Symmetry behavior in the Einstein universe: Gaussian approximation in the Schrödinger picture

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The effective potential for a scalar theory in the static Einstein universe is calculated using the Gaussian approximation in a field-theoretic Schrödinger picture. There occurs an infrared divergence in the slope of the effective potential at the would-be symmetry-breaking minimum and the effective potential is complex in a region including this would-be minimum. Spontaneous symmetry breaking, therefore, cannot occur in the static Einstein universe.

I. INTRODUCTION

Symmetry breaking in a static Einstein universe has been studied by a number of authors: Ford[1] considered instabilities in a $\lambda\phi^4$ theory; Toms[2] and Denardo and Spallucci[3] calculated the effective potential of scalar fields for conformally coupled field; and O'Connor, Hu, and Shen[4] computed the one-loop effective potential of a $\lambda\phi^4$ theory by ζ-function regularization both for conformally and minimally coupled cases. Anderson and Holman[5] calculated the effective potential for a scalar theory with $O(N)$ symmetry using the $1/N$ approximation and showed that the finite size of the spatial sections leads to an infrared divergence which, in turn, prevents spontaneous symmetry breaking (SSB) from occurring. This lack of SSB is in line with that of the $O(N)$ model in one and two spacetime dimensions in flat space,[6] and in four-dimensional flat space with nontrivial topology if two or more of the dimensions are compact.[7] In the one-loop approximation, however, SSB seems to occur although a similar infrared divergence is present for the massless minimally coupled scalar fields.[4] This discrepancy in symmetry behavior was discussed in Ref. 5.

Considering the importance of phase transitions in the early Universe which could have possessed strong curvature we believe that it is worthwhile to examine the above issue using different methods as a first step toward a systematic investigation of the effect of curvature on symmetry behavior. In this paper we employ the Gaussian approximation with a field-theoretic Schrödinger picture[8,9] to study SSB in the static Einstein universe. This method is particularly useful for some cosmological problems,[10] and as an approximation it goes beyond one-loop computations. Our results agree with those of Anderson and Holman,[5] and we present an improved form of the effective potential so that the analysis is more explicit and transparent. Furthermore, we show that the effective potential is complex at the symmetry-breaking minimum point, and therefore there cannot arise SSB in the Einstein universe.

In Sec. II we formulate the Schrödinger-picture Gaussian approximation of the effective potential in a general curved space. In Sec. III we calculate and renormalize the effective potential in the static Einstein universe, and then investigate the symmetry behavior in the $\lambda\phi^4$ theory. In the last section we discuss the relation between the Gaussian method and one-loop approximation.

II. SCHRÖDINGER PICTURE AND GAUSSIAN APPROXIMATION IN CURVED SPACE

In dealing with time-dependent field-theoretic problems, the Schrödinger picture[8] is more suitable than the usual Green's-function method, and it is particularly useful in studying the evolution of the early Universe in connection with the inflationary cosmology.[10] Renormalization in the Schrödinger picture has also been established both for static[11] and time-dependent problems.[9,12] Most work in the field-theoretic Schrödinger picture has been done only in flat spaces, and it is necessary to extend the formalism in order to study SSB in a curved space. In this section we develop a general formalism so that we can use it in the specific space, namely, the Einstein universe.

For the purpose of streamlining we introduce the following notation:

$$\int_{\mathbf{x}} \equiv \int d^3x\sqrt{g} \ , \tag{2.1}$$

$$\delta_c(\mathbf{x},\mathbf{y}) \equiv \frac{1}{\sqrt{g}}\delta(\mathbf{x}-\mathbf{y}) = \langle \mathbf{x} \mid \mathbf{y} \rangle \ , \tag{2.2}$$

$$\frac{\delta\phi(\mathbf{x})}{\delta_c\phi(\mathbf{y})}\equiv\frac{1}{\sqrt{g}}\frac{\delta\phi(\mathbf{x})}{\delta\phi(\mathbf{y})}=\delta_c(\mathbf{x},\mathbf{y}) , \tag{2.3}$$

where $\mathbf{x}$ and $\mathbf{y}$ denote three-dimensional space vector, and g is the determinant of the spatial metric tensor. The last equation is particularly useful since we can treat the curved-space formulas as if they were flat-space ones.

