_0'' = E_0 \frac{\sin(\psi_1 - \psi_2)}{\sin(\psi_1 + \psi_2)} \hat{\mathbf{m}} = \frac{\sin(\psi_1 - \psi_2)}{\sin(\psi_1 + \psi_2)} (-\cos 2\psi_1 \mathbf{E}_0 + \sin 2\psi_1 \mathbf{B}_0). \quad (5.7)$$

The expression in the second line also holds when the incident wave is normal to the xy plane. The corresponding magnetic fields can be obtained from the relation $\mathbf{B} = \frac{\mathbf{k}}{w} \times \mathbf{E}$. The reflection and the transmission coefficients defined as $E_0'' = rE_0$, $E_0' = tE_0$ are however independent of the details of the incident wave and are always given by

$$r = \left|\frac{\sin(\psi_1 - \psi_2)}{\sin(\psi_1 + \psi_2)}\right|, \qquad t = \left|\frac{\sin 2\psi_1}{\sin(\psi_1 + \psi_2)}\right|. \tag{5.8}$$

B. Gauge fields of a single dyon in the Abelian limit

For simplicity, we consider the SU(2) gauge theory, which is broken spontaneous to U(1) subgroup by the Higgs expectation values at the vacuum

$$\langle \tilde{X}_1 \rangle = \frac{\sigma_3}{\sqrt{2}}\tilde{v}. \tag{5.9}$$

The diagonal components of the fields will be massless, and off-diagonal ones will be massive. We solve the $1/2$ BPS Eq. (4.5) in the Abelian limit where the non-Abelian core size vanishes. For definiteness, we work with the first condition of Eq. (4.1). For a single dyon with charge (q, g) at $z = a > 0$, electric and magnetic fields would have the form

$$\mathbf{B} = \begin{cases} \frac{g}{4\pi}\frac{(x,y,z-a)}{r_+^3} + \frac{g'}{4\pi}\frac{(x,y,z+a)}{r_-^3}, & z > 0 \\ \frac{g''}{4\pi}\frac{(x,y,z-a)}{r_+^3}, & z < 0, \end{cases}$$
$$\mathbf{E} = \begin{cases} \frac{e_1^2 q}{4\pi}\frac{(x,y,z-a)}{r_+^3} + \frac{e_1^2 q'}{4\pi}\frac{(x,y,z+a)}{r_-^3}, & z > 0 \\ \frac{e_2^2 g''}{4\pi}\frac{(x,y,z-a)}{r_+^3}, & z < 0, \end{cases} \tag{5.10}$$

where $r_\pm^2 = x^2 + y^2 + (z \mp a)^2$, and the group factor $\sigma_3/\sqrt{2}$ is omitted for simplicity. The image charges (q', g') and (q'', g'') are to be determined from the BPS equations and the boundary conditions. The configuration of scalar field $\tilde{X}_1$ may be obtained by integrating the electric field through $E_i = D_i\tilde{X}_1$. From the BPS equation $E_i = \tan\psi B_i$, it immediately follows that

$$q = \frac{g}{e_1^2}\tan\psi_1, \qquad q' = \frac{g'}{e_1^2}\tan\psi_1, \qquad q'' = \frac{g''}{e_2^2}\tan\psi_2. \tag{5.11}$$

In addition, we have four equations from the boundary conditions Eq. (5.2). Since we have only four unknowns it may look over constrained. However, with the help of the relation Eq. (2.7) between e^2 and θ, we can find a solution satisfying all the equations,

$$\begin{aligned} g' &= -g\frac{\sin(\psi_1 - \psi_2)}{\sin(\psi_1 + \psi_2)} \\ g'' &= 2g\frac{\sin\psi_1\cos\psi_2}{\sin(\psi_1 + \psi_2)} \\ q' &= -2gD\frac{\sin^2\psi_1\sin(\psi_1 - \psi_2)}{\sin(\psi_1 + \psi_2)} \\ q'' &= 4gD\frac{\sin\psi_1\sin^2\psi_2\cos\psi_2}{\sin(\psi_1 + \psi_2)}. \end{aligned} \tag{5.12}$$

