

Low Energy Dynamics of Monopoles in Supersymmetric Yang-Mills Theories with Hypermultiplets

Chanju Kim

Department of Physics, Ewha Womans University, Seoul 120-750, Korea
cjkim@ewha.ac.kr

Abstract

We derive the low energy dynamics of monopoles and dyons in $N = 2$ supersymmetric Yang-Mills theories with hypermultiplets in arbitrary representations by utilizing a collective coordinate expansion. We consider the most general case that Higgs fields both in the vector multiplet and in the hypermultiplets have nonzero vacuum expectation values. The resulting theory is a supersymmetric quantum mechanics which has been obtained by a nontrivial dimensional reduction of two-dimensional (4,0) supersymmetric sigma models with potentials.

1 Introduction

In supersymmetric Yang-Mills theories (SYM) with extended supersymmetry, there are many BPS monopole and dyon states. At weak coupling, their low-energy dynamics can be understood semiclassically by studying the moduli space of classical BPS monopole solutions. It turns out the dynamics is governed by some kind of a supersymmetric quantum mechanics [1].

The simplest case, where only a single adjoint Higgs field has a nonvanishing vacuum expectation value, was analyzed in Refs. [2]-[10]. When a second adjoint Higgs is also nonvanishing, there are BPS states with electric and magnetic charge vectors that are not parallel [11]-[17]. In this case, the low-energy dynamics is governed by a supersymmetric quantum mechanics with potential terms [18]-[23], which can be obtained by a non-trivial ‘‘Scherk-Schwarz’’ dimensional reduction of two-dimensional (4,0) supersymmetric sigma models [24]. This has been studied in both $N = 2$ and $N = 4$ theories through direct derivation using collective coordinate approach and/or indirect argument based on supersymmetric considerations. In particular, in [24], the low-energy dynamics was derived in $N = 2$ and $N = 4$ SYM with hypermultiplets when the two adjoint Higgs fields are nonvanishing.

One can further investigate the theory with hypermultiplets by considering the case that the scalars in the hypermultiplets also acquire nonzero expectation values while maintaining a nontrivial Coulomb branch. This is possible when the hypermultiplets are massless and the representations contain zero-weight vectors. The corresponding supersymmetric quantum mechanics was derived when the hypermultiplets are in real representations [24]. In deriving this, it was crucial that there are three complex structures on the index bundle associated with the matter fermions in the real representation. When the representation is not real, however, the index bundle is equipped with only a single complex structure in general and the low-energy dynamics was not considered. In this paper, we will address this issue and obtain the supersymmetric quantum mechanics, which will complete the derivation of the most general low-energy dynamics of BPS monopoles and dyons in $N = 2$ and $N = 4$ SYM with hypermultiplets in arbitrary representations.

The plan of the rest of this paper is as follows. In section 2, we briefly review the monopole dynamics in pure $N = 2$ SYM to fix notations. Section 3 is the main part of the paper. We consider $N = 2$ SYM with hypermultiplets in arbitrary representations and derive the low-energy dynamics of monopoles when scalars in the hypermultiplets additionally have nonzero vacuum expectation values while maintaining a nontrivial Coulomb branch. We conclude in section 4.

2 Monopole Dynamics in Pure $N = 2$ Super Yang-Mills Theory

In this section, we briefly review the dynamics of monopoles in pure $N = 2$ SYM. Details can be found in [23, 24]. The Lagrangian of $N = 2$ SYM is

$$L_0 = -\text{Tr} \left\{ -\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} D_M \Phi^I D^M \Phi^I - \frac{1}{2} [\Phi^1, \Phi^2]^2 - i \bar{\chi} \gamma^M D_M \chi + i \bar{\chi} [\Phi^1, \chi] - \bar{\chi} \gamma_5 [\Phi^2, \chi] \right\}, \quad (1)$$

where Φ^I , $I = 1, 2$ denote the two real Higgs fields, $D_M \Phi^I = \partial_M \Phi^I + [A_M, \Phi^I]$, χ is a Dirac spinor and all fields are in the adjoint representation of the gauge group G . The anti-hermitian generators of the Lie algebra \mathcal{G} are normalised so that $\text{Tr } t^{ab} t^{cd} = -\delta^{ab} \delta^{cd}$. Our metric has mostly minus signature and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. The classical vacuum satisfy $[\Phi^1, \Phi^2] = 0$ and thus Φ^I lie in the Cartan subalgebra of G : $\Phi^I = \phi^I \cdot \mathbf{H}$. We will only consider vacua where the symmetry is maximally broken to $U(1)^r$ where $r = \text{rank } G$. For a given vacuum electric and magnetic charge two-vectors are defined by

$$\begin{aligned} Q_e^I &= -\text{Tr} \oint \hat{n} \cdot \vec{E} \Phi^I = \phi^I \cdot \mathbf{q}, \\ Q_m^I &= -\text{Tr} \oint \hat{n} \cdot \vec{B} \Phi^I = \phi^I \cdot \mathbf{g}, \end{aligned} \quad (2)$$

where we have introduced the electric and magnetic charge vectors,

$$\begin{aligned} \mathbf{q} &= n_e^m \beta^m, \\ \mathbf{g} &= 4\pi n_m^m \beta_m^*, \end{aligned} \quad (3)$$

respectively. β^m are the simple roots and β_m^* are the simple co-roots of \mathcal{G} , and n_m^m are the topological winding numbers and n_e^m are, in the quantum theory, the electric quantum numbers.

There is a classical mass bound given by [25, 16]

$$\begin{aligned} M &\geq \max |Z_{\pm}| \\ &\equiv \max |(Q_e^1 - Q_m^2) \pm i(Q_m^1 + Q_e^2)|. \end{aligned} \quad (4)$$

Only Z_- appears as a central charge in the $N = 2$ supersymmetry algebra and half-BPS states satisfy $M = |Z_-|$ [26, 23]. Thus BPS solitons can only have charges satisfying $|Z_-| \geq |Z_+|$. This bound is saturated when

$$\begin{aligned} \vec{E} &= \pm \vec{D}a, \\ \vec{B} &= \vec{D}b, \end{aligned} \quad (5)$$

where we have defined the rotated Higgs fields via

$$\begin{aligned} a &= \cos \alpha \Phi^1 - \sin \alpha \Phi^2, \\ b &= \sin \alpha \Phi^1 + \cos \alpha \Phi^2, \end{aligned} \quad (6)$$

and the angle α is constrained to be

$$\tan \alpha = \frac{Q_m^1 \mp Q_e^2}{Q_m^2 \pm Q_e^1}. \quad (7)$$

The second equation in (5) is the usual BPS equation for a single Higgs field of which the solutions are usual BPS monopoles. For a given solution of the the second equation, the first equation has a unique solution for specified asymptotic behavior of a . The solutions to the general equations can thus be viewed as electrically dressed solutions to the BPS monopoles.