In the field-theoretic Schrödinger picture a quantum-mechanical wave function $\psi(x,t)$ is replaced by a wave functional $\Psi(\phi,t)$ which is a functional of a c-number field $\phi(x)$ at a fixed time t. Since it is impossible to find the exact wave functional it is useful to take an ansatz, the Gaussian form of the wave functional. In this paper we will use this Gaussian trial wave functional and apply the time-dependent variational technique to derive equations.[8]

The Gaussian wave functional is given as

$$\Psi(\phi,t)=N\exp\left[-\int_{\mathbf{x},\mathbf{y}}\tilde{\phi}(\mathbf{x})B(\mathbf{x},\mathbf{y})\tilde{\phi}(\mathbf{y})+\frac{i}{\hbar}\int_{\mathbf{x}}\hat{\pi}(\mathbf{x})\tilde{\phi}(\mathbf{x})-i\int_{\mathbf{x},\mathbf{y}}G(\mathbf{x},\mathbf{y})\Sigma(\mathbf{y},\mathbf{x})\right] , \tag{2.4}$$

where N is a normalization constant, and

$$\tilde{\phi}(\mathbf{x})\equiv\phi(\mathbf{x})-\hat{\phi}(\mathbf{x},t) , \tag{2.5}$$

$$B(\mathbf{x},\mathbf{y})\equiv\frac{1}{4\hbar}G^{-1}(\mathbf{x},\mathbf{y},t)-\frac{i}{\hbar}\Sigma(\mathbf{x},\mathbf{y},t) . \tag{2.6}$$

The meaning of other notations is seen from the following expectation values:

$$\langle\phi(\mathbf{x})\rangle=\hat{\phi}(\mathbf{x},t) , \tag{2.7a}$$

$$\left\langle -i\hbar\frac{\delta}{\delta_c\phi(\mathbf{x})}\right\rangle=\hat{\pi}(\mathbf{x},t) , \tag{2.7b}$$

$$\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle=\hat{\phi}(\mathbf{x},t)\hat{\phi}(\mathbf{y},t)+\hbar G(\mathbf{x},\mathbf{y},t) , \tag{2.7c}$$

$$\left\langle i\hbar\frac{\partial}{\partial t}\right\rangle=\int_{\mathbf{x}}\hat{\pi}(\mathbf{x},t)\dot{\hat{\phi}}(\mathbf{x},t)+\hbar\int_{\mathbf{x},\mathbf{y}}\Sigma(\mathbf{x},\mathbf{y},t)\dot{G}(\mathbf{y},\mathbf{x},t) . \tag{2.7d}$$

The above expectation values are taken by functional integration which is slightly modified from the flat-space case as

$$\int D\phi(\mathbf{x})\rightarrow\int D\phi(\mathbf{x})g(\mathbf{x})^{1/4} . \tag{2.8}$$

This was pointed out by Toms.[13]

The effective action in this picture is given by[8]

$$\Gamma=\int dt\langle\Psi(t)\,|\,i\hbar\partial_t-H\,|\,\Psi(t)\rangle , \tag{2.9}$$

and one obtains the functional Schrödinger equations by varying the effective action with respect to the parameter functions. In a static situation, as in our problem in the Einstein universe, the effective action reduces to the effective potential times a constant. The Hamiltonian, in a curved space, for a scalar field is

$$H=\int_{\mathbf{x}}\left[\frac{1}{2}\left[i\frac{\delta}{\delta_c\phi}\right]^2+\tfrac{1}{2}(\partial_i\phi)(\partial_j\phi)g^{ij}+V(\phi)\right] , \tag{2.10}$$

where g_{ij} is the spatial metric tensor and g^{ij} is its inverse.

