Scalar-tensor theory in higher-dimensional space-time with torsion

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The modified Brans-Dicke theory with a torsion field is discussed in five-dimensional space of Kaluza-Klein type. It is shown that the electromagnetic field and the scalar field appear during the reduction of five-dimensional action. A conformally mapped metric is also used to see the relation between the scalar field and the tlaplon field which is induced for the electromagnetic field to be invariant under the gauge transformation and their equivalence.

Recently the modified Brans-Dicke theory, which shows that the torsion field can be generated by the scalar field ϕ, was investigated.[1] This scalar field ϕ is given by the inverse value of the gravitational "constant" which is assumed not to be a constant but a spacetime-dependent field as usual in the Brans-Dicke theory.[2] If the electromagnetic field is introduced into this modified Brans-Dicke theory, then the usual procedure of minimal coupling of replacing ordinary derivatives with generally covariant ones will give rise to a theory which is not gauge invariant under the usual gauge transformations,[3] since the base space of our modified Brans-Dicke theory is the Einstein-Cartan manifold. One of two possibilities for avoiding this problem is to take the definition of the electromagnetic field-strength tensor as[4]

$$F_{\mu\nu}=A_{\nu;\mu}-A_{\mu;\nu}, \quad A_{\mu;\nu}=A_{\mu,\nu}-\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}A_\lambda , \tag{1}$$

where the semicolon denotes the covariant derivative using the Christoffel symbols as in the case of conventional general relativity. In this way we retain gauge invariance at the cost of restricting the minimal-coupling procedure.[3] To maintain the gauge invariance the second definition is better than the first in Eq. (1) (Refs. 5 and 6):

$$F_{\mu\nu}=A_{\nu|\mu}-A_{\mu|\nu}, \quad A_{\mu|\nu}=A_{\mu,\nu}-\Gamma^{\lambda}{}_{\mu,\nu}A_\lambda , \tag{2}$$

where the bar symbols denote the covariant derivative with respect to the linear affine connection $\Gamma^{\lambda}{}_{\mu\nu}$ which is written with the torsion field $S^{\lambda\mu\nu}=\Gamma^{\lambda}{}_{\mu\nu}-\Gamma^{\lambda}{}_{\nu\mu}$ as

$$\Gamma^{\lambda}{}_{\mu\nu}=\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}-K^{\lambda}{}_{\mu\nu}$$

$$=\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}-\tfrac{1}{2}(-S^{\lambda}{}_{\mu\nu}+S_{\mu}{}^{\lambda}{}_{\nu}+S_{\nu}{}^{\lambda}{}_{\mu}) . \tag{3}$$

By requiring the invariance of the matter Lagrangian which contains a complex scalar field $\Psi(x)$ that transforms according to a U(1) gauge group,

$$\Psi(x)\rightarrow e^{iq\Lambda}\Psi(x) \tag{4}$$

under the local transformations, we are led to introduce the gauge potential A_μ through the minimal-coupling procedure:

$$A_\mu\rightarrow A_\mu+e^{\Phi}\Lambda_{,\nu} . \tag{5}$$

The second definition in Eq. (2) of the electromagnetic field gives the useful result that the torsion field is determined by the gradient of the scalar field Φ, the so-called "tlaplon,"[5,7] in order to have minimal coupling and gauge invariance coexisting in a consistent way:

$$S^{\lambda}{}_{\mu\nu}=\delta^{\lambda}_{\mu}\Phi_{,\nu}-\delta^{\lambda}_{\nu}\Phi_{,\mu} . \tag{6}$$

The structure of the torsion field by the tlaplon is precisely the same form as one by Φ of our modified Brans-Dicke theory.