In terms of the vectors \mathbf{a}, \mathbf{b} , which are defined through (6), the mass bound is given by

$$M \geq \max(\pm \mathbf{a} \cdot \mathbf{q} + \mathbf{b} \cdot \mathbf{g}), \quad (8)$$

which can be obtained by noting that (7) can be recast as

$$\mathbf{b} \cdot \mathbf{q} = \pm \mathbf{a} \cdot \mathbf{g}. \quad (9)$$

In deriving the low-energy dynamics, we treat these dyons as particular excited states of the monopole dynamics. We thus begin with a given magnetic charge vector \mathbf{g} and fixed Higgs expectation values Φ^I . Setting $\mathbf{q} = 0$ then fixes the angle α and the fields a, b defined in (6). From (9), it also means that \mathbf{a} is orthogonal to the magnetic charge,

$$\mathbf{a} \cdot \mathbf{g} = 0. \quad (10)$$

The collective coordinate expansion then begins with a static purely magnetic solution to the equation $B_i = D_i b$. The dynamical effect of the second Higgs field is treated as a perturbation of this solution. The collective coordinate expansion can be considered to be an expansion in $n = n_\partial + \frac{1}{2}n_f$, where n_∂ is the number of time derivatives and n_f is the number of fermions. The equations of motion of the low-energy effective action will be of order $n = 2$, so we will solve the equations of motion of the field theory to order $n = 0$, $n = \frac{1}{2}$ and $n = 1$. To incorporate the affects of the second Higgs field we will also assume that a is of order $n = 1$.

Since the collective coordinate expansion is constructed about solutions of the ordinary BPS equation for a single Higgs field $B_i = D_i b$, we summarize some aspects of the geometry of the moduli spaces of solutions following [27, 2].

We first define a connection W_μ on R^4 that is translationally invariant in the four direction via $W_\mu = (A_i, b)$ and field strength $G_{\mu\nu} = [D_\mu, D_\nu]$ with $D_\mu = \partial_\mu + [W_\mu, \cdot]$. then the BPS equations can be recast as self-duality equations for W_μ ,

$$G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}. \quad (11)$$

Denote the moduli space of solutions to the BPS equations within a given topological class k by \mathcal{M}_k . A natural set of coordinates is provided by the moduli z^m

that specify the most general gauge equivalence class of solutions $W_\mu(x, z)$. The zero modes $\delta_m W_\mu$ about a given solution satisfy the linearized BPS equation

$$D_{[\mu} \delta_m W_{\nu]} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} D_\rho \delta_m W_\sigma, \quad (12)$$

as well as

$$D_\mu \delta_m W_\mu = 0. \quad (13)$$

A natural metric on \mathcal{M}_k is

$$g_{mn} = - \int d^3x \text{Tr} (\delta_m W_\mu \delta_n W_\mu). \quad (14)$$

Then (13) implies that the zero mode is orthogonal to gauge modes.

The zero modes are in general written as

$$\delta_m W_\mu = \partial_m W_\mu - D_\mu \eta_m, \quad (15)$$

where the gauge parameters $\eta_m(x, z)$ are chosen to satisfy (13). Then, on \mathcal{M}_k , η_m define a natural connection with covariant derivative

$$s_m = \partial_m + [\eta_m, \], \quad (16)$$

and field strength

$$\phi_{mn} = [s_m, s_n]. \quad (17)$$

The pair $(W_\mu(x, z), \eta_m(x, z))$ defines a natural connection on $R^4 \times \mathcal{M}_k$. The components of the field strength are given by $G_{\mu\nu}$, ϕ_{mn} and the mixed components are given by

$$[s_m, D_\mu] = \delta_m W_\mu. \quad (18)$$

They satisfy the following identities:

$$\begin{aligned} s_m G_{\mu\nu} &= 2D_{[\mu} \delta_m W_{\nu]}, \\ D_\mu \phi_{mn} &= -2s_{[a} \delta_{b]} W_\mu, \\ \phi_{mn} &= 2(D_\mu D_\mu)^{-1} [\delta_m W_\nu, \delta_n W_\nu]. \end{aligned} \quad (19)$$

The Christoffel connection associated with the metric (14) can be written in the form:

$$\Gamma_{mnk} = g_{ml} \Gamma^l_{nk} = - \int d^3x \text{Tr} (\delta_m W_\mu s_k \delta_n W_\mu). \quad (20)$$

The hyper-Kähler structure on R^4 gives rise to a hyper-Kähler structure on \mathcal{M}_k . The three complex structures can be written

$$J_m^{(s)n} = -g^{np} \int d^3x J^{(s)}_{\mu\nu} \text{Tr} (\delta_m W_\mu \delta_p W_\nu), \quad (21)$$

which implies

$$J_m^{(s)n} \delta_n W_\mu = -J_{\mu\nu}^{(s)} \delta_m W_\nu. \quad (22)$$

Now we turn to the zero modes of the adjoint fermions. On the Euclidean space R^4 , we introduce Hermitian gamma matrices,

$$\Gamma_i = \gamma_0 \gamma_i, \quad \Gamma_4 = \gamma_0, \quad (23)$$

satisfying $\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$ and define $\Gamma_5 = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4$. The fermion zero modes are time independent solutions of the Dirac equation in the presence of a BPS monopole,

$$\Gamma_\mu D_\mu \chi = 0. \quad (24)$$

They are necessarily anti-chiral. The monopole breaks 1/2 of the supersymmetry and the unbroken supersymmetry can be used to pair the bosonic and fermionic zero modes via

$$\chi_m = \delta_m W_\mu \Gamma^\mu \epsilon_+, \quad (25)$$

where ϵ_+ is a c-number chiral spinor that can be chosen to satisfy

$$\epsilon_+^\dagger \epsilon_+ = 1, \quad J_{\mu\nu}^{(3)} = -i \epsilon_+^\dagger \Gamma_{\mu\nu} \epsilon_+. \quad (26)$$

Using (22) we deduce that the fermionic zero modes satisfy

$$J_m^{(3)n} \chi_n = i \chi_m, \quad (27)$$

and hence that two bosonic zero modes are paired with one fermionic zero mode [28].