In the Gaussian approximation the effective action can be explicitly given as

$$\begin{aligned}\Gamma=\int dt\,\Bigg[&\int_{\mathbf{x}}[\hat{\pi}\dot{\hat{\phi}}-\tfrac{1}{2}(\hat{\pi})^2-\tfrac{1}{2}g^{ij}\partial_i\hat{\phi}\partial_j\hat{\phi}-V(\hat{\phi})]\\&+\hbar\left[\int_{\mathbf{x},\mathbf{y}}\Sigma\dot{G}-2\int_{\mathbf{x},\mathbf{y},\mathbf{z}}\Sigma G\Sigma-\tfrac{1}{8}\int_{\mathbf{x}}[G^{-1}(\mathbf{x},\mathbf{x})-\tfrac{1}{2}g^{ij}\partial_i^{\mathbf{x}}\partial_j^{\mathbf{y}}G(\mathbf{x},\mathbf{y})\,|_{\mathbf{x}=\mathbf{y}}-\tfrac{1}{2}V^{(2)}(\hat{\phi})G(\mathbf{x},\mathbf{x})]\right]\\&-\frac{\hbar^2}{8}\int_{\mathbf{x}}V^{(4)}(\hat{\phi})G(\mathbf{x},\mathbf{x})G(\mathbf{x},\mathbf{x})\Bigg] ,\end{aligned} \tag{2.11}$$

where $V^{(n)}(\hat{\phi})\equiv d^nV(\hat{\phi})/d\hat{\phi}^n$. It is presented in $\hbar$ order: the zeroth-order terms are the classical action, the first-order terms are one-loop quantum effects, which we will discuss in the last section. The $\hbar^2$ term is the improvement over the one-loop approximation and it allows to capture some of the nonlinear features of the quantum field theory.

III. RENORMALIZATION OF THE EFFECTIVE POTENTIAL

In a static space the effective action is reduced to an effective potential as

$$\Gamma=-\int d^4x\,V_{\text{eff}} , \tag{3.1a}$$

$$V_{\text{eff}}=V(\hat{\phi})+\hbar\frac{V^{(2)}(\hat{\phi})}{2}G(\mathbf{x},\mathbf{x})+\hbar^2\frac{V^{(4)}(\hat{\phi})}{8}[G(\mathbf{x},\mathbf{x})]^2+\frac{\hbar}{8}G^{-1}(\mathbf{x},\mathbf{x})+\tfrac{1}{2}g^{ij}[\partial_i\hat{\phi}(\mathbf{x})\partial_j\hat{\phi}(\mathbf{y})+\hbar\partial_i^{\mathbf{x}}\partial_j^{\mathbf{y}}G(\mathbf{x},\mathbf{y})]\,|_{\mathbf{x}=\mathbf{y}} . \tag{3.1b}$$

In a further simplified situation, i.e., a stationary static space we can set $\hat{\phi}(\mathbf{x})=\hat{\phi}=$const, and $V_{\rm eff}$ becomes

$$V_{\rm eff}=V(\hat{\phi})+\frac{\hbar}{4}G^{-1}(\mathbf{x},\mathbf{x})-\frac{\hbar^2}{8}V^{(4)}(\hat{\phi})[G(\mathbf{x},\mathbf{x})]^2 . \quad (3.2)$$

Specifying our investigation to the static Einstein universe whose metric is

$$ds^2=dt^2-a^2(d\chi^2+\sin^2\chi\, d\theta^2+\sin^2\chi\sin^2\theta\, d\phi^2) \quad (3.3)$$

and to the potential

$$V(\hat{\phi})=\tfrac{1}{2}\mu^2\hat{\phi}^2+\frac{\xi R}{12}\hat{\phi}^2+\frac{\lambda}{4!}\hat{\phi}^4$$
$$\equiv\tfrac{1}{2}\hat{\mu}^2\hat{\phi}^2+\frac{\lambda}{4!}\hat{\phi}^4 , \quad (3.4)$$

where $\hat{\mu}^2\equiv\mu^2+\xi/6R=\mu^2+\xi/a^2$, we only need to compute the effective potential

$$V_{\rm eff}=\tfrac{1}{2}\hat{\mu}^2\hat{\phi}^2+\frac{\lambda}{4!}\hat{\phi}^4+\frac{\hbar}{4}G^{-1}(\mathbf{x},\mathbf{x})-\frac{\hbar^2}{8}\lambda[G(\mathbf{x},\mathbf{x})]^2 . \quad (3.5)$$

