Zero modes of the nonrelativistic SU(2) Chern-Simons solitons

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It is demonstrated that $4n$ parameters are required to specify a nonrelativistic n-soliton solution in the nonlinear planar Schrödinger equation coupled to the SU(2) Chern-Simons gauge fields. This result implies that the most general soliton solutions of the theory can be obtained from the Toda ansatz.

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I. INTRODUCTION

U(1)-invariant gauge theories in $2+1$ space-time dimensions with a Chern-Simons term possess static soliton solutions that satisfy self-dual equations [1]. Especially, in the nonrelativistic limit the self-dual equations produce the Liouville equation, all of whose solutions are known [2]. The resulting U(1)-invariant n-soliton solutions depend on $4n$ parameters, interpreted as $2n$ locations, n scales, and n phases, with one overall phase being irrelevant. An index theorem confirms that $4n$ is indeed the correct number [3]. The problem can be generalized to the case where the invariance group is non-Abelian [4,5].

Dunne, Jackiw, Pi, and Trungenberger [6] (DJPT) made a systematic analysis of the nonlinear planar Schrödinger equation with a coupling to a non-Abelian Chern-Simons term and its self-dual equation. In the nonrelativistic limit with the Toda ansatz DJPT have found special solutions of the self-dual equations of the theory and have shown that the Chern-Simons theory provides a unified framework for a variety of nonlinear equations (Toda, affine Toda, sinh-Gordon, nonlinear σ model CP^N model, Bullough-Dodd, etc.). However, general solutions of the self-dual equations of the theory are not yet known. It will be interesting then to count the degrees of freedom needed to describe the most general configuration of the n-soliton system of the theory. The degrees of freedom within the Toda ansatz have been computed with the result of $4n$ [7]. A main result of this paper is that this number of parameters entering the most general soliton solution is $4n$.

The system we are studying is governed by the Hamiltonian

$$H=\int d^2x\left\{\frac{1}{2m}(\mathbf{D}\Phi)^\dagger(\mathbf{D}\Phi)-\frac{g^2}{2mk}(\Phi^\dagger T^a\Phi)(\Phi^\dagger T^a\Phi)\right\},\quad k<0 \tag{1.1}$$

where Φ is the matter field matrix transforming under local gauge transformations according to the adjoint representation of the SU(2) gauge group,

$$\begin{aligned}&[T^a,T^b]=i\epsilon_{abc}T^c,\quad (T^a)_{bc}=-i\epsilon_{abc}\ ,\\ &\mathbf{D}\Phi=\nabla\Phi-ig[\mathbf{A},\Phi]\ ,\end{aligned} \tag{1.2}$$

and $\mathbf{A}$ is the vector field matrix which is determined by the Chern-Simons equation

$$B=\partial_1A_2-\partial_2A_1-ig[A_1,A_2]=-\frac{g}{k}[\Phi,\Phi^\dagger]\ . \tag{1.3}$$

The Hamiltonian achieves its minimum when Φ satisfies the static self-dual equation

$$(D_1+iD_2)\Phi=0\ . \tag{1.4}$$

Using the notation $\partial_\pm=\partial_1\pm i\partial_2$, $A_\pm=A_1\pm iA_2$, we rewrite Eqs. (1.3) and (1.4) in the form

$$\partial_+A_--\partial_-A_+-ig[A_+,A_-]=\frac{2ig}{k}[\Phi,\Phi^\dagger]\ , \tag{1.5}$$

$$\partial_+\Phi-ig[A_+,\Phi]=0\ . \tag{1.6}$$

In the Cartan-Weyl basis, the field $\Phi,\Phi^\dagger$, and A can be decomposed as

$$\Phi=\phi^{(3)}T_3+\phi^{(-)}T_++\phi^{(+)}T_-\ , \tag{1.7}$$

$$A_\pm = A_\pm^{(3)}T_3 + A_\pm^{(-)}T_+ + A_\pm^{(+)}T_- \;, \tag{1.8}$$

where $T_\pm = (1/\sqrt{2})(T_1 \pm iT_2)$. Note that $\phi^{(-)}$ and $\phi^{(+)}$ are not complex conjugate to each other unless $\Phi = \Phi^\dagger$. The self-dual equation (1.6) then yields three equations:

$$\partial_+\phi^{(3)} - ig(A_+^{(-)}\phi^{(+)} - A_+^{(+)}\phi^{(-)}) = 0 \;, \tag{1.9}$$