For later use, we discuss more on the complex structures. The charge conjugation of the spinor χ is defined as

$$\chi^c \equiv C \bar{\chi}^T = C (\gamma^0)^T \chi^* \quad (28)$$

where the charge-conjugation matrix C satisfies,

$$C C^* = -1, \quad C \Gamma_M^T = -\Gamma_M C. \quad (29)$$

Then with the c-number spinor $\epsilon'_+ \equiv C \epsilon_+^*$, we see that $\delta_m W_\mu \Gamma_\mu \epsilon'_+$ are also zero modes and can be expressed as a linear combination of original zero modes since χ (and W) is in a real representation of the gauge group, i.e.,

$$\delta_m W_\mu \Gamma_\mu \epsilon'_+ = \mathcal{C}_m^k \delta_k W_\mu \Gamma_\mu \epsilon_+, \quad (30)$$

where the matrix \mathcal{C} can be chosen to be anti-symmetric and unitary so that $\mathcal{C}^2 = -1$. By taking the complex conjugate of (27), it follows that \mathcal{C} anticommutes with $J^{(3)}$,

$$\mathcal{C} J^{(3)} = -J^{(3)} \mathcal{C}. \quad (31)$$

This matrix \mathcal{C} generates a second complex structure on the moduli space which we also denote by $J^{(2)}$. Defining $J^{(1)} = J^{(2)} J^{(3)}$ we obtain the hyper-Kähler structure of the monopole moduli space which can be taken to be the same as (21) by an appropriate choice of complex structures on R^4 .

With the above formalism on moduli space, it is now quite a simple matter to derive the low-energy effective action of pure $N = 2$ SYM. First, we rewrite the Lagrangian in terms of b, a rather than Φ^1, Φ^2 ,

$$L = -\text{Tr} \left\{ -\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} D_M a D^M a + \frac{1}{2} D_M b D^M b - \frac{1}{2} [a, b]^2 - i \bar{\chi} \gamma^M D_M \chi + i \bar{\chi} [b, \chi] + \bar{\chi} \gamma_5 [a, \chi] \right\}, \quad (32)$$

where χ has now been redefined as the field rotated by the angle $(\alpha - \pi/2)/2$. Then the following ansatz solve the equations of motion to order $n = 1$ [24]:

$$\begin{aligned} W_\mu &= W_\mu(x, z(t)), \\ \chi &= \delta_m W_\mu \Gamma^\mu \epsilon_+ \tilde{\lambda}^m(t), \\ A_0 &= \dot{z}^m \eta_m - i \phi_{mn} \tilde{\lambda}^{\dagger m} \tilde{\lambda}^n, \\ a &= \bar{a} + i \phi_{mn} \tilde{\lambda}^{\dagger m} \tilde{\lambda}^n, \end{aligned} \quad (33)$$

where

$$D_\mu \bar{a} = -G^m \delta_m W_\mu, \quad (34)$$

and G^m is a linear combination of the r tri-holomorphic Killing vector fields \mathbf{K} on \mathcal{M}_k corresponding to the $U(1)^r$ gauge transformations

$$G = \mathbf{a} \cdot \mathbf{K}. \quad (35)$$

Because of (27) the complex fermionic Grassmann odd collective coordinates $\tilde{\lambda}^m$ are not independent and satisfy

$$-i \tilde{\lambda}^m J_m^{(3)n} = \tilde{\lambda}^n. \quad (36)$$

Real independent λ^m can be defined via

$$\lambda^m = \sqrt{2} (\tilde{\lambda}^m + (\tilde{\lambda}^m)^\dagger). \quad (37)$$

After substituting the ansatz into the action, one finds that the low-energy effective action becomes [24]

$$S = \frac{1}{2} \int dt [\dot{x}^m \dot{x}^n g_{mn} + i g_{mn} \lambda^m D_t \lambda^n - G^m G^n g_{mn} - i D_m G_n \lambda^m \lambda^n] - \mathbf{b} \cdot \mathbf{g}, \quad (38)$$

which was first given in [23] based on supersymmetry considerations.

3 Inclusion of Hypermultiplets

We now consider the low-energy dynamics of monopoles in $N = 2$ SYM with a hypermultiplet in an arbitrary representation. This was first studied in [5, 6, 7] in the case that only a single adjoint Higgs field has a non-trivial expectation value. It

was then generalized in [24] to the case that both of the adjoint Higgs in the vector multiplet have non-vanishing expectation values. In [24], the low-energy dynamics was also derived when additional scalar vevs in the hypermultiplet are turned on. In deriving this, it was necessary to assume that the hypermultiplet is in a real representation to utilize complex structures of the index bundle associated with the matter fermions. Here, we derive the low-energy effective theory for the most general case, namely when the hypermultiplet is in an arbitrary representation and nontrivial scalar vevs of the hypermultiplet are turned on.

The massless hypermultiplet contribution to the Lagrangian is given by

$$\begin{aligned}
L_H = & \frac{1}{2}D_K M^\dagger D^K M + i\bar{\Psi}\gamma^K D_K \Psi - \bar{\Psi}(-i\Phi_1 - \gamma_5\Phi_2)\Psi \\
& + M^{\dagger 1}\bar{\chi}\Psi + \bar{\Psi}\chi M_1 + iM^{\dagger 2}\bar{\chi}^c\gamma_5\Psi + i\bar{\Psi}\gamma_5\chi^c M_2 \\
& + \frac{1}{2}M^\dagger(\Phi_1^2 + \Phi_2^2)M + \frac{1}{8}(M^\dagger t^\alpha \tau_s M)^2,
\end{aligned} \tag{39}$$

where M is a doublet of complex scalars $(M_1, M_2)^T$, t^α are anti-hermitian generators in the matter representation, τ_s are Pauli matrices, and χ^c is the charge conjugation of χ . Since we will assume that M_i 's have nonzero vevs, the hypermultiplet is necessarily massless and its representation should contain a zero-weight vector so that the $U(1)$ gauge symmetries of the Coulomb phase are left intact by turning on the vevs.