In order to find those relations between the scalar field ϕ, the electromagnetic field, and the tlaplon field Φ, it is necessary to extend the modified Brans-Dicke theory into the Kaluza-Klein[8] five-dimensional space. The aim of five-dimensional Kaluza-Klein theory is to unify the gravitational and electromagnetic fields into a single classical, geometric theory. The fifth component of the metric gives the additional electromagnetic Lagrangian, while for the common four-dimensional case the electromagnetic field is contained simply in the matter Lagrangian term. Its generalization to the higher-dimensional space was achieved by giving general non-Abelian gauge fields instead of the electromagnetic field.[9] Since then many applications to the Kaluza-Klein theory have streamed out constantly. Among the applications the torsion field of higher dimension[7,10] and higher-dimensional Brans-Dicke theory[11] are very relevant to our motivations, as the unification of Kaluza-Klein theory and Einstein-Cartan theory, and the unification of Brans-Dicke theory and Kaluza-Klein theory, respectively. The scalar fields in extra dimensions are used to be a function of some variables[12] or constants.[13] Five-dimensional Kaluza-Klein theory[14] treating the scalar field with various physical quantities are also interesting.

The four-dimensional modified Brans-Dicke theory is necessarily introduced in order to be extended into five-dimensional space. The modified Brans-Dicke theory in four-dimensional spacetime with torsion starts off with the action[15]

$$I=\int \sqrt{-g}\,(\phi R-\omega\phi_{,\mu}\phi^{,\mu}/\phi)d^4x \tag{7}$$

in the absence of matter fields. Here ω denotes an arbitrary dimensionless parameter. Then the field equations are

$$G_{\mu\nu}=\omega(\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda})/\phi^2\ , \tag{8}$$

$$S^{\lambda}{}_{\mu\nu}=(\phi_{,\mu}\delta^{\lambda}_{\nu}-\phi_{,\nu}\delta^{\lambda}_{\mu})/2\phi\ , \tag{9}$$

$$R-\omega\phi_{,\mu}\phi^{,\mu}/\phi^2+2\omega\Delta\phi/\phi-2\omega S^{\lambda}{}_{\mu\lambda}\phi^{,\mu}/\phi=0\ . \tag{10}$$

The d'Alembertian Δ in the Einstein-Cartan manifold is defined by

$$\Delta\phi=\phi^{,\mu}{}_{|\mu}=\phi^{,\mu}{}_{,\mu}+\Gamma^{\mu}{}_{\lambda\mu}\phi^{,\lambda}\ . \tag{11}$$

If we extend the spacetime into the five-dimensional space, then the action (7) and the field equations (8)–(10) must be[16]

$$\int \sqrt{-\lambda}(\phi\,{}^{(5)}R-\omega\phi_{,M}\phi^{,M}/\phi)d^{\,4}x\,d\theta \tag{12}$$

and

$$G_{MN}=\omega(\phi_{,M}\phi_{,N}-\tfrac{1}{2}\gamma_{MN}\phi_{,L}\phi^{,L})/\phi^2\ , \tag{13}$$

$$S^{L}{}_{MN}=(\phi_{,M}\delta^{L}_{N}-\phi_{,N}\delta^{L}_{M})/3\phi\ , \tag{14}$$

$${}^{(5)}R-\omega\phi_{,M}\phi^{,M}/\phi^2+2\omega\phi^{,M}{}_{|M}/\phi-2\omega S^{L}{}_{ML}\phi^{,M}/\phi=0\ , \tag{15}$$

where γ is the determinant of the five-dimensional metric γ_{AB} and ${}^{(5)}R$ is the corresponding scalar curvature. Discarding the surface terms the action (12) becomes

$$\int \sqrt{-\gamma}\{\phi[{}^{(5)}R(\{\ \})-\tfrac{1}{2}S_{LMN}S^{NML}-\tfrac{1}{4}S_{LMN}S^{LMN}+S_{ML}{}^{M}S^{NL}{}_{N}-\omega\phi_{,M}\phi^{,M}/\phi^2] +(\phi_{,L}{}^{;L}+2\phi^{,L}{}_{,L}-2\phi_{,L}S^{N}{}_{N})\}d^4x\,d\theta\ , \tag{16}$$