The magnetic and electric fluxes to the northern and southern hemispheres are, respectively,

$$\begin{aligned} \Phi_M^N &= \tfrac{1}{2}(g + g'), \qquad & \Phi_M^S &= \tfrac{1}{2}g'', \\ \Phi_E^N &= \tfrac{1}{2}(q + q'), \qquad & \Phi_E^S &= \tfrac{1}{2}q'', \end{aligned} \tag{5.13}$$

and the charges Q_M and Q_E in Eqs. (4.10) and (4.11) are given by the total fluxes

$$\begin{aligned} Q_M &= \tfrac{1}{2}(g + g' + g'') = g, \\ Q_E &= \tfrac{1}{2}(q + q' + q'') = 2gD\sin\psi_1\sin\psi_2\cos(\psi_1 - \psi_2), \end{aligned} \tag{5.14}$$

which satisfy Eq. (4.11) as it should be.

One may wonder whether this object has nonzero field angular momentum. At a first look this has to be the case because a dyon produces nonzero angular momentum in the background of an axionic domain wall [18] for which θ changes from zero to 2π, while the coupling e^2 remains a constant. For the present case, however, the electric field is proportional to the magnetic field thanks to the half-BPS equation and hence the field angular momentum defined by

$$\mathbf{M} = \int d^3x\mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \tag{5.15}$$

is identically zero.

The magnetic flux through the xy plane is

$$\Phi_M^{a>0} = -\Phi_M^S = -\tfrac{1}{2}g'', \tag{5.16}$$

which induces electric charge on the boundary as in Eq. (5.3). If the dyon is on the negative z axis ($a < 0$), the corresponding magnetic flux on the xy plane would be

$$\Phi_M^{a<0} = \tfrac{1}{2}g''|_{1\leftrightarrow 2}. \tag{5.17}$$

Let us now consider the situation that a dyon with charge $(\tilde{q}, g)$ at $z < 0$ region passes from the xy plane to $z > 0$, where $\tilde{q} = g\tan\psi_2/e_2^2$ as given in (5.11). Since the coupling constants change from (e_2, θ_2) to (e_1, θ_1), the charge should change to (q, g), accordingly. It is interesting to check how the conservation of electric charge works. In fact, as the dyon passes the xy plane the induced electric charge (5.3) should also change due to the change of magnetic flux, which is given by

$$\begin{aligned} \Delta\Phi &= \Phi_M^{a>0}(q, g) - \Phi_M^{a<0}(\tilde{q}, g) \\ &= -g\frac{\sin\psi_1\cos\psi_2}{\sin(\psi_1 + \psi_2)} - g\frac{\sin\psi_2\cos\psi_1}{\sin(\psi_1 + \psi_2)} = -g. \end{aligned} \tag{5.18}$$

Then,

$$\Delta Q_E^{\text{induced}} = \frac{\theta_1 - \theta_2}{8\pi^2} g. \tag{5.19}$$

This is to be compared with the change of the electric charge of the dyon

$$\Delta q = q - \tilde{q} = \frac{g}{e_1^2}\tan\psi_1 - \frac{g}{e_2^2}\tan\psi_2, \tag{5.20}$$

where we have used Eq. (5.11). On using Eq. (2.7), this is precisely cancelled by $\Delta Q_E^{\text{induced}}$ in Eq. (5.19). Usually the induced charge on the axion domain wall is due to the polarization of fermions, which led the photon-axion interaction. In our case, a further study is needed to clarify the exact nature of the sharp interface.

If the 1/4-BPS configurations are considered, the electric and magnetic fields are not parallel to each other in general, and more richer configurations with nonzero angular momentum would appear near a sharp interface. The detail will be left as an exercise.

VI. ADDITIONAL SUSY BREAKING JANUS

Let us consider the further supersymmetry breaking Janus configurations. This could happen two ways. First is to have additional terms in the Lagrangian while keeping the coupling constant $\tau(z)$ depending only on one spatial coordinate. Another is to introduce additional space-time dependence to the coupling constant and then correct the Lagrangian. We have done a full classification in Ref. [5] without the θ term, and the same classification works for the present case as long as we keep the supersymmetric condition on ϵ_0, which we describe in the following.