For the evaluation of G and G^{-1} we use the following equation derived by the variation of Γ [Eqs. (2.11)] with respect to G:

$$\tfrac{1}{8}G^{-2}(\mathbf{x},\mathbf{y})-\tfrac{1}{2}\left[g^{ij}\partial_i^x\partial_j^y+V^{(2)}(\hat{\phi})+\frac{\hbar}{2}V^{(4)}(\hat{\phi})G(\mathbf{x},\mathbf{x})\right]\delta_c(\mathbf{x},\mathbf{y})=0 . \quad (3.6)$$

Since the space is S^3 we use Gegenbauer polynomials $Y_N(\mathbf{x})\equiv Y_{n,l,m}(\chi,\theta,\phi)$ which are normalized as

$$\sum_N Y_N^*(\mathbf{x})Y_N(\mathbf{y})=\delta_c(\mathbf{x},\mathbf{y}) , \quad (3.7a)$$

$$\sum_{l,m} Y_{n,l,m}^*(\mathbf{x})Y_{n,l,m}(\mathbf{x})=\frac{(n+1)^2}{2\pi^2} , \quad (3.7b)$$

where the eigenvalues $N=(n,l,m)$ are

$$n=0,1,2,\ldots , \quad (3.8a)$$

$$l=0,1,2,\ldots,n , \quad (3.8b)$$

$$m=0,1,\ldots,\pm l . \quad (3.8c)$$

We can express $G(\mathbf{x},\mathbf{y})$ in terms of Gegenbauer polynomials as

$$G(\mathbf{x},\mathbf{y})=\sum_N Y_N(\mathbf{x})\hat{G}(n)Y_N^*(\mathbf{y}) , \quad (3.9a)$$

$$G(\mathbf{x},\mathbf{x})\equiv G(0)=\sum_{n=0}^{\infty}\frac{(n+1)^2}{2\pi^2}\hat{G}(n) , \quad (3.9b)$$

and inserting these relations into (3.6) we can solve it easily as

$$\hat{G}(n)=\tfrac{1}{2}\left[\frac{n(n+2)}{a^2}+V^{(2)}(\hat{\phi})+\frac{\hbar}{2}V^{(4)}(\hat{\phi})G(0)\right]^{-1/2} , \quad (3.10a)$$

$$G(0)=\sum_{n=0}^{\infty}\frac{(n+1)^2}{4\pi^2}\left[\frac{n(n+2)}{a^2}+m^2\right]^{-1/2} , \quad (3.10b)$$

$$G^{-1}(\mathbf{x},\mathbf{x})=\sum_{n=0}^{\infty}\frac{(n+1)^2}{\pi^2}\left[\frac{n(n+2)}{a^2}+m^2\right]^{1/2} . \quad (3.10c)$$

Here the effective mass m^2 is defined as

$$m^2\equiv V^{(2)}(\hat{\phi})+\frac{\hbar}{2}V^{(4)}(\hat{\phi})G(0) \quad (3.11a)$$

$$=\hat{\mu}^2+\frac{\lambda}{2}\hat{\phi}^2+\hbar\frac{\lambda}{2}G(0) . \quad (3.11b)$$

Since $G(0)$ has a divergence in it, m^2 and $V_{\rm eff}$ need renormalization, for which we follow the prescription used in the flat space by Pi and Samiullah.[9] In order to express m^2 in terms of renormalized $\hat{\mu}_R^2$ and λ_R we define

$$\frac{\hat{\mu}_R^2}{\lambda_R}=\frac{\hat{\mu}^2}{\lambda}+\frac{1/2}{4\pi^2a^2}I_1 \quad (3.12a)$$

and

$$\frac{1}{\lambda_R}=\frac{1}{\lambda}+\frac{1/2}{4\pi^2a^2}I_2(M) , \quad (3.12b)$$

where

$$I_1\equiv\frac{2}{\pi}\int_0^{\infty}dk\left[\left[\sum_{n=0}^{\infty}\right]+\frac{1}{2}-\frac{\pi}{2}k+\frac{\pi}{2}\frac{1}{a^2M^2-1}[(k^2+a^2M^2-1)^{1/2}-k)\right] \quad (3.13)$$

and

$$I_2(M)\equiv\frac{a^2}{a^2M^2-1}\int_0^{\infty}dk[(k^2+a^2M^2-1)^{1/2}-k] . \quad (3.14)$$