where ${}^{(5)}R(\{\ \})$ is calculated from the five-dimensional metric γ_{AB} without the five-dimensional torsion fields $S^{L}{}_{MN}$. With the help of the torsion field (14) the action (16) is given by

$$\int \sqrt{-\lambda}\{\phi[{}^{(5)}R(\{\ \})-(\omega+\tfrac{4}{3})\phi_{,M}\phi^{,M}/\phi^2]+(\phi_{,L}{}^{;L}+2\phi^{,L}{}_{,L})\}d^4x\ d\theta\ . \tag{17}$$

Assuming the five-dimensional metric is given by

$$\gamma_{AB}=\begin{bmatrix} g_{\mu\nu}+A_\mu A_\nu & A_\mu \\ A_\nu & 1\end{bmatrix} \quad\text{or}\quad \gamma^{AB}=\begin{bmatrix} g^{\mu\nu} & -g^{\mu\lambda}A_\lambda \\ -g^{\nu\lambda}A_\lambda & 1+A_\lambda A^\lambda\end{bmatrix}\ , \tag{18}$$

the electromagnetic potential A_μ allows the gauge transformations

$$A_\mu\rightarrow A_\mu+\Lambda,_{\mu} \tag{19}$$

under the general coordinate transformation

$$x'^{\mu}=x^{\mu},\quad x'^{5}=x^{5}+\Lambda(x^{\mu})\ . \tag{20}$$

Working out the curvature scalar ${}^{(5)}R(\{\ \})$ for the five-dimensional space and an analogous quantity $R(\{\ \})$ for the four-dimensional space, we find

$${}^{(5)}R(\{\ \})=R(\{\ \})-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\ . \tag{21}$$

Here the electromagnetic field is defined by the first covariant derivative, i.e., Eq. (1). When the electromagnetic field is defined as Eq. (2) during the reduction of the dimension, then the coupled terms of A_μ and the derivative of scalar field generating torsion field according to Eq. (9) should appear. Since the transformation (19) is not the form of Eq. (5) for the case of metric (18), we can insist on Eq. (21) for simplicity. Then the action (17) with the assumption of $\phi_{,5}=0$ becomes

$$V\int \sqrt{-g}\,\{\phi[R(\{\ \})-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-(\omega+\tfrac{4}{3})\phi_{,\mu}\phi^{,\mu}/\phi^2]+(\phi_{,\mu}{}^{;\mu}+2\phi^{,\mu}{}_{,\mu})\}d^4x =V\int \sqrt{-g}\,\phi(R-\omega\phi_{,\mu}\phi^{,\mu}/\phi^2-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{6}\phi_{,\mu}\phi^{,\mu}/\phi^2)d^4x\ , \tag{22}$$

where V is the volume of the fifth dimension. The four scalar curvature R in the second line is determined by reuniting in reverse order of the method of derivation Eq. (16). The fourth term $\frac{1}{6}\phi_{,\mu}\phi^{,\mu}/\phi^2$ of the integrand Eq. (22) is produced in reducing the dimension of the torsion field, while $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is due to the scalar curvature ${}^{(5)}R(\{\ \})$. The fact that the scalar field ϕ appeared before the electromagnetic field shows that scalar field ϕ

interacts with the electromagnetic field in the same order of the interaction with R. It will also be easily found from ${}^{(5)}R(\{\ \})$ of the field equation for ϕ in five-dimensional space [see Eq. (15)]. However, in the case of the usual four-dimensional Brans-Dicke theory the electromagnetic field exists independently, not being coupled with the scalar field, since the matter Lagrangian of the electromagnetic field is merely added to the gravitational Lagrangian. The action (22) induces the conservation law for the electromagnetic field including the scalar field as

$$(\phi F^{\mu\nu})_{;\nu}=0 \ . \tag{23}$$

The action also gives the same field equations as Eqs. (8) and (10) of the four-dimensional case, except that the value of ω is changed into $\omega-\frac{1}{6}$ which is the dimensional-reduction effect for the torsion field. This type of action (22) is conformally equivalent to a steady-state continuous creation model with the electromagnetic Lagrangian since the exponents of the coupled scalar with R and the matter Lagrangian are 1 identically.[17]