A. The $\tau(z)$ case

In this subsection we are still interested in the case where the coupling constant $e^2(z)$, $\theta(z)$ depends only on one spatial coordinate. We can impose additional constraints on the SUSY parameters ϵ_0, which is compatible with what we have already imposed. We then find the corrections to the Lagrangian and SUSY transformation, which needs several undetermined parameters. Depending on the value of these parameters, the number of preserved supersymmetry would be 8, 4, or 2.

We impose on the ten-dimensional Majorana Weyl spinor ϵ, the four conditions including one in (2.10),

$$\begin{aligned} \Gamma^{3456}\epsilon_0 &= \epsilon_0, & \Gamma^{3489}\epsilon_0 &= -\epsilon_0, \\ \Gamma^{3597}\epsilon_0 &= -\epsilon_0, & \Gamma^{3678}\epsilon_0 &= -\epsilon_0. \end{aligned} \tag{6.1}$$

As the product of the above four conditions is an identity, there are only three independent conditions, breaking the supersymmetry to 1/8th or two supersymmetries.

To cancel the supersymmetric variation (2.5) we add to the Lagrangian the following three terms:

$$\begin{aligned} \mathcal{L}_1 &= i\frac{\psi'}{4e^2}\bar{\Psi}\Big[-\Gamma^{012} + \frac{1}{\sin\psi}(c_0\Gamma^{456} - c_1\Gamma^{489} - c_2\Gamma^{597} \\ &\quad - c_3\Gamma^{678}) - \frac{1}{\cos\psi}(c_0\Gamma^{789} - c_1\Gamma^{567} - c_2\Gamma^{648} \\ &\quad - c_3\Gamma^{459})\Big]\Psi, \\ \mathcal{L}_2 &= -i\frac{2\psi'}{e^2\sin\psi}\mathrm{Tr}(c_0\phi_4[\phi_5,\phi_6] - c_1\phi_4[\phi_8,\phi_9] \\ &\quad - c_2\phi_5[\phi_9,\phi_7] - c_3\phi_6[\phi_7,\phi_8]) \\ &\quad + i\frac{2\psi'}{e^2\cos\psi}\mathrm{Tr}(c_0\phi_7[\phi_8,\phi_9] - c_1\phi_5[\phi_6,\phi_7] \\ &\quad - c_2\phi_6[\phi_4,\phi_8] - c_3\phi_4[\phi_5,\phi_9]), \\ \mathcal{L}_3 &= \sum_{I=4}^{9} r_I\,\mathrm{Tr}\phi_I^2, \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} r_I &= D\psi'^2[c_I(c_I + \sin^2\psi)\cot\psi \\ &\quad + (1-c_I)(1-c_I+\cos^2\psi)\tan\psi] \\ &\quad - D\psi''(c_I - \sin^2\psi), \end{aligned} \tag{6.3}$$

and c_I's are real constants satisfying

$$\begin{aligned} &c_0 + c_1 + c_2 + c_3 = 1, \qquad c_4 = c_0 + c_1, \\ &c_5 = c_0 + c_2, \qquad c_6 = c_0 + c_3, \qquad c_7 = c_2 + c_3, \\ &c_8 = c_1 + c_3, \qquad c_9 = c_1 + c_2. \end{aligned} \tag{6.4}$$

The correction of the supersymmetric transformation is

$$\delta_1\Psi = -\psi'(\cot\psi\,\Gamma\cdot X - \tan\psi\,\Gamma\cdot Y)\Gamma^3\epsilon, \tag{6.5}$$

where

$$\begin{aligned} \Gamma\cdot X &\equiv c_0\sum_{a=4,5,6}\Gamma^a\phi_a + c_1\sum_{a=4,8,9}\Gamma^a\phi_a + c_2\sum_{a=5,9,7}\Gamma^a\phi_a \\ &\quad + c_3\sum_{a=6,7,8}\Gamma^a\phi_a, \\ \Gamma\cdot Y &\equiv c_0\sum_{p=7,8,9}\Gamma^p\phi_p + c_1\sum_{p=5,6,7}\Gamma^p\phi_p + c_2\sum_{p=6,4,8}\Gamma^p\phi_p \\ &\quad + c_3\sum_{p=4,5,9}\Gamma^p\phi_p. \end{aligned} \tag{6.6}$$

Then the total Lagrangian $\mathcal{L}_0 + \mathcal{L}_\theta + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ is invariant under the corrected supersymmetric transformation. For a generic values of constants c_I, the number of supersymmetry is two. If two of c_0, c_1, c_2, c_3 vanish, it is enhanced to four. If only one of them is nonvanishing, we will have eight supersymmetries as in the previous sections.