Before discussing the low-energy dynamics of the system, we briefly summarize some aspects of the geometry of the index bundle defined by the fermion zero modes. The zero modes of matter fermion Ψ satisfy the Dirac equation in the background of a monopole configuration

$$\Gamma_\mu D_\mu \gamma_5 \Psi = 0, \tag{40}$$

and are chiral. Let $\Psi_A(x, z)$, $A = 1 \dots l$ be a basis of the fermion zero modes in monopole background specified by the moduli z satisfying

$$\int d^3x \Psi_A^\dagger \Psi_B \equiv \langle \Psi_{\bar{A}} | \Psi_B \rangle = \delta_{\bar{A}B}, \tag{41}$$

where $\Psi_{\bar{A}}^\dagger \equiv (\Psi_A)^\dagger$. Note that the following completeness relation holds:

$$|\Psi_A\rangle \delta^{A\bar{B}} \langle \Psi_{\bar{B}}| + \Pi + \frac{1 - \Gamma_5}{2} = 1, \tag{42}$$

where the operator Π projects onto the chiral non-zero modes and has the form

$$\Pi = \gamma_5 \not{D} \frac{1}{D_\mu D_\mu} \not{D} \gamma_5 \frac{1 + \Gamma_5}{2}. \tag{43}$$

The fermion zero modes define an index bundle with a connection

$$A_{m\bar{A}B} = \langle \Psi_{\bar{A}} | s_m \Psi_B \rangle, \tag{44}$$

and the corresponding field strength is written in the form

$$F_{mn\bar{A}B} = \langle s_m \Psi_{\bar{A}} | \Pi s_n \Psi_B \rangle - \langle s_n \Psi_{\bar{A}} | \Pi s_m \Psi_B \rangle + \langle \Psi_{\bar{A}} | \phi_{mn} \Psi_B \rangle. \tag{45}$$

Since the connection one-form is unitary, the structure group of the index bundle is generically $U(l)$. The index bundle thus admits a covariantly constant complex structure I with Kähler form

$$I_{A\bar{B}} = i\delta_{A\bar{B}}. \quad (46)$$

Now the collective coordinate expansion can be done. After a suitable chiral rotation as in the previous section, the ansatz solving the equations of motion to order $n = 1$ is [24]

$$\begin{aligned} W_\mu &= W_\mu(x, z(t)), \\ \chi &= \delta_m W_\mu \Gamma^\mu \epsilon_+ \tilde{\lambda}^m(t), \\ A_0 &= \dot{z}^m \eta_m - i\phi_{mn} \tilde{\lambda}^{\dagger m} \tilde{\lambda}^n + \frac{i}{D^2} (\Psi^\dagger t^\alpha \Psi t^\alpha), \\ a &= \bar{a} + i\phi_{mn} \tilde{\lambda}^{\dagger m} \tilde{\lambda}^n + \frac{i}{D^2} (\Psi^\dagger t^\alpha \Psi t^\alpha), \\ \Psi &= \psi^A(t) \Psi_A, \\ M_1 &= \bar{M}_1 - \frac{2}{D^2} (\bar{\chi} \Psi), \\ M_2 &= \bar{M}_1 - \frac{2i}{D^2} (\bar{\chi}^c \gamma_5 \Psi), \end{aligned} \quad (47)$$

where $\psi^A(t)$ is the Grassmann odd complex collective coordinates for the matter fermion zero modes. \bar{a} satisfies (34) and $\bar{M}_{1,2}$ are order $n = 1$ and solve

$$D^2 \bar{M}_{1,2} = 0. \quad (48)$$

After substituting this ansatz into the field theory action, the \bar{M} -independent terms give rise to the supersymmetric quantum mechanics derived in [24]:

$$\begin{aligned} \mathcal{L}_1 &= \frac{1}{2} \left(g_{mn} \dot{z}^m \dot{z}^n + i g_{mn} \lambda^m D_t \lambda^n - g^{mn} G_m G_n - i D_m G_n \lambda^m \lambda^n \right. \\ &\quad \left. + i \psi^a \mathcal{D}_t \psi^a + \frac{1}{2} F_{mnab} \lambda^m \lambda^n \psi^a \psi^b - iT_{ab} \psi^a \psi^b \right) - \mathbf{b} \cdot \mathbf{g}. \end{aligned} \quad (49)$$

where we traded off complex ψ^A 's in favor of real ψ^a 's as in (37) and

$$\mathcal{D}_t \psi^a = \dot{\psi}^a + A_m^a{}_b \dot{z}^m \psi^b. \quad (50)$$

T is defined by

$$T_{\bar{A}B} = \langle \Psi_{\bar{A}} | \bar{a} \Psi_B \rangle, \quad (51)$$

and is anti-Hermitian in the real basis, $T_{ab} = -T_{ba}$. Furthermore, it satisfies [24]

$$T_{\bar{A}B;m} = F_{mn\bar{A}B} G^m. \quad (52)$$

In the following we derive \bar{M} -dependent terms which are the main result of this paper.

3.1 Bosonic potential

First we note that, given (48), $\not{D}\bar{M}_i\epsilon_+$ and $\not{D}\bar{M}_i\epsilon'_+$ are fermion zero modes and hence can be expanded in terms of the basis Ψ_A :

$$\begin{aligned}\not{D}\bar{M}_i\epsilon_+ &= -\gamma_5\sqrt{2}K_i^A(z)\Psi_A, \\ \not{D}\bar{M}_i\epsilon'_+ &= -\gamma_5\sqrt{2}K_i'^A(z)\Psi_A.\end{aligned}\tag{53}$$

The quantities K_i^A and $K_i'^A$ define sections on the dual of the index bundle over the monopole moduli space. Then K_i^A and $K_i'^A$ are orthogonal to each other,

$$K_i^{*A}K_j'^A = 0.\tag{54}$$

To see this, write

$$2K_i^{*A}K_j'^A = \int d^3x (\not{D}\bar{M}_i\epsilon_+)^\dagger (\not{D}\bar{M}_j\epsilon'_+),\tag{55}$$

where we used the orthogonality of the zero modes Ψ_A . In the right hand side of the equation, we have the expression

$$\epsilon_+^\dagger \Gamma_\mu \Gamma_\nu \epsilon'_+ = \epsilon_+^\dagger (\delta_{\mu\nu} + \Gamma_{\mu\nu}) \epsilon'_+.\tag{56}$$