Here M is an arbitrary large constant mass which provides the renormalization scale. Using these relations we rewrite m^2 as

$$m^2=\hat{\mu}_R^2+\frac{\lambda_R}{2}\hat{\phi}^2+\hbar\frac{\lambda_R}{2}\left[G(0)-\frac{1}{4\pi^2a^2}I_1+\frac{m^2}{4\pi^2a^2}I_2(M)\right] . \quad (3.15)$$

From Eq. (3.10b) we would show that (3.15) is a finite expression, i.e., the divergence in $G(0)$ must be canceled by I_1 and I_2. For this we use the following two relations in order to extract the finite parts of $G(0)$:

$$\frac{1}{\omega}=\frac{1}{\pi}\int_{-\infty}^{\infty}dk\frac{1}{k^2+\omega^2} \quad (3.16)$$

and

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+A^2}=\frac{\pi}{A}+\frac{2\pi}{A}\frac{1}{\exp(2\pi A)-1} . \quad (3.17)$$

We rearrange $G(0)$ [Eq. (3.10b)] as

$$
\begin{aligned}
G(0)=&\frac{1}{4\pi^2a^2}\frac{2}{\pi}\int_0^{\infty}dk\left\{\left[\left[\sum_{n=0}^{\infty}\right]+\frac{1}{2}-\frac{\pi}{2}k\right]+\frac{\pi/2}{a^2M^2-1}[(k^2+a^2M^2-1)^{1/2}-k]\right\}\\
&+\frac{1}{4\pi^2a^2}\int_0^{\infty}dk\left[[k-(k^2+a^2m^2-1)^{1/2}]+\frac{a^2m^2-1}{a^2M^2-1}[(k^2+a^2M^2-1)^{1/2}-k]\right]\\
&+\frac{1}{4\pi^2a^2}2\int_0^{\infty}dk\left[-\frac{(k^2+a^2m^2-1)^{1/2}}{\exp[2\pi(k^2+a^2m^2-1)^{1/2}]-1}\right]-\frac{1}{4\pi^2a^2}m^2\frac{a^2}{a^2M^2-1}\int_0^{\infty}dk[(k^2+a^2M^2-1)^{1/2}-k]\\
=&\frac{1}{4\pi^2a^2}[I_1-m^2I_2(M)]+\frac{1}{4\pi^2a^2}\frac{a^2m^2-1}{4}\ln\left[\frac{a^2m^2-1}{a^2M^2-1}\right]\\
&+\frac{2}{4\pi^2a^2}\int_0^{\infty}dk\frac{(k^2+a^2m^2-1)^{1/2}}{\exp[2\pi(k^2+a^2m^2-1)^{1/2}]-1}\;, \qquad (3.18)
\end{aligned}
$$

where the last two terms are the desired finite parts. Inserting (3.18) into (3.15) we obtain

$$
m^2=\hat{\mu}_R^2+\frac{\lambda_R}{2}\hat{\phi}^2+\frac{\hbar\lambda_R}{32\pi^2}(m^2-1/a^2)\ln\left[\frac{a^2m^2-1}{a^2M^2-1}\right]-\frac{\hbar\lambda_R}{4\pi^2a^2}F_1\;, \qquad (3.19)
$$

where

$$
F_1=\int_0^{\infty}dk\frac{(k^2+a^2m^2-1)^{1/2}}{\exp[2\pi(k^2+a^2m^2-1)^{1/2}]-1}\;. \qquad (3.20)
$$

Note that the third term on the right-hand side is only a slight modification of the finite part of the same model in the flat space,[9] while the F_1 term is a genuinely new one in the Einstein universe. In the $a\to\infty$ limit we recover the flat-space result.