We can also treat the five-dimensional metric Eq. (18) as the conformal type:

$$\gamma_{AB}=\begin{bmatrix} g_{\mu\nu}+\chi^{-1}A_\mu A_\nu & A_\mu \\ A_\nu & \chi \end{bmatrix}$$

or

$$\gamma_{AB}=\begin{bmatrix} g^{\mu\nu} & -\chi^{-1}A^\nu \\ -\chi^{-1}A^\nu & \chi^{-1}(1+\chi^{-1}A^\lambda A_\lambda) \end{bmatrix} , \tag{24}$$

writing $\chi=\gamma_{55}$ which, unlike Eq. (18), will not be assumed to be a constant. This is a Jordan-type metric.[18] Define

$$\bar g_{\mu\nu}=\chi^{-1}g_{\mu\nu} \ \text{ or } \ \bar g^{\mu\nu}=\chi g^{\mu\nu} \ ; \tag{25}$$

i.e., carry out a conformal mapping in the four-dimensional space. Lowering the index on $\bar A^\mu=A^\mu$ with $\bar g_{\mu\nu}$ gives

$$\bar A_\mu=\chi^{-1}A_\mu \ . \tag{26}$$

Then, in terms of new quantities, the five-dimensional space metric is

$$\gamma_{AB}=\chi\begin{bmatrix} \bar g_{\mu\nu}+\bar A_\nu \bar A_\nu & \bar A_\mu \\ \bar A_\nu & 1 \end{bmatrix}$$

or

$$\gamma^{AB}=\chi^{-1}\begin{bmatrix} \bar g^{\mu\nu} & -\bar A^\mu \\ -\bar A^\nu & 1+\bar A_\lambda \bar A^\lambda \end{bmatrix} . \tag{27}$$

Therefore defining

$$\bar\gamma_{AB}=\chi^{-1}\gamma_{AB} \ , \tag{28}$$

we see that $\bar\gamma_{AB}$ expressed in terms of $\bar g_{\mu\nu}$ and $\bar A_\mu$ has precisely the same form as the five-dimensional metric of Kaluza-Klein theory, Eq. (18).

Under the transformation Eq. (20), $\bar A_\mu$ admits the gauge transformation

$$\bar A_\mu \rightarrow \bar A_\mu+\Lambda_{,\mu} \ \text{ or } \ A_\mu \rightarrow A_\mu+\chi^{\Lambda}{}_{,\mu} \ . \tag{29}$$

These transformations define the electromagnetic field as

$$F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}+\chi^{-1}(\chi_{,\nu}A_\mu-\chi_{,\mu}A_\nu) \ , \tag{30}$$

which is the second definition of the electromagnetic field, Eq. (2), assuming that the torsion field is given by

$$S^{\lambda}{}_{\mu\nu}=\chi^{-1}(\delta^\lambda_\mu \chi_{,\nu}-\delta^\lambda_\nu \chi_{,\mu}) \ . \tag{31}$$

This definition of torsion shows that χ should be $\phi^{-1/2}$ up to a constant factor compared with Eq. (9). Equations (5) and (6) also relate the value of χ with the tlaplon field Φ:

$$\chi=\phi^{-1/2}=e^{\Phi} \ . \tag{32}$$

Using the conformal transformation laws[19] of curvature scalars ${}^{(5)}R(\{\ \})$ and ${}^{(5)}\bar R(\{\ \})$ constructed from γ_{AB}, Eq. (27), and $\bar\gamma_{AB}$, Eq. (28), respectively, the action (17) can be written as

$$V\int\sqrt{-g}\,\{\phi[R(\{\ \})-\tfrac14\chi^{-1}F_{\mu\nu}F^{\mu\nu}+\chi^{-1}\chi_{,\mu}{}^{;\mu}-\tfrac12\chi^{-2}\chi_{,\mu}\chi^{,\mu}] +[-(\omega+\tfrac43)\phi^{-1}\phi_{,\mu}\phi^{,\mu}+\phi_{,\mu}{}^{;\mu}+2\phi^{,\mu}{}_{,\mu}+\tfrac12\chi^{-1}\chi_{,\mu}\phi^{,\mu}]\}d^4x \ . \tag{33}$$