Since the c-number chiral spinor ϵ_+ is orthogonal to its charge-conjugated one ϵ'_+ , which can easily be verified by using (26), only the antisymmetric part survives and hence (55) becomes

$$2K_i^{*A}K_j'^A = \frac{1}{2}\epsilon_+^\dagger \Gamma_{\mu\nu} \epsilon'_+ \int d^3x D_{[\mu}\bar{M}_i^\dagger D_{\nu]}\bar{M}_j.\tag{57}$$

After an integration by parts, this can be written as a sum of a vanishing boundary integral and the term containing the field strength $G_{\mu\nu} = [D_\mu, D_\nu]$. But this is self-dual and goes to zero when multiplied by $\epsilon_+^\dagger \Gamma_{\mu\nu} \epsilon'_+$ since ϵ_+ is chiral. This establishes (54). Furthermore, it is clear from the definition that K_1 and K_2 have the same magnitude as K_1' and K_2' , i.e.,

$$K_1^{*A}K_1^A = K_1'^{*A}K_1'^A, \quad K_2^{*A}K_2^A = K_2'^{*A}K_2'^A.\tag{58}$$

Now we are ready to deal with the \bar{M} -dependent bosonic potential terms, which arise from the kinetic terms of M in (39). Using the similar line of argument as above, we find that they reduce to

$$\begin{aligned}\mathcal{L}_{Hb} &\equiv \frac{1}{2} \int d^3x D_\mu \bar{M}_i^\dagger D_\mu \bar{M}_i \\ &= \frac{1}{2} \int d^3x (\not{D}\bar{M}_i\epsilon_+)^\dagger \not{D}\bar{M}_i\epsilon_+ \\ &= K_i^{*A}K_i^A.\end{aligned}\tag{59}$$

(The cross terms which are linear in \bar{M} vanish due to (48).)

It turns out to be convenient to define

$$v^A = K_1^A - K_2^A. \quad (60)$$

Then using (54) and (58), we can rewrite (59) as

$$\mathcal{L}_{Hb} = v^{*A}v^A = \frac{1}{2}v^a v_a, \quad (61)$$

where, as before, we rewrote the complex quantities v^A in terms of real quantities v^a by expanding

$$v^A = \frac{1}{\sqrt{2}}(v^{2A-1} + iv^{2A}). \quad (62)$$

3.2 Fermion bilinear terms

Fermion bilinear terms are

$$\mathcal{L}_{Hf} \equiv \int M^{\dagger 1} \bar{\chi} \Psi + \bar{\Psi} \chi M_1 + iM^{\dagger 2} \bar{\chi} \gamma_5 \Psi + i\bar{\Psi} \gamma_5 \chi M_2. \quad (63)$$

With the relation $\delta_m W_\mu = [s_m, D_\mu]$, we find

$$\chi \bar{M}_1 = \tilde{\lambda}^m s_m \not{D} \bar{M}_1 \epsilon_+ + \dots, \quad (64)$$

where the ellipsis denote terms of the form $\not{D}(\dots)$ which do not contribute any new terms in the low energy dynamics since Ψ in the Lagrangian is chiral and satisfies the Dirac equation. Using the relation (30), this can also be written as

$$\chi \bar{M}_1 = -\tilde{\lambda}^m \mathcal{C}_m{}^n s_n \not{D} \bar{M}_1 \epsilon'_+ + \dots. \quad (65)$$

The existence of two alternative expressions for the same quantity is basically related to the hyper-Kähler structure of the moduli space, as mentioned in section 2. We will see shortly that this plays crucial roles in constraining the form of the effective Lagrangian so that it becomes supersymmetric. Using (53), we see that the term containing \bar{M}_1 in (63) becomes

$$\begin{aligned} \int \bar{\Psi} \chi \bar{M}_1 &= i\sqrt{2} \tilde{\lambda}^m \psi^{\bar{A}} \nabla_m K_{1\bar{A}} \\ &= -i\sqrt{2} \tilde{\lambda}^m \mathcal{C}_m{}^n \psi^{\bar{A}} \nabla_n K'_{1\bar{A}}. \end{aligned} \quad (66)$$

Similarly, for $-i\gamma_5 \chi^c \bar{M}$,

$$\begin{aligned} -i\gamma_5 \chi^c \bar{M}_2 &= \tilde{\lambda}^{\dagger m} s_m \not{D} M_2 \epsilon'_+ + \dots \\ &= \tilde{\lambda}^{\dagger m} \mathcal{C}_m{}^n s_n \not{D} M_2 \epsilon_+ + \dots, \end{aligned} \quad (67)$$

which gives

$$\begin{aligned} \int i\bar{\Psi} \gamma_5 \chi^c \bar{M}_2 &= -i\sqrt{2} \tilde{\lambda}^{\dagger m} \psi^{\bar{A}} \nabla_m K'_{2\bar{A}} \\ &= -i\sqrt{2} \tilde{\lambda}^{\dagger m} \psi^{\bar{A}} \mathcal{C}_m{}^n \nabla_n K_{2\bar{A}}. \end{aligned} \quad (68)$$

Taking into account the complex conjugates of (66) and (68), we find that the fermion bilinear terms become

$$\mathcal{L}_{Hf} = i\sqrt{2}(\tilde{\lambda}^m\psi^{\bar{A}}\nabla_m K_{1\bar{A}} + \tilde{\lambda}^{\dagger m}\psi^A\nabla_m K_{1A} - \tilde{\lambda}^m\psi^A\nabla_m K'_{2A} - \tilde{\lambda}^{\dagger m}\psi^{\bar{A}}\nabla_m K'_{2\bar{A}}), \quad (69)$$

where we used the expression without $\mathcal{C}_m{}^n$. In terms of the real quantities v^a defined in (60) and λ^m , it can be reshuffled to

$$\begin{aligned} \mathcal{L}_{Hf} = & i\lambda^m\psi^a\nabla_m v_a \\ & -i\sqrt{2}(\tilde{\lambda}^m\psi^A\nabla_m K_{1A} - \tilde{\lambda}^m\psi^{\bar{A}}\nabla_m K'_{2\bar{A}} - \tilde{\lambda}^{\dagger m}\psi^A\nabla_m K'_{2A} + \tilde{\lambda}^{\dagger m}\psi^{\bar{A}}\nabla_m K_{1\bar{A}}). \end{aligned} \quad (70)$$

Now we are going to show that each of the last four terms in the above equation is actually zero. First, note that (66) gives a nontrivial relation

$$(1 - iJ^{(3)})\nabla\psi^{\bar{A}}K_{1\bar{A}} = -(1 - iJ^{(3)})\mathcal{C}\nabla\psi^{\bar{A}}K'_{1\bar{A}}, \quad (71)$$

where we used (36).