Having expressed m^2 in terms of renormalized parameters we may reexpress V_{eff} [Eq. (3.5)] as a function of m^2 using the definition (3.11), then it becomes easier to show the finiteness of V_{eff}. First, we note that

$$
V_{\text{eff}}=-\frac{m^4}{2\lambda}+\frac{\hat{\mu}^2}{\lambda}m^2+\frac{\hat{\phi}^2}{2}m^2+\frac{\hbar}{4}G^{-1}-\frac{\lambda}{12}\hat{\phi}^4\;, \qquad (3.21)
$$

where we ignored the irrelevant constant $-\mu^4/(2\lambda)$. After replacing λ and $\hat{\mu}^2$ by λ_R and $\hat{\mu}_R^2$ we have

$$
\begin{aligned}
V_{\text{eff}}=&-\frac{m^4}{2\lambda_R}+\frac{\hat{\mu}_R^2}{\lambda_R}m^2+\frac{\hat{\phi}^2}{2}m^2-\frac{\lambda}{12}\hat{\phi}^4+\left[\frac{\hbar}{4}G^{-1}(\mathbf{x},\mathbf{x})+\frac{\hbar}{4\pi^2a^2}\frac{m^4}{4}I_2(M)-\frac{m^4\hbar/2}{4\pi^2a^2}I_1\right]\\
=&-\frac{m^4}{2\lambda_R}+\frac{\hat{\mu}_R^2}{\lambda_R}m^2+\frac{\hat{\phi}^2}{2}m^2+\frac{1}{4\pi^2a^2}\frac{\hbar a^2}{16}\left[m^2-\frac{1}{a^2}\right]^2\left[\ln\left[\frac{a^2m^2-1}{a^2M^2-1}\right]-\frac{1}{2}\right]+\frac{1}{4\pi^2a^2}F_2(am)\;, \qquad (3.22)
\end{aligned}
$$

where

$$
F_2(am)=\frac{\hbar}{2}\int_{1/a^2}^{m^2}dm^2\int_0^{\infty}dk\frac{(k^2+a^2m^2-1)^{1/2}}{\exp[2\pi(k^2+a^2m^2-1)^{1/2}-1}\;. \qquad (3.23)
$$

In deriving this equation we used the relation $\partial G^{-1}(x,x)/\partial m^2=2G(0)$, and the lower limit of the integration variable m^2 is taken conveniently. In the limit of infinite cutoff, λ goes to 0_- as λ_R is kept to be constant in (3.12b).

We can now study whether SSB can occur in the Einstein space. The minimum of the potential (3.22) is determined by

$$
\frac{dV_{\text{eff}}}{d\hat{\phi}}=\frac{\partial V_{\text{eff}}}{\partial\hat{\phi}}+\frac{\partial V_{\text{eff}}}{\partial m^2}\frac{dm^2}{d\hat{\phi}}=\frac{\partial V_{\text{eff}}}{\partial\hat{\phi}}=\hat{\phi}m^2=0\;. \qquad (3.24)
$$

Here we used $\partial V_{\text{eff}}/\partial m^2=0$, which follows from the defining equation of m^2 [Eq. (3.19)]. There are two possible minima:

$$
\hat{\phi}=0 \ \text{ and } \ m^2=0\;, \qquad (3.25)
$$

where the point $\hat{\phi}=0$ keeps the symmetry, and the point $m^2=0$ would break the symmetry spontaneously.

Now we will show that the point $m^2=0$ is not the minimum of the effective potential because of the infrared divergence. From (3.19) if we set $m^2=0$ we find that

$$\hat{\phi}^2=-\frac{\hat{\mu}_R^2}{\lambda_R}-\frac{1}{4\pi^2 a^2}F_1(0) , \tag{3.26}$$

where $F_1(0)$ is divergent because the integral behaves, near $k \simeq 0$, as

$$F_1(0)\sim\frac{-1}{\pi}\int_0 \frac{dk}{k^2} .$$

Therefore at $m^2=0$, $\hat{\phi}^2\to-\infty$ which is unphysical. This prevention of SSB via infrared divergence was pointed out by Anderson and Holman,[5] and it is similar to the lack of SSB in the one- and two-dimensional cases.[6]