$$\partial_+\phi^{(-)} - ig(A_+^3\phi^{(-)} - A_+^{(-)}\phi^{(+)}) = 0 \;, \tag{1.10}$$

$$\partial_+\phi^{(+)} - ig(A_+^{(+)}\phi^{(3)} - A_+^{(3)}\phi^{(+)} =)0 \;, \tag{1.11}$$

and the Chern-Simons equation (1.5) yields

$$\partial_+A_-^{(3)} - \partial_-A_+^{(3)} - ig(A_+^{(-)}A_-^{(+)} - A_+^{(+)}A_-^{(-)}) = \frac{2ig}{\kappa}(\phi^{(-)}\phi^{(-)*} - \phi^{(+)}\phi^{(+)*}) \;, \tag{1.12}$$

$$\partial_+A_-^{(-)} - \partial_-A_+^{(+)} - ig(A_+^{(3)}A_-^{(-)} - A_+^{(-)}A_-^{(3)}) = \frac{2ig}{\kappa}(\phi^{(3)}\phi^{(+)*} - \phi^{(-)}\phi^{(3)*}) \;, \tag{1.13}$$

$$\partial_+A_-^{(+)} - \partial_-A_+^{(+)} - ig(A_+^{(+)}A_-^{(3)} - A_+^{(3)}A_-^{(+)}) = \frac{2ig}{\kappa}(\phi^{(+)}\phi^{(3)*} - \phi^{(3)}\phi^{(-)*}) \;. \tag{1.14}$$

Various ansatz can be made for soliton-type solutions of these equations. One simple soliton-type solution can be obtained from the Toda ansatz ($\Phi = \phi^{(-)}T_+, A = A^{(3)}T_3$). With this ansatz, the self-dual equation and the Chern-Simons equation become, respectively,

$$\partial_+\phi^{(-)} - igA_+^{(3)}\phi^{(-)} = 0 \;, \tag{1.15}$$

$$\partial_+A_-^{(3)} - \partial_-A_+^{(3)} = \frac{2ig}{k}|\phi^{(-)}|^2 \;. \tag{1.16}$$

The soliton-type solutions of these two equations have been studied in detail by Jackiw and Pi [2]. For later use, we present the spherically symmetric n-soliton solution of Eqs. (1.15) and (1.16):

$$\phi^{(-)}(r,\theta) = e^{i(n-1)\theta}\frac{2n}{\sqrt{\alpha}r}\left[\frac{1}{r^n + r^{-n}}\right], \quad \alpha = \frac{g^2}{|k|} \;, \tag{1.17}$$

$$g\,\mathbf{A} = \frac{n}{r}\frac{2r^n}{r^n + r^{-n}}(-\mathbf{i}\sin\theta + \mathbf{j}\cos\theta) \;. \tag{1.18}$$

This paper is organized as follows. In Sec. II we derive eigenvalue equations for zero modes of the SU(2) Chern-Simons solitons. In Sec. III, we solve these eigenvalue equations to exhibit the $4n$ zero modes explicitly. Concluding remarks and discussions are given in the last section.

II. EIGENVALUE EQUATIONS FOR ZERO MODES

In order to count the parameters entering the general static soliton solutions of the SU(2) Chern-Simons system, we take infinitesimal variations of Eqs. (1.5) and (1.6). On the Toda backgrounds, infinitesimal fluctuations preserving self-duality satisfy the equations

$$\partial_+\delta\phi^{(3)} + ig\phi^{(-)}\delta A_+^{(+)} = 0 \;, \tag{2.1}$$

$$(\partial_+ - igA_+^{(3)})\delta\phi^{(-)} - ig\phi^{(-)}\delta A_+^{(3)} = 0 \;, \tag{2.2}$$

$$(\partial_+ + igA_+^{(3)})\delta\phi^{(+)} = 0 \;, \tag{2.3}$$

$$(\partial_+\delta A_-^{(3)} - \partial_-\delta A_+^{(3)}) - \frac{2ig}{k}(\phi^{(-)}\delta\phi^{(-)*} + \phi^{(-)*}\delta\phi^{(-)} = 0 \;, \tag{2.4}$$

$$(\partial_+ + igA_+^{(3)})\delta A_-^{(+)} - (\partial_- + igA_-^{(3)})\delta A_+^{(+)} + \frac{2ig}{k}\phi^{(-)*}\delta\phi^{(3)} = 0 \;. \tag{2.5}$$