Note the operators $(1 \mp iJ^{(3)})\nabla$ appearing in (71) are holomorphic and anti-holomorphic covariant derivatives with respect to the third complex structure $J^{(3)}$ on \mathcal{M}_k . The reason that $J^{(3)}$ appears in this equation is because we used the c-number spinors ϵ_+, ϵ'_+ associated $J^{(3)}$ in considering the fermion zero modes. There are, however, three complex structures on \mathcal{M}_k and hence we can obtain similar relations if we use the c-number spinors associated with the other complex structures $J^{(2)} = \mathcal{C}$ and $J^{(1)} = J^{(2)}J^{(3)}$. The corresponding c-number spinors $\epsilon_+^{(s)}$, $s = 1, 2, 3$ are defined by

$$\epsilon_+^{(s)\dagger}\epsilon_+^{(s)} = 1, \quad J_{\mu\nu}^{(s)} = -i\epsilon_+^{(s)\dagger}\Gamma_{\mu\nu}\epsilon_+^{(s)}, \quad J_{\mu\nu}^{(s)}\Gamma_\nu\epsilon_+^{(s)} = i\Gamma_\mu\epsilon_+^{(s)}, \quad (72)$$

which generalize the relation (26). The corresponding zero modes are denoted as

$$\chi_m^{(s)} = \delta_m W_\mu \Gamma^\mu \epsilon_+^{(s)}. \quad (73)$$

After some calculation, the explicit form of $\epsilon_+^{(s)}$ can be obtained by using (72):

$$\begin{aligned} \epsilon_+^{(1)} &= -\frac{e^{i\pi/4}}{\sqrt{2}}(\epsilon_+ + \epsilon'_+), \\ \epsilon_+^{(2)} &= -\frac{e^{-i\pi/4}}{\sqrt{2}}(\epsilon_+ - i\epsilon'_+), \end{aligned} \quad (74)$$

where phases of the spinors are carefully chosen so that the cyclicity for the label of the complex structures are manifest in various relations shown below. (We will continue omit the superscript label for quantities associated with $J^{(3)}$.) We also define $\epsilon_+^{\prime(s)} = \mathcal{C}\epsilon_+^{(s)*}$.

Now let us consider the expansion

$$\begin{aligned} \mathcal{D}\bar{M}_i\epsilon_+^{(2)} &= -\gamma_5\sqrt{2}K_i^{(2)A}(z)\Psi_A, \\ \mathcal{D}\bar{M}_i\epsilon_+^{\prime(2)} &= -\gamma_5\sqrt{2}K_i^{\prime(2)A}(z)\Psi_A. \end{aligned} \quad (75)$$

From the relation (74), we can express $K_i^{(2)A}$ and $K_i^{(2)A'}$ as

$$\begin{aligned} K_i^{(1)A} &= -\frac{e^{i\pi/4}}{\sqrt{2}}(K_i + K_i'), & K_i^{(1)A'} &= -\frac{e^{-i\pi/4}}{\sqrt{2}}(K_i' - K_i)^A, \\ K_i^{(2)A} &= -\frac{e^{-i\pi/4}}{\sqrt{2}}(K_i - iK_i')^A, & K_i^{(2)A'} &= -\frac{e^{i\pi/4}}{\sqrt{2}}(K_i' - iK_i)^A. \end{aligned} \quad (76)$$

With the expansion (75) for $s = 2$, the condition corresponding to (71) now takes the form

$$(1 - iJ^{(2)})\nabla\psi^{\bar{A}}K_{1\bar{A}}^{(2)} = -(1 - iJ^{(2)})J^{(1)}\nabla\psi^{\bar{A}}K_{1\bar{A}}^{(2)'}, \quad (77)$$

which, on using (71) and (76), can be simplified to

$$\nabla\psi^{\bar{A}}K_{1\bar{A}} = -iJ^{(1)}\nabla\psi^{\bar{A}}K_{1\bar{A}}'. \quad (78)$$

Similarly, for $s = 1$ we obtain the relation

$$(1 + iJ^{(3)})\nabla\psi^{\bar{A}}K_{1\bar{A}} = -(1 + iJ^{(3)})J^{(2)}\nabla\psi^{\bar{A}}K_{1\bar{A}}'. \quad (79)$$

Combining (78) and (79), we actually find that each side of (79) is zero separately. In other words,

$$\tilde{\lambda}^{\dagger m}\nabla_m\psi^{\bar{A}}K_{1\bar{A}} = 0, \quad \tilde{\lambda}^{\dagger m}\nabla_m\psi^{\bar{A}}K_{1\bar{A}}' = 0. \quad (80)$$

Finally, exactly the same kind of analysis with (67) shows that

$$\tilde{\lambda}^{\dagger m}\nabla_m\psi^A K_{2\bar{A}}' = 0, \quad \tilde{\lambda}^{\dagger m}\nabla_m\psi^{\bar{A}}K_{2\bar{A}} = 0. \quad (81)$$

(80) and (81) complete the proof that the second line of (70) is zero, i.e, the effective action coming from the fermion bilinear terms become

$$\mathcal{L}_{Hf} = i\lambda^m\psi^a\nabla_m v_a. \quad (82)$$

From (61) and (82), the contribution to the effective Lagrangian from nonvanishing hypermultiplet vevs is

$$\mathcal{L}_2 = \frac{1}{2}v^a v_a + i\lambda^m\psi^a\nabla_m v_a. \quad (83)$$