Our presentation of the $V_{\rm eff}$ (3.22) is better suited for analysis than that of Anderson and Holman[5] in the sense it is more explicit and transparent, and the flat-space limit ($a\to\infty$) is immediately recoverable. One further important point is that the allowed region of m^2 is manifest in this form. The effective potential becomes complex if $m^2<1/a^2$, therefore $m^2=0$ is not allowed from the beginning. A similar thing in the flat space is that the $m^2<0$ was not allowed, and $m^2=0$ was the allowed minimum of $V_{\rm eff}$ (Ref. 6). This limit of allowed region of $m^2(m^2>1/a^2)$ is only indirectly seen in the ϕ^2 and X relation [Eq. (26) in Ref. 5, our m^2 is their X] of Anderson and Holman.[5] We conclude that there is neither spontaneous symmetry breaking nor infrared divergence in the static Einstein universe because the effective potential is complex at $m^2=0$. Thus we see that the result of the Gaussian approximation is qualitatively equivalent to that of the large-N approximation of Ref. 5. We note, however, that the effective potential (3.22), which is computed for $N=1$, is one-third of the large-N result (or, the Gaussian result for $N=1$ is equivalent to the large-N approximation for $N=3$).

IV. DISCUSSION

One of the nice features of the Gaussian approximation is that it includes one-loop approximation as a first order of $\hbar$. This is in contrast with the $1/N$ expansion method whose relation with one-loop approximation is not manifest. If we ignore the $\hbar^2$ term in (3.5) we have

$$V_{\rm eff}=\tfrac{1}{2}\hat{\mu}^2\hat{\phi}^2+\frac{\lambda}{4!}\hat{\phi}^4+\frac{\hbar}{4}G^{-1}(\mathbf{x},\mathbf{x}) , \tag{4.1}$$

and neglecting the $\hbar$ term in (3.10) and (3.11b) we get

$$G^{-1}(\mathbf{x},\mathbf{x})=\sum_{n=0}^{\infty}\frac{(n+1)^2}{\pi^2}\left[\frac{n(n+2)}{a^2}+m^2\right]^{1/2} , \tag{4.2}$$

$$m^2=\hat{\mu}^2+\frac{\lambda}{2}\hat{\phi}^2 . \tag{4.3}$$

Now we compare this $V_{\rm eff}$ with the one-loop effective potential $V^{(1)}$:

$$V^{(1)}=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\sum_{n,l,m}\ln\left[\frac{k_0^2+\frac{n(n+2)}{a^2}+m^2}{\mu^2}\right] . \tag{4.4}$$

After replacing the constant μ^2 by $k_0^2+n(n+2)/a^2$ and performing the integration we have

$$V^{(1)}=\frac{\hbar}{4\pi^2}\sum_{n,l,m}\left[\frac{n(n+2)}{a^2}+m^2\right]^{1/2} , \tag{4.5}$$

where an infinite constant is neglected. The result indeed, agrees with the $\hbar$ term in (4.1).

Let us examine the symmetry behavior and the infrared divergence in the one-loop approximate potential. The minimum is located at

$$\frac{dV_{\rm eff}}{d\hat{\phi}}=\left[\hat{\mu}^2+\frac{\lambda}{6}+\frac{\hbar\lambda}{2}G(0)\right]\hat{\phi}=0 . \tag{4.6}$$

The symmetric minimum $\hat{\phi}=0$ is the same as the Gaussian case, but another minimum occurs at $\hat{\phi}=\hat{\phi}_0$,

$$\hat{\phi}_0^2=-\frac{6}{\lambda}\hat{\mu}^2-3\hbar G(0) . \tag{4.7}$$

At this point the effective mass becomes

$$m^2|_{\hat{\phi}^2=\hat{\phi}_0^2}=-2\hat{\mu}^2-\frac{3\lambda\hbar}{2}G(0) . \tag{4.8}$$

From (4.2) we only need $m^2>0$ in order to have real $V_{\rm eff}$, therefore we cannot obtain the limit $m^2>1/a$ in this one-loop approximation.

Since we are interested in the infrared divergence we will ignore the ultraviolet divergence problem simply by cutting off the "momentum" n by some large finite number N. Then

$$G(0)=\sum_{n=0}^{N}\frac{1}{4\pi^2}\frac{(n+1)^2}{\left[\frac{n(n+2)}{a^2}+m^2\right]^{1/2}} . \tag{4.9}$$

which is clearly divergent due to the infrared divergence ($n=0$) only if $m^2=0$. However, the point $m^2=0$ is not consistent with the potential minimum condition (4.8). Therefore, if we take only the one-loop approximation, the SSB could occur, and no infrared divergence conspiracy at the minimum of $V_{\rm eff}$ does occur, and $m^2>1/a$ is not seen.

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