These equations will admit an infinite family of solutions connected with gauge transformations. In order to remove gauge degrees of freedom from Eqs. (2.1)–(2.5), we impose a gauge condition. There are two convenient gauges: the Coulomb gauge and background gauge, where the latter requires the fluctuations to be orthogonal to the gauge transformations whose parameter vanishes at spatial infinity. We will consider both gauge conditions by introducing a parameter in the equation

$$[\partial_i - i\epsilon g A_i, g\delta A_i] + \frac{i}{2}\epsilon[\delta\Phi, \Phi^\dagger] + \frac{i}{2}\epsilon[\delta\Phi^\dagger, \Phi] = 0 \;. \tag{2.6}$$

The Coulomb and the background gauges correspond to $\epsilon = 0$ and $\epsilon = 1$, respectively.

With the Toda backgrounds, Eq. (2.6) implies

$$g[\partial_+\delta A_-^{(3)} + \partial_-\delta A_+^{(3)}] + i\epsilon[\phi^{(-)*}\delta\phi^{(-)} - \phi^{(-)}\delta\phi^{(-)*}] = 0 \;, \tag{2.7}$$

$$g[\partial_+\delta A_-^{(+)} + \partial_-\delta A_+^{(+)}] + ig^2\epsilon[A_+^{(3)}\delta A_-^{(+)} + A_-^{(3)}\delta A_+^{(+)}] - i\epsilon\phi^{(-)*}\delta\phi^{(3)} = 0 \;. \tag{2.8}$$

If we write

$$\phi^{(a)} = \phi_R^{(a)} + i\phi_I^{(a)}, \quad a = +,-,3 \;, \qquad A_i^{(+)} = A_{iR}^{(+)} + iA_{iI}^{(+)}, \quad A_+^{(3)} = A_1^{(3)} + iA_2^{(3)} \;, \tag{2.9}$$

Eqs. (2.1)–(2.6) take the matrix form, which is fully reducible as

$$D\eta = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = 0 \;, \tag{2.10}$$

where

$$D_1\eta_1=\begin{bmatrix} \partial_1+gA_2^{(3)} & -\partial_2+gA_1^{(3)} & \phi_I^{(-)} & \phi_R^{(-)} \\ \partial_2-gA_1^{(3)} & \partial_1+gA_2^{(3)} & -\phi_R^{(-)} & \phi_I^{(-)} \\ 2\alpha\phi_R^{-} & 2\alpha\phi_I^{-} & \partial_2 & -\partial_1 \\ \epsilon\phi_I^{(-)} & -\epsilon\phi_R^{(-)} & \partial_1 & \partial_2 \end{bmatrix}\begin{bmatrix} \delta\phi_R^{(-)} \\ \delta\phi_I^{(-)} \\ g\delta A_1^{(3)} \\ g\delta A_2^{(3)} \end{bmatrix}=0 , \tag{2.11}$$

$$D_2\eta_2=\begin{bmatrix} \partial_1-gA_2^{(3)} & -\partial_2-gA_1^{(3)} \\ \partial_2+gA_1^{(3)} & \partial_1-gA_2^{(3)} \end{bmatrix}\begin{bmatrix} \delta\phi_R^{(+)} \\ \delta\phi_I^{(+)} \end{bmatrix}=0 , \tag{2.12}$$

$$D_3\eta_3=\begin{bmatrix} \partial_1 & \partial_2 & -\epsilon gA_1^{(3)} & -\epsilon gA_2^{(3)} & -\frac{\epsilon}{2}\phi_I^{(-)} & -\frac{\epsilon}{2}\phi_R^{(-)} \\ \epsilon gA_1^{(3)} & \epsilon gA_2^{(3)} & \partial_1 & \partial_2 & -\frac{\epsilon}{2}\phi_R^{(-)} & -\frac{\epsilon}{2}\phi_I^{(-)} \\ -\phi_I^{(-)} & -\phi_R^{(-)} & -\phi_R^{(-)} & \phi_I^{(-)} & \partial_1 & -\partial_2 \\ \phi_R^{(-)} & -\phi_I^{(-)} & -\phi_I^{(-)} & -\phi_R^{(-)} & \partial_2 & \partial_1 \\ \partial_2 & -\partial_1 & -gA_2^{(3)} & gA_1^{(3)} & -\alpha\phi_R^{(-)} & -\alpha\phi_I^{(-)} \\ -gA_2^{(3)} & gA_1^{(3)} & -\partial_2 & \partial_1 & -\alpha\phi_I^{(-)} & \alpha\phi_R^{(-)} \end{bmatrix}\begin{bmatrix} g\delta A_{1R}^{(+)} \\ g\delta A_{2R}^{(+)} \\ g\delta A_{1I}^{(+)} \\ g\delta A_{2I}^{(+)} \\ \delta\phi_R^{(3)} \\ \delta\phi_R^{(3)} \end{bmatrix}=0 . \tag{2.13}$$