B. The $\tau(y, z)$ case

In our previous work [5], we classified all supersymmetric theories with compatible intersecting Janus interface, both in the two-dimensional and three-dimensional case. One can work out a similar analysis to the current case with the nontrivial θ term. Let us first focus on the case where the coupling constants are functions of two coordinates $e^2(y, z)$ and $\theta(y, z)$. As before, compatible supersymmetric conditions can be expressed by introducing a constant Majorana Weyl spinor ϵ_0 defined by

$$\epsilon(y, z) = e^{-(1/2)\psi(y,z)\Gamma^{0123}}\epsilon_0. \tag{6.7}$$

One can impose three compatible supersymmetry conditions

$$\Gamma^{2789}\epsilon_0 = \Gamma^{3456}\epsilon_0 = \epsilon_0, \tag{6.8}$$

$$-\Gamma^{2459}\epsilon_0 = \Gamma^{3456}\epsilon_0 = \epsilon_0, \tag{6.9}$$

$$-\Gamma^{2567}\epsilon_0 = \Gamma^{3456}\epsilon_0 = \epsilon_0, \tag{6.10}$$

which implies

$$\Gamma^{2648}\epsilon_0 = \Gamma^{3489}\epsilon_0 = \Gamma^{3597}\epsilon_0 = \Gamma^{3678}\epsilon_0 = -\epsilon_0. \tag{6.11}$$

Each condition breaks the supersymmetry to $1/2$ and imposing the three at the same time breaks the supersymmetry to the minimal one $1/16$.

With these supersymmetric conditions we consider the following interface Lagrangian:

$$\begin{aligned}\mathcal{L}_1 = &-i\frac{\partial_3\psi}{4e^2}\mathrm{Tr}(\bar{\Psi}\Gamma^{012}\Psi) + i\frac{\partial_2\psi}{4e^2}\mathrm{Tr}(\bar{\Psi}\Gamma^{013}\Psi)\\ &+ i\frac{\partial_3\psi}{4e^2}\mathrm{Tr}(\bar{\Psi}(\csc\psi M_3 - \sec\psi N_3)\Psi)\\ &+ i\frac{\partial_2\psi}{4e^2}\mathrm{Tr}(\bar{\Psi}(\csc\psi M_2 - \sec\psi N_2)\Psi),\end{aligned} \tag{6.12}$$

and

$$\begin{aligned}\mathcal{L}_2 = &-2i\frac{\partial_3\psi}{e^2}\csc\psi\,\mathrm{Tr}(c_0\phi_4[\phi_5,\phi_6] - c_1\phi_4[\phi_8,\phi_9] - c_2\phi_5[\phi_9,\phi_7] - c_3\phi_6[\phi_7,\phi_8])\\ &+ 2i\frac{\partial_3\psi}{e^2}\sec\psi\,\mathrm{Tr}(c_0\phi_7[\phi_8,\phi_9] - c_1\phi_5[\phi_6,\phi_7] - c_2\phi_6[\phi_4,\phi_8] - c_3\phi_4[\phi_5,\phi_9])\\ &- 2i\frac{\partial_2\psi}{e^2}\csc\psi\,\mathrm{Tr}(b_0\phi_7[\phi_8,\phi_9] - b_1\phi_5[\phi_6,\phi_7] - b_2\phi_6[\phi_4,\phi_8] - b_3\phi_4[\phi_5,\phi_9])\\ &+ 2i\frac{\partial_2\psi}{e^2}\sec\psi\,\mathrm{Tr}(b_0\phi_4[\phi_5,\phi_6] - b_1\phi_4[\phi_8,\phi_9] - b_2\phi_5[\phi_9,\phi_7] - b_3\phi_6[\phi_7,\phi_8]),\end{aligned} \tag{6.13}$$