3.3 Low-energy effective theory

Here we summarize the general low-energy dynamics of monopoles in $N = 2$ SYM with hypermultiplets. Collecting the terms (49) and (83), we find that the Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(g_{mn}\dot{z}^m\dot{z}^n + ig_{mn}\lambda^m D_t\lambda^n - g^{mn}G_m G_n - iD_m G_n\lambda^m\lambda^n \\ &\quad + i\psi^a\mathcal{D}_t\psi^a + \frac{1}{2}F_{mnab}\lambda^m\lambda^n\psi^a\psi^b - iT_{ab}\psi^a\psi^b \\ &\quad + v^a v_a + 2i\lambda^m\psi^a\nabla_m v_a) - \mathbf{b} \cdot \mathbf{g}. \end{aligned} \quad (84)$$

This Lagrangian has the same form as that obtained by a non-trivial Scherk-Schwarz dimensional reduction on a two dimensional (4,0) supersymmetric sigma model with potential [29]. It is invariant under $N = 4$ supersymmetry transformation given by

$$\begin{aligned}\delta z^m &= -i\epsilon\lambda^m + i\epsilon_s J^{(s)m}{}_n \lambda^n, \\ \delta\lambda^m &= (z^m - G^m)\epsilon + J^{(s)m}{}_n (\dot{z}^n - G^n)\epsilon_s - i\epsilon_s \lambda^k \lambda^n J^{(s)l}{}_k \Gamma_{ln}^m, \\ \delta\psi^a &= -A_m{}^a{}_b \delta z^m \psi^b + \epsilon v^a + \epsilon_s t_{(s)}^a,\end{aligned}\tag{85}$$

where $\epsilon, \epsilon_s, s = 1, 2, 3$ are Grassmann odd parameters, provided that the sections $t_{(s)}^a, s = 1, 2, 3$ can be found satisfying [24]:

$$\begin{aligned}J_m^{(s)n} \nabla_n v^a &= -\nabla_m t_{(s)}^a, \\ (v_a t_{(s)}^a)_{;m} &= 0, \\ G^n v_{;n}^a &= T_{ab} v^b, \\ G^m t_{(s);n}^a &= T_{ab} t_{(s)}^b.\end{aligned}\tag{86}$$

In addition, the following constraints should be met: The first is the well-known requirements that the moduli space is hyper-Kähler and the curvature F is of (1,1) type with respect to all three complex structures of the manifold. Also G must be a tri-holomorphic Killing vector field, and the two form on the bundle T must satisfy

$$T_{ab;m} = F_{mnab} G^m.\tag{87}$$

(87) is already discussed in (52) and, in the following, we will show that the relations in (86) are indeed satisfied.

To establish the first line of (86), we consider the consequences of the relations (80) and (81). Similar relations should also hold for other complex structures. In terms of real quantities, (80), (81) and the corresponding relations for other complex structures can be written

$$\begin{aligned}(1 + iJ^{(s)})\nabla(1 + iI)K_i^{(s)} &= 0, \\ (1 + iJ^{(s)})\nabla(1 - iI)K_i^{(s)'} &= 0, \quad (\text{no sum over } s)\end{aligned}\tag{88}$$

where $i = 1, 2, s = 1, 2, 3$ and I is the complex structure of the index bundle introduced in (46). Since all quantities in (88) are now real, the real and the imaginary parts should hold separately and we have the following 12 relations,

$$\begin{aligned}\nabla K_i^{(s)} - J^{(s)}\nabla(IK_i^{(s)}) &= 0, \\ \nabla K_i^{(s)'} + J^{(s)}\nabla(IK_i^{(s)'}) &= 0. \quad (\text{no sum over } s)\end{aligned}\tag{89}$$

These equations are, however, not all independent. Using the relation (76), we find after some algebra that all the relations in (89) can be recast into the three equations:

$$J^{(s)}\nabla v^a = -\nabla t_{(s)}^a,\tag{90}$$

where $v = K_1 - K'_2$ as before and

$$\begin{aligned} t_{(1)} &= I(K'_1 - K_2), \\ t_{(2)} &= -K'_1 - K_2, \\ t_{(3)} &= I(K_1 + K'_2). \end{aligned} \tag{91}$$

This is precisely the first equation of (86).

With the above identification for $t_{(s)}$, the section v^a turns out to be orthogonal to $t_{(s)}^a$, i.e.,

$$v_a t_{(s)}^a = 0, \tag{92}$$

which automatically satisfy the second line of (86). This can be shown by using the similar arguments as in (55) and the details are omitted.

As for the last two relations in (86), consider the identity

$$G^m \nabla_m K_{i\bar{A}} = \frac{1}{\sqrt{2}} \int d^3x \Psi_{\bar{A}}^\dagger \gamma^0 G^m s_m \not{D} \bar{M}_i \epsilon_+, \tag{93}$$

which can be easily seen from (53). Using the relation $[s_m, \not{D}] = \delta_m W_\mu \Gamma_\mu$, \not{D} can move to the left and kills $\Psi_{\bar{A}}^\dagger$ since it is a zero mode. Then

$$\begin{aligned} G^m \nabla_m K_{i\bar{A}} &= \frac{1}{\sqrt{2}} \int d^3x \Psi_{\bar{A}}^\dagger \gamma^0 G^m \delta_m W_\mu \Gamma_\mu \bar{M}_i \epsilon_+ \\ &= -\frac{1}{\sqrt{2}} \int d^3x \Psi_{\bar{A}}^\dagger \gamma^0 D_\mu \bar{a} \Gamma_\mu \bar{M}_i \epsilon_+, \end{aligned} \tag{94}$$

where (34) was used in the second line. From (53), we see that the right hand side is proportional to $T_{\bar{A}B}$ defined in (51), i.e.,

$$G^m \nabla_m K_{i\bar{A}} = T_{\bar{A}B} K_i^B. \tag{95}$$

Since the sections v and $t_{(s)}$ are linear combinations of K_i , this proves that the last two identities of (86) hold.

In summary, the low-energy effective Lagrangian (84) has all the right properties to have $N = 4$ supersymmetry.