Equation (2.13) can be written in a simpler form if we define η_e as

$$\eta_3=Q\xi , \tag{2.14}$$

where

$$Q\xi=\begin{bmatrix} -\partial_1 & gA_1^{(3)} & \alpha\phi_I^{(-)} & -\alpha\phi_R^{(-)} & 0 & \partial_2-gA_1^{(3)} \\ -\partial_2 & gA_2^{(3)} & \alpha\phi_R^{(-)} & \alpha\phi_I^{(-)} & 0 & -\partial_1-gA_2^{(3)} \\ -gA_-^{(3)} & -\partial_1 & \alpha\phi_R^{(-)} & \alpha\phi_I^{(-)} & \partial_2-gA_1^{(3)} & 0 \\ -gA_2^{(3)} & -\partial_2 & -\alpha\phi_I^{(-)} & \alpha\phi_R^{(-)} & -\partial_1-gA_2^{(3)} & 0 \\ -\phi_I^{(-)} & -\phi_R^{(-)} & -\partial_1 & -\partial_2 & 0 & 0 \\ \phi_R^{(-)} & -\phi_I^{(-)} & \partial_2 & -\partial_1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} . \tag{2.15}$$

Defining three complex fields $\xi^{(1)}=\xi_1+i\xi_2,\xi^{(2)}=\xi_3+i\xi_4$, and $\xi^{(3)}=\xi_5-i\xi_6$, we can write Eq. (2.13) in the three complex equations

$$\partial_+[(\partial_-+igA_-^{(3)})\xi^{(3)}]=0 , \tag{2.16}$$

$$(\nabla^2+2\alpha|\phi^{(-)}|^2)\xi^{(2)}-i\phi^{(-)}(\partial_+-igA_+^{(3)})\xi^{(3)}=0 , \tag{2.17}$$

$$[\nabla^2-\epsilon/2|\phi^-|^2-\varepsilon g^2|\mathbf{A}^{(3)}|^2+(1+\varepsilon)ig\,\mathbf{A}^{(3)}\cdot\nabla]\xi^{(1)}-[i\alpha(\partial_-+ig\varepsilon A_-^{(3)})\phi^{(-)*}+(i\alpha+1\varepsilon/2)\phi^{(-)*}\partial_-]\xi^{(2)}$$
$$-\tfrac{1}{2}[(\partial_{(-)}+ig\varepsilon A_1^{(3)})(\partial_+-igA_+^{(3)})-(\partial_++ig\varepsilon A_+^{(3)})(\partial_-+igA_-^{(3)})]\xi^{(3)}=0 . \tag{2.18}$$

The fluctuations are then determined through the relations

$$g\delta A_+^{(+)}=-(\partial_++igA_+^{(3)})\xi^{(1)}+2i\alpha\phi^{(-)*}\xi^{(2)}$$
$$+(\partial_+-igA_+^{(3)})\xi^{(3)} , \tag{2.19}$$

$$g\delta A_-^{(+)}=-(\partial_-+igA_-^{(3)})(\xi^{(1)}+\xi^{(2)}) , \tag{2.20}$$

$$\delta\phi^{(3)}=i\phi^{(-)}\xi^{(1)}-\partial_-\xi^{(2)} . \tag{2.21}$$

Normalizable zero modes of Eq. (2.10) can be obtained if we solve the three equations $D_i\eta_i=0$, $i=1,2,3$. Before trying to solve them, we will first investigate the index of the operator D [8].