where M_m, N_m, $(m = 2, 3)$ are matrices defined by

$$\begin{aligned}M_3 &\equiv c_0\Gamma^{456} - c_1\Gamma^{489} - c_2\Gamma^{597} - c_3\Gamma^{678}\\ N_3 &\equiv c_0\Gamma^{789} - c_1\Gamma^{567} - c_2\Gamma^{648} - c_3\Gamma^{459}\\ M_2 &\equiv b_0\Gamma^{789} - b_1\Gamma^{567} - b_2\Gamma^{648} - b_3\Gamma^{459}\\ N_2 &\equiv -(b_0\Gamma^{456} - b_1\Gamma^{489} - b_2\Gamma^{597} - b_3\Gamma^{678}),\end{aligned} \tag{6.14}$$

and c_i, b_i are real parameters satisfying

$$\sum_{i=0}^{3} c_i = \sum_{i=0}^{3} b_i = 1, \tag{6.15}$$

so that there are six independent parameters. For convenience we also denote $c_a^{(2)} = b_a$ and $c_a^{(3)} = c_a$. Note that the following properties hold for M_m and N_m:

$$\Gamma^{0123}M_m\epsilon = N_m\epsilon, \qquad (\cos\psi M_m + \sin\psi N_m)\epsilon = \Gamma^m\epsilon. \tag{6.16}$$

Now we define the correction to the supersymmetric transformation (2.2) $\delta_1\Psi$ as

$$\delta_1\Psi = -\partial_3\psi B_3\Gamma^3\epsilon - \partial_2\psi B_2\Gamma^2\epsilon, \tag{6.17}$$

where

$$B_m = \cot\psi\sum_{a=4}^{9} c_a^{(m)}\Gamma^a\phi_a - \tan\psi\sum_{a=4}^{9} c_{a+3}^{(m)}\Gamma^a\phi_a, \quad (m = 2, 3). \tag{6.18}$$

In this expression $c_4^{(m)}, \ldots, c_9^{(m)}$ are given in terms of $c_1^{(m)}$, $c_2^{(m)}$, $c_3^{(m)}$ as

$$\begin{aligned}c_4 &\equiv c_0 + c_1, & c_5 &\equiv c_0 + c_2, & c_6 &\equiv c_0 + c_3,\\ c_7 &\equiv c_2 + c_3, & c_8 &\equiv c_1 + c_3, & c_9 &\equiv c_1 + c_2,\end{aligned} \tag{6.19}$$

and

$$\begin{aligned}b_4 &\equiv b_2 + b_3, & b_5 &\equiv b_1 + b_3, & b_6 &\equiv b_1 + b_2,\\ b_7 &\equiv b_0 + b_1, & b_8 &\equiv b_0 + b_2, & b_9 &\equiv b_0 + b_3.\end{aligned} \tag{6.20}$$

The index a is understood to be cyclic in $4, 5, \ldots, 9$, i.e., $c_{10} = c_4$, $c_{11} = c_5$, and so on. With these, it is not difficult to show that

$$(\cos\psi M_m - \sin\psi N_m)B_n\epsilon = \left[-B_n + 2\sum_{a=4}^{9} c_a^{(m)}(c_a^{(n)}\cot\psi - c_{a+3}^{(n)}\tan\psi)\Gamma^a\phi_a\right]\Gamma^m\epsilon. \tag{6.21}$$

Also note that for $a = 4, 5, \ldots, 9$,

$$c_a + c_{a+3} = 1, \qquad c_a b_{a+3} - b_a c_{a+3} = c_a - b_a, \qquad c_{a+3} - b_{a+3} = -(c_a - b_a). \tag{6.22}$$