With the help of Eq. (32) the Lagrangian density of the action (33) becomes

$$\sqrt{-g}\,\phi[R(\{\ \})-(\omega+\tfrac{11}{24})\phi^{-2}\phi_{,\mu}\phi^{,\mu}+\phi^{-1}(\phi_{,\mu}{}^{;\mu}+2\phi^{,\mu}{}_{,\mu})]-\sqrt{-g}\,\tfrac14\phi^{3/2}F_{\mu\nu}F^{\mu\nu} \ . \tag{34}$$

Along the same method of Eq. (22), the above Lagrangian density Eq. (34) can be also rewritten as

$$\sqrt{-g}\,\phi[R-(\omega-\tfrac{25}{24})\phi^{-2}\phi_{,\mu}\phi^{,\mu}-\tfrac14\phi^{1/2}F_{\mu\nu}F^{\mu\nu}] \ . \tag{35}$$

It is easily shown that Maxwell's equation for the electromagnetic field of Eq. (30) is conformally invariant taking the same form as Eq. (23) under the conformal transformation Eq. (28):

$$F^{\mu\nu}{}_{;\nu}=F^{\lambda\nu}S^{\mu}{}_{\lambda\nu}=\phi^{-1}(\phi_{,\lambda}F^{\lambda\mu}) \ \text{ or } \ (\phi F^{\mu\nu})_{;\nu}=0 \ . \tag{36}$$

Although the scaling factor $\chi\rightarrow\chi^n$ in Eq. (24), the form of coupling ϕ and electromagnetic field does not vary satisfying conformal invariance of Maxwell's equation. Other field equations of Eq. (35) are transformed only varying the value of ω into $\omega-\frac{25}{24}$ by the effects of dimensional reduction of the torsion field and conformal transformation.

When $\omega - \frac{25}{24}$ with the gauge transformation Eq. (29), the definition of the electromagnetic field Eq. (30), and the relation Eq. (32), then the Lagrangian density Eq. (35) becomes that of Hojman, Rosenbaum, Ryan, and Shepley,[5] except for the coupling of the scalar field and the electromagnetic field. That is to say, if our modified Brans-Dicke theory with a torsion field having appropriate value of parameter ω is extended into five-dimensional space of metric

$$\gamma_{AB} = \begin{bmatrix} g_{\mu\nu} + \phi^{1/2} A_\mu A_\nu & A_\mu \\ A_\nu & \phi^{-1/2} \end{bmatrix} = \phi^{-1/2} \begin{bmatrix} \overline{g}_{\mu\nu} + \overline{A}_\mu \overline{A}_\nu & \overline{A}_\mu \\ \overline{A}_\nu & 1 \end{bmatrix}, \tag{37}$$

then the scalar field ϕ generating the torsion field makes the role of minimal coupling of the vector potential A_μ equal to the tlaplon field ϕ with the relation Eq. (32) and it gives the interaction with electromagnetic field via Eq. (36).

The value of $\gamma_{55} = \phi^{-1/2}$ allows the smallness of the extra manifold not to be seen, since the average value $\langle \phi \rangle$ is given as the inverse value of the gravitational constant G. Instead of $\phi^{-1/2}$ it is applicable to other physical quantities[12–14] for γ_{55}. It can be also extended into six- or higher-dimensional space using the generalized Kaluza-Klein theory with non-Abelian gauge fields.

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