The index of D is defined as

$$I(D)=\dim(\mathrm{Ker}D)-\dim(\mathrm{Ker}D^\dagger)$$
$$=\dim(\mathrm{Ker}D^\dagger D)-\dim(\mathrm{Ker}DD^\dagger) . \tag{2.22}$$

From Eqs. (2.10) and (2.22) we find

$$I(D)=\sum_{i=1}^{3} I(D_i)\ , \tag{2.23}$$

$$\mathrm{kim}(\mathrm{Ker}D)=\sum_{i=1}^{3}\dim(\mathrm{Ker}D_i)\ , \tag{2.24}$$

Using the formula

$$I(D)=\mathrm{Tr}\left[\frac{M^2}{D^\dagger D+M^2}\right]-\mathrm{Tr}\left[\frac{M^2}{DD^\dagger+M^2}\right]\ , \tag{2.25}$$

it is straightforward to calculate

$$\begin{aligned} &I(D_3)=0\ ,\\ &I(D_1)=-I(D_2)=4n\ ,\\ &I(D)=0\ . \end{aligned} \tag{2.26}$$

The result $I(D)=0$ implies that a straightforward application of an index theorem is not so useful here as in other cases [9] in computing the number of zero eigenmodes of D.

III. EXPLICIT ZERO EIGENMODES

We first calculate dim $(\mathrm{Ker}D_3)$ by solving Eqs. (2.16), (2.17), and (2.18). By setting

$$\xi^{(3)}(r,\theta)=\sum_m e^{im\theta}H_m(r) \tag{3.1}$$

in Eq. (2.16) we find the most general solution for $\xi^{(3)}(r,\theta)$ as

$$\xi^{(3)}=\sum_m [A_m^{(1)}H_m^{(1)}(r)+A_m^{(2)}H_m^{(2)}(r)]\ , \tag{3.2}$$

where

$$H_m^{(1)}=\frac{r^{-m}}{r^{2n}+1}\underset{r\to 0}{\sim} r^{-m} \underset{r\to\infty}{\sim} r^{-m-2n}\ , \tag{3.3}$$

$$H_{m\neq 0}^{(2)}=r^m\left[\frac{2m+n}{n}-\frac{r^n-r^{-n}}{r^n+r^{-n}}\right]\underset{r\to 0}{\sim} r^m \underset{r\to\infty}{\sim} r^m\ , \tag{3.4}$$

$$H_0^{(2)}=\frac{1}{r^{2n}+1}\left[\frac{1}{2}r^{2n}+n\ln r\right]\underset{r\to 0}{\sim}\ln r \underset{r\to\infty}{\sim} r^0\ . \tag{3.5}$$

Substituting Eq. (3.2) into Eq. (2.17) yields

$$\left[\nabla^2+\frac{8n^2}{r^2(r^n+r^{-n})^2}\right]\xi^{(2)}(r,\theta)=\sum_m e^{i(n+m)\theta}h_m(r)\ , \tag{3.6}$$

where

$$h_{m\neq 0}(r)=\frac{4mni}{\sqrt{\alpha}}\frac{1}{r^2(r^n+r^{-n})^2}\times[-A_m^{(1)}r^{-m-n}+A_m^{(2)}r^{m+n}]\ , \tag{3.7}$$

$$h_0(r)=\frac{2n^2 i}{\sqrt{\alpha}}\frac{A_0^{(2)}}{r^2(r^n+r^{-n})}\ . \tag{3.8}$$

For the homogeneous part of Eq. (3.6), setting

$$\xi_h^{(2)}(r,\theta)=\sum_m e^{im\theta}F_m^{(h)}(r) \tag{3.9}$$

in Eq. (3.6) yields

$$\left[r\frac{d}{dr}r\frac{d}{dr}-m^2+\frac{8n^2}{r^n+r^{-n})^2}\right]F_m^{(h)}(r)=0\ . \tag{3.10}$$

The solution of Eq. (3.10) is (see Ref. [3])

$$F_m^h(r)=B_m^{(1)}f_m(r)+B_m^{(2)}f_{-m}(r)\ , \tag{3.11}$$

where

$$f_m(r)=\frac{r^m}{r^{2n}+1}[(m+n)+(m-n)r^{2n}]\ . \tag{3.12}$$

For $m=0,n,f_m$ and f_{-m} coincide and thus we need second solutions to obtain the most general solutions. The most general solution for $\xi_h^{(2)}(r,\theta)$ can be written in the form

$$\xi_h^{(2)}(r,\theta)=\sum_{m=-\infty}^{\infty} e^{im\theta}F_m^{(h)}(r)\ , \tag{3.13}$$

where

$$F_0^{(h)}(r)=B_0^{(1)}f_0(r)+B_0^{(2)}[f_0(r)\ln r-1]\ , \tag{3.14}$$

$$F_n^{(h)}(r)=F_{-n}^{*}(r)=B_n^{(1)}\left[f_n(r)\ln r+\frac{r^{2n}-1}{2r^n}\right]\ . \tag{3.15}$$