We are ready to find the supersymmetric Langrangian. First, one can show that the zeroth variation $\delta_0(\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2)$ is cancelled by the first order terms from $\delta_1\mathcal{L}_1$. The only nonvanishing terms are $\delta_1\mathcal{L}_1$ and the second order terms from $\delta_1\mathcal{L}_0$, which should be cancelled by introducing another term $\mathcal{L}_3$ in the Lagrangian. Utilizing the above properties among the parameters and matrices, it can readily be shown that the desired term is

$$\begin{aligned}\mathcal{L}_3 = &-\sum_{m=2,3}\frac{(\partial_m\psi)^2}{e^2}\sum_{a=4}^{9}[c_a^{(m)}(1 + c_a^{(m)}\csc^2\psi) + c_{a+3}^{(m)}(1 + c_{a+3}^{(m)}\sec^2\psi)]\phi_a^2 \\ &+ \sum_{m=2,3}\frac{\partial_m^2\psi}{2e^2}\sum_{a=4}^{9}[c_a^{(m)}\cot\psi - c_{a+3}^{(m)}\tan\psi]\phi_a^2 - \frac{\partial_2\psi\,\partial_3\psi}{e^2}(\csc^2\psi - \sec^2\psi) \\ &\times\sum_{a=4,5,6}(c_a - b_a)\,\mathrm{Tr}(\phi_a\phi_{a+3}) + \frac{\partial_2\partial_3\psi}{e^2}(\cot\psi + \tan\psi)\sum_{a=4,5,6}(c_a - b_a)\,\mathrm{Tr}(\phi_a\phi_{a+3}).\end{aligned} \tag{6.23}$$

Then the full Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\theta + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ is invariant under the suupersymmetric transformation $(\delta_0 + \delta_1)\mathcal{L} = 0$.

C. The $\tau(x, y, z)$ case

When the coupling constants depend on all three coordinates, $e^2(x, y, z)$ and $\theta(x, y, z)$, there are two independent supersymmetry conditions:

$$\begin{aligned}&\Gamma^{1467}\epsilon_0 = \Gamma^{2475}\epsilon_0 = \Gamma^{3456}\epsilon_0 = \epsilon_0, \\ &\Gamma^{1458}\epsilon_0 = \Gamma^{2468}\epsilon_0 = \Gamma^{3478}\epsilon_0 = \epsilon_0.\end{aligned} \tag{6.24}$$

As in the previous section, we define

$$\begin{aligned}&M_1 \equiv a_1\Gamma^{467} + a_2\Gamma^{458}, \qquad N_1 \equiv a_1\Gamma^{589} + a_2\Gamma^{679}, \\ &M_2 \equiv b_1\Gamma^{475} + b_2\Gamma^{468}, \qquad N_2 \equiv b_1\Gamma^{689} + b_2\Gamma^{597}, \\ &M_3 \equiv c_1\Gamma^{456} + c_2\Gamma^{478}, \qquad N_3 \equiv c_1\Gamma^{789} + c_2\Gamma^{569},\end{aligned} \tag{6.25}$$

where parameters satisfy the relation

$$a_1 + a_2 = b_1 + b_2 = c_1 + c_2 = 1. \tag{6.26}$$

The correction to the supersymmetric transformation is

$$\delta_1\lambda = -\partial_1\psi B_1\Gamma^1\epsilon - \partial_2\psi B_2\Gamma^2\epsilon - \partial_3\psi B_3\Gamma^3\epsilon, \tag{6.27}$$

where B_m ($m = 1, 2, 3$) are given by

$$B_m = \cot\psi\left(\Gamma^4\phi_4 + \sum_{i=5}^{8}c_i^{(m)}\Gamma^i\phi_i\right) - \tan\psi\left(\Gamma^9\phi_9 + \sum_{i=5}^{8}(1 - c_i^{(m)})\Gamma^i\phi_i\right), \tag{6.28}$$

and here we introduced the notations as before,

$$\begin{aligned}&a_5 = a_2, \quad a_6 = a_1, \quad a_7 = a_1, \quad a_8 = a_2, \\ &b_5 = b_1, \quad b_6 = b_2, \quad b_7 = b_1, \quad b_8 = b_2, \\ &c_5 = c_1, \quad c_6 = c_1, \quad c_7 = c_2, \quad c_8 = c_2, \\ &c_i^{(1)} \equiv a_i, \quad c_i^{(2)} \equiv b_i, \quad c_i^{(3)} \equiv c_i \quad (i = 5, 6, 7, 8).\end{aligned} \tag{6.29}$$