4 Conclusions

In this paper, we have given a detailed derivation of the effective action governing the low-energy dynamics of monopoles and dyons in $N = 2$ SYM with hypermultiplets in arbitrary representations by generalizing the techniques developed in [24]. This includes the case that not only adjoint Higgs fields in the $N = 2$ vector multiplet but also Higgs fields in the hypermultiplets have non-vanishing expectation values while maintaining a non-trivial Coulomb branch, which is possible when the matter representation contains a zero weight vector. The improvement over the earlier work is that the hypermultiplets are not necessarily in real representations. Thus we have

obtained the low-energy effective action in the most general case. It is given by a supersymmetric quantum mechanics with potential terms which was obtained by a non-trivial “Scherk-Schwarz” dimensional reduction of (4,0) sigma models in two dimensions, which might have a more direct stringy origin along the line of [30].

It would be interesting to study the supersymmetric quantum mechanics derived in this paper and check the results in the context of Seiberg-Witten theories [31]. This has been done for example in [23] and [32] for pure SYM case. In particular, it is an interesting problem to study the BPS spectrum using the effective action derived in this paper.

Acknowledgments

We would like to thank Piljin Yi for encouragement and reading the manuscript, and S. Sethi for useful comments. This work was supported by the Science Research Center Program of the Korea Science and Engineering Foundation through the Center for Quantum Spacetime(CQUeST) of Sogang University with grant number R11-2005-021.

References

- [1] For a recent review, see D. Tong, “TASI lectures on solitons,” hep-th/0509216.
- [2] J.P. Gauntlett, Nucl. Phys. **B411** (1994) 443, hep-th/9305068.
- [3] J. Blum, Phys. Lett. **B333** (1994) 92, hep-th/9401133.
- [4] A. Sen, Phys. Lett. **B329** (1994) 217, hep-th/9402032.
- [5] S. Sethi, M. Stern and E. Zaslow, Nucl. Phys. **B457** (1995) 484, hep-th/9508117.
- [6] M. Cederwall, G. Ferretti, B. E. Nilsson and P. Salomonson, Mod. Phys. Lett. **A11** (1996) 367, hep-th/9508124.
- [7] J. P. Gauntlett and J. A. Harvey, Nucl. Phys. **B463** (1996) 287, hep-th/9508156.
- [8] J.P. Gauntlett and D.A. Lowe, Nucl. Phys. **B472** (1996) 194, hep-th/9601085.
- [9] K. Lee, E.J. Weinberg, and P. Yi, Phys. Lett. **B376** (1996) 97, hep-th/9601097; Phys. Rev. **D54** (1996) 1633, hep-th/9602167.
- [10] G. W. Gibbons, Phys. Lett. **B382** (1996) 53, hep-th/9603176.
- [11] O. Bergman, Nucl. Phys. **B525** (1998) 104, hep-th/9712211.
- [12] K. Hashimoto, H. Hata, and N. Sasakura, Phys. Lett. **B431** (1998) 303, hep-th/9803127.

- [13] O. Bergman and B. Kol, Nucl. Phys. **B536** (1998) 149, hep-th/9804160.
- [14] T. Kawano and K. Okuyama, Phys. Lett. **B432** (1998) 338, hep-th/9804139.
- [15] K. Hashimoto, H. Hata, and N. Sasakura, Nucl. Phys. **B535** (1998) 83, hep-th/9804164.
- [16] K. Lee and P. Yi, Phys. Rev. **D58** (1998) 066005, hep-th/9804174.
- [17] D. Bak, K. Hashimoto, B. Lee, H. Min, and N. Sasakura, Phys. Rev. **D60** (1999) 046005, hep-th/9901107.
- [18] D. Tong, Phys. Lett. **B460** (1999) 295, hep-th/9902005.
- [19] D. Bak, C. Lee, K. Lee and P. Yi, Phys. Rev. **D61** (2000) 025001, hep-th/9906119;
- [20] D. Bak, K. Lee and P. Yi, Phys. Rev. **D61** (2000) 045003, hep-th/9907090.
- [21] D. Bak and K. Lee, Phys. Lett. **B468** (1999) 76, hep-th/9909035.
- [22] D. Bak, K. Lee, P. Yi, Phys. Rev. **D62** (2000) 025009, hep-th/9912083.
- [23] J. P. Gauntlett, N. Kim, J. Park and P. Yi, Phys. Rev. **D61** (2000) 125012, hep-th/9912082.
- [24] J. P. Gauntlett, C. Kim, K. Lee and P. Yi, Phys. Rev. **D63** (2001) 065020, hep-th/0008031.
- [25] C. Fraser and T.J. Hollowod, Phys. Lett. **B402** (1997) 106, hep-th/9704011.
- [26] E. Witten and D. Olive, Phys. Lett. **78B** (1978) 97.
- [27] J. A. Harvey and A. Strominger, Commun. Math. Phys. **151** (1993) 221, hep-th/9108020.
- [28] C. Callias, Commun. Math. Phys. **62** (1978) 213.
- [29] C. M. Hull, G. Papadopoulos and P. K. Townsend, Phys. Lett. **B316** (1993) 291, hep-th/9307013.
- [30] A. Mikhailov, N. Nekrasov and S. Sethi, Nucl. Phys. B **531** (1998) 345, hep-th/9803142.
- [31] N. Seiberg and E. Witten, Nucl. Phys. B **426** (1994) 19 [Erratum-ibid. B **430** (1994) 485], hep-th/9407087; N. Seiberg and E. Witten, Nucl. Phys. B **431** (1994) 484, hep-th/9408099; P. C. Argyres and A. E. Faraggi, Phys. Rev. Lett. **74** (1995) 3931, hep-th/9411057; A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. B **344** (1995) 169, hep-th/9411048; A. Klemm, W. Lerche and S. Theisen, Int. J. Mod. Phys. A **11** (1996) 1929, hep-th/9505150;

A. Hanany and Y. Oz, Nucl. Phys. B **452** (1995) 283, hep-th/9505075; P. C. Argyres, M. R. Plesser and A. D. Shapere, Phys. Rev. Lett. **75** (1995) 1699, hep-th/9505100; P. C. Argyres and A. D. Shapere, Nucl. Phys. B **461** (1996) 437, hep-th/9509175; P. C. Argyres, M. R. Plesser and N. Seiberg, Nucl. Phys. B **471** (1996) 159, hep-th/9603042; G. Carlino, K. Konishi and H. Murayama, Nucl. Phys. B **590** (2000) 37, hep-th/0005076.

[32] M. Stern and P. Yi, Phys.Rev. **D62** (2000) 125006, hep-th/0005275.