By using the solutions of the homogeneous part of Eq. (3.6) and their Wronskian, $W[f_{n+m}(r),f_{-n-m}(r)]$, we can obtain particular solutions of the inhomogeneous equation (3.6). The most general solution of Eq. (3.6) is then given in the form

$$\xi^{(2)}(r,\theta)=\sum_m e^{im\theta}F_m^{(h)}+\sum_m e^{i(n+m)\theta}F_m^{(p)}\ , \tag{3.16}$$

where

$$F_{m=0}^{(p)}=\frac{i}{\sqrt{\alpha}}\frac{1}{r^n+r^{-n}}[A_m^{(1)}r^{-m}-A_m^{(2)}r^m]\ , \tag{3.17}$$

$$F_0^{(p)}=\frac{i}{\sqrt{\alpha}}A_0^{(2)}\left[\frac{r^n}{2}-\frac{1}{r^n+r^{-n}}+\frac{n}{r^n+r^{-n}}\ln r\right]\ . \tag{3.18}$$

By substituting the results (3.2) and (3.16) into Eq. (2.18), we can obtain the most general solution of $\xi^{(1)}$ for arbitrary values of ϵ and α. We however, found that the most general solution of Eq. (2.18) is given in a very simple closed form in the background gauge with a particular value of $\alpha,\alpha=1/2$. In this case, setting

$$\xi^{(1)}(r,\theta)=\sum_m e^{im\theta}R_m(r) \tag{3.19}$$

in Eq. (2.18) yields

$$\left[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}-\frac{m^2}{r^2}-\frac{4n(m+n)r^n}{r^2(r^n+r^{-n})}\right]R_m(r)=-8n^2[A_m^{(1)}+2n\sqrt{\alpha}iB_{n+m}^{(2)}]\frac{r^{-n-m}}{r^2(r^n+r^{-n})^3}$$
$$+4[A_m^{(2)}+2n\sqrt{\alpha}iB_{n+m}^{(1)}]\frac{r^m}{r^2(r^n+r^{-n})}\times\left[\frac{2n(m+n)}{r^n+r^{-n}}-\frac{2n^2r^n}{(r^n+r^{-n})^2}+m(m+n)r^n\right], \tag{3.20}$$

where we set $A_0^{(2)}=B_0^{(2)}=B_n^{(2)}=0$ for simplicity since it can be easily shown that the fluctuation fields are unnormalizable for the log-dependent terms.

By introducing a variable u as

$$u=\frac{r^{-n}}{r^n+r^{-n}} \tag{3.21}$$

and by setting

$$R_m(r)=\left[\frac{1}{r^n+r^{-n}}\right]r^{-m-n}G_m(r), \tag{3.22}$$

we may write the homogeneous part of Eq. (3.20) in the form

$$u(u-1)\frac{d^2G_m}{du^2}+\left[4u-\left[3+\frac{n}{m}\right]\right]\frac{dG_m}{du}+2G_m=0, \tag{3.23}$$

whose solutions are recognized as hypergeometric functions. The most general solution of Eq. (3.23) is

$$G_m(r)=C_m^{(1)}G_m^{(1)}+C_m^{(2)}G_m^{(2)}, \tag{3.24}$$

where

$$G_m^{(1)}={}_2F_1\left[1,2,3+\frac{m}{n};\frac{r^{-n}}{r^n+r^{-n}}\right],$$
$$G_m^{(2)}=r^{2(m+n)}(r^n+r^{-n})^2. \tag{3.25}$$

By using the solutions of the homogeneous part of Eq. (3.20), we can construct particular solutions of the inhomogeneous part of Eq. (3.20). We present the most general solution of Eq. (3.20) in the form

$$\xi^{(1)}(r,\theta)=\sum_m e^{im\theta}[C_m^{(1)}R_m^{(1)}+C_m^{(2)}R_m^{(2)}]+e^{-in\theta}[A_{-n}^{(1)}+A_{-n}^{(2)}+2n\sqrt{\alpha}iB_0^{(1)}]\frac{1}{r^n+r^{-n}}$$
$$+\sum_{m\neq n}e^{im\theta}\left[[A_m^{(1)}+2n\sqrt{\alpha}iB_{n+m}^{(2)}]\frac{-m-n}{r^n+r^{-n}}-[A_m^{(2)}+2n\sqrt{\alpha}iB_{n+m}^{(1)}]\left[\frac{mr^n}{m+n}+\frac{r^{m-n}}{r^n+r^{-n}}\right]\right], \tag{3.26}$$