Then the correction terms to the Lagrangian again turn out to consist of three terms:

$$\begin{aligned}\mathcal{L}_1 = -\frac{i}{4e^2}\mathrm{Tr}[&\partial_1\psi\bar{\lambda}\Gamma^{023}\lambda + \partial_2\psi\bar{\lambda}\Gamma^{031}\lambda + \partial_3\psi\bar{\lambda}\Gamma^{012}\lambda - \partial_1\psi\bar{\lambda}(\csc\psi M_1 - \sec\psi N_1)\lambda \\ &- \partial_2\psi\bar{\lambda}(\csc\psi M_2 - \sec\psi N_2)\lambda - \partial_3\psi\bar{\lambda}(\csc\psi M_3 - \sec\psi N_3)\lambda],\end{aligned} \tag{6.30}$$

$$\begin{aligned}\mathcal{L}_2 = &-2i\frac{\partial_1\psi}{e^2}\,\mathrm{Tr}\{\csc\psi(a_1\phi_4[\phi_6,\phi_7]+a_2\phi_4[\phi_5,\phi_8])-\sec\psi(a_1\phi_5[\phi_8,\phi_9]+a_2\phi_6[\phi_7,\phi_9])\}\\ &-2i\frac{\partial_2\psi}{e^2}\,\mathrm{Tr}\{\csc\psi(b_1\phi_4[\phi_7,\phi_5]+b_2\phi_4[\phi_6,\phi_8])-\sec\psi(b_1\phi_6[\phi_8,\phi_9]+b_2\phi_5[\phi_9,\phi_7])\}\\ &-2i\frac{\partial_3\psi}{e^2}\,\mathrm{Tr}\{\csc\psi(c_1\phi_4[\phi_5,\phi_6]+c_2\phi_4[\phi_7,\phi_8])-\sec\psi(c_1\phi_7[\phi_8,\phi_9]+c_2\phi_5[\phi_6,\phi_9])\},\end{aligned} \tag{6.31}$$

and

$$\begin{aligned}\mathcal{L}_3 = &-\sum_{m=1}^{3}\frac{(\partial_m\psi)^2}{2e^2}\,\mathrm{Tr}[(1+\csc^2\psi)\phi_4^2+(1+\sec^2\psi)\phi_9^2]\\ &-\sum_{m=1}^{3}\frac{(\partial_m\psi)^2}{2e^2}\sum_{i=5}^{8}\mathrm{Tr}[1+(c_i^{(m)})^2\csc^2\psi+(1-c_i^{(m)})^2\sec^2\psi]\phi_i^2\\ &+\sum_{m=1}^{3}\frac{\partial_m^2\psi}{2e^2}\,\mathrm{Tr}\Big[\cot\psi\Big(\phi_4^2+\sum_{i=5}^{8}c_i^{(m)}\phi_i^2\Big)-\tan\psi\Big(\phi_9^2+\sum_{i=5}^{8}(1-c_i^{(m)})^2\phi_i^2\Big)\Big]\\ &-\frac{1}{e^2}[\partial_1\psi\partial_2\psi(\csc^2\psi-\sec^2\psi)-\partial_1\partial_2\psi(\cot\psi+\tan\psi)]\mathrm{Tr}[(a_2-b_1)\phi_5\phi_6+(a_1-b_1)\phi_7\phi_8]\\ &-\frac{1}{e^2}[\partial_2\psi\partial_3\psi(\csc^2\psi-\sec^2\psi)-\partial_2\partial_3\psi(\cot\psi+\tan\psi)]\mathrm{Tr}[(b_2-c_1)\phi_6\phi_7+(b_1-c_1)\phi_5\phi_8]\\ &-\frac{1}{e^2}[\partial_3\psi\partial_1\psi(\csc^2\psi-\sec^2\psi)-\partial_3\partial_1\psi(\cot\psi+\tan\psi)]\mathrm{Tr}[(c_2-a_1)\phi_5\phi_7+(c_1-a_1)\phi_6\phi_8].\end{aligned} \tag{6.32}$$

It is straightforward to show that the full Lagrangian is invariant,

$$(\delta_0+\delta_1)(\mathcal{L}_0+\mathcal{L}_\theta+\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3)=0. \tag{6.33}$$

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