where

$$R_m^{(1)}=\frac{r^{-m-n}}{r^n+r^{-n}}{}_2F_1\left[1,2,3+\frac{m}{n};\frac{r^{-n}}{r^n+r^{-n}}\right],$$
$$R_m^{(2)}=r^{m+n}(r^n+r^{-n}).$$

If we substitute the solutions for $\xi^{(i)}$'s (3.2), (3.16), and (3.26) into Eqs. (2.19), (2.20), and (2.21) we can determine fluctuation fields as the solutions of Eq. (2.13). Without a writing explicit form, we present only the result that there are no normalizable modes for $\eta_3(=\delta A_+^+,\delta A_-^+,\delta\phi^{(3)})$, i.e.,

$$\dim(\mathrm{Ker}D_3)=0. \tag{3.27}$$

In Ref. [3], Eq. (2.11) was solved exactly. The result is

$$\dim(\mathrm{Ker}D_1)=4n. \tag{3.28}$$

These $4n$ parameters are interpreted as $2n$ locations, n scales, and an $n-1$ relative U(1) phase of an n-soliton system which is invariant under the rotation about the T_3 axis. The remaining one is an irrelevant overall phase.

We finally consider Eq. (2.12). With the background (1.18), Eq. (2.12) can be written in the complex form as

$$\left[\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\theta}-\frac{2nr^n}{r(r^n+r^{-n})}\right]\delta\phi^{(+)}=0. \tag{3.29}$$

The most general solution of Eq. (3.29) is easily obtained in the form

$$\delta\phi^{(+)}=\sum_m e^{im\theta}C_m(r^{m+2n}+r^m), \tag{3.30}$$

which is unnormalizable for any value of m. We thus conclude that

$$\dim(\mathrm{Ker}D_2)=0. \tag{3.31}$$

From Eqs. (2.26), (3.27), (3.28), and (3.31), we find that the SU(2) Chern-Simons n-soliton solutions depend on $4n$ parameters, which include one overall U(1) phase.

IV. DISCUSSIONS

Our results imply that the Toda ansatz leads to the most general soliton solutions of the nonlinear planar Schrödinger equation coupled to the SU(2) Chern-Simons fields. The resulting equations (1.15) and (1.16) are of the same form as that obtained from the U(1) invariant planar gauge theory with a Chern-Simons term [2], therefore, their solutions coincide exactly. Under the infinitesimal change of global SU(2) phase $\delta\omega$, the fields Φ and A_i change by the amount.

$$\delta\Phi = i[\delta\omega, \Phi] , \tag{4.1}$$

$$\delta A_i = i[\delta\omega, A_i] , \tag{4.2}$$

where

$$\delta\omega = \delta\omega^{(3)}T_3 + \delta\omega^{(+)}T_- + \delta\omega^{(-)}T_+ . \tag{4.3}$$

With the Toda backgrounds

$$\Phi = \phi^{(-)}T_+ ,$$

$$A_i = A_i^{(3)}T_3 ,$$

the fluctuations become

$$\delta\Phi = i\phi^{(-)}[\delta\omega^{(3)}T_+ - \delta\omega^{(+)}T_3] , \tag{4.4}$$

$$\delta A_i = iA_i[\delta\omega^{(+)}T_- - \delta\omega^{(-)}T_+] . \tag{4.5}$$

For the radially symmetric backgrounds presented in (1.17) and (1.18), $\delta\Phi$ is normalizable but δA_i is unnormalizable. When $\delta\omega$ is taken in the T_3 direction,

$$\delta\omega = \delta\omega^{(3)}T_3 , \tag{4.6}$$

$$\delta\Phi = i\phi^{(-)}\delta\omega^{(3)}T_+ , \tag{4.7}$$

$$\delta A_i = 0 , \tag{4.8}$$

the fluctuations are normalizable. But the soliton solutions of Eqs. (1.15) and (1.16) are invariant under the global SU(2) transformations. We therefore conclude that the nonrelativistic SU(2) Chern-Simons soliton solution depends on $4n+2$ parameters including the global SU(2) phase.

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