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Zero-point field in a circular-motion frame. II. Spinor and vector fields

J. S. Kim and K. S. Soh
Department of Physics Education, Seoul National University, Seoul 151, Korea

S. K. Kim
Department of Physics, Ehwa Women's University, Seoul 120, Korea

J. H. Yee
Department of Physics, Yonsei University, Seoul 120, Korea
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The energy spectrum of zero-point fields of massless spinor and vector fields in a circular-motion frame is investigated. In the nonrelativistic limit it exhibits half-integer harmonics of the circular-motion frequency. The density-of-states factor is of the same form as the uniform-acceleration case, but the energy spectrum is not of the thermal form.

I. INTRODUCTION

Previously we studied the energy density of a zero-point field in a circular-motion frame. In this paper we extend the previous work[1] (hereafter denoted as paper I) from scalar fields to fields with spin $\frac{1}{2}$ and 1. Fields with any spin in uniformly accelerated frames have been investigated by Hacyan[2] and others.[3] In this paper we adopt Hacyan's formalism and apply it to a circular-motion case.

As was shown in the paper I and other works[4,5] the spectrum of zero-point energy of massless scalar fields in a circular-motion frame does not have the thermal character of uniformly accelerated frames. In this paper we reaffirm this result in the fields with spin. In the nonrelativistic limit the spectrum exhibits the harmonics of the circular-motion frequency, and in the extremely relativistic limit the energy spectrum is mostly determined by a pole in the complex proper-time plane. The essential features of the involved calculations are similar to but more complicated than those of scalar field cases.

There are, however, distinctive features in the cases of fields with spin. In the nonrelativistic limit there appear harmonics of half the circular-motion frequency. This is due to the interfering term $u^\alpha x_\alpha$ (u^α is a velocity four-vector) which is absent in the scalar field case. Other features are the extra factors $[1+(a/2)^2/\omega^2]$ (spinor) and $(1+a^2/\omega^2)$ (vector) which are multiplied to the thermal energy spectrum. These factors agree with the results obtained by Hacyan,[2] Candelas and Deutsch,[3] and Boyer.[6]

In this paper we restrict our attention to massless fields with spin $\frac{1}{2}$ and 1 in flat four-dimensional space-time. We adopt the formalism of Hacyan:[2] A zero-rest-mass field with spin S can be defined by a spinor[7] $\phi_{A_1\cdots A_{2S}}$ which transforms as the $D(S,0)$ representation of the Lorentz group. The Wightman functions are defined as the following vacuum expectation values:

$$D^{+}_{A\dot{A}}(\chi,\chi')\equiv\langle\phi_A(\chi)\bar{\phi}_{\dot{A}}(\chi')\rangle\ ,\qquad D^{-}_{A\dot{A}}(\chi,\chi')\equiv\langle\phi_A(\chi')\bar{\phi}_{\dot{A}}(\chi)\rangle,\quad s=\tfrac{1}{2} \tag{1}$$

and

$$D^{+}_{AB,\dot{A}\dot{B}}(\chi,\chi')\equiv\langle\phi_{AB}(\chi)\bar{\phi}_{\dot{A}\dot{B}}(\chi')\rangle\ ,\qquad D^{-}_{AB,\dot{A}\dot{B}}(\chi,\chi')\equiv\langle\phi_{AB}(\chi')\bar{\phi}_{\dot{A}\dot{B}}(\chi)\rangle,\quad s=1\ . \tag{2}$$

Here, the indices A, $\dot{B}$, etc., are spinorial ones which are related to the tensorial indices by the Rindler convention; thus for a vector we write

$$u^\alpha=u^{A\dot{A}}\sigma^\alpha_{A\dot{A}}\ , \tag{3}$$

where it is understood that $u^{A\dot{A}}=\sigma^{A\dot{A}}_\alpha u^\alpha$.

The Wightman functions for a scalar field are

$$D^{\pm}(\chi-\chi')=\frac{-1}{4\pi^2}\frac{1}{[(t-t'\mp i\epsilon)^2-|\mathbf{x}-\mathbf{x}'|^2]}\ , \tag{4}$$

where $\chi=(t,\mathbf{x})$. Those for the fields with spin are

$$D^{\pm}_{A\dot{A}}(\chi-\chi')=n'_{1/2}\frac{\partial}{\partial\chi^{A\dot{A}}}D^{\pm}(\chi-\chi'),\quad s=\tfrac{1}{2} \tag{5a}$$

and

$$D^{\pm}_{AB,\dot{A}\dot{B}}(\chi-\chi')=n'_1\frac{\partial}{\partial\chi^{A\dot{A}}}\frac{\partial}{\partial\chi^{B\dot{B}}}D^{\pm}(\chi-\chi'),\quad s=1\ , \tag{5b}$$

where n'_s are appropriate normalization constants.

In this paper the two points χ and χ' are considered as points on a given world line of a detector, i.e., $\chi^\alpha=\chi^\alpha(\tau+\frac{1}{2}\sigma)$ and $\chi^\alpha=\chi^\alpha(\tau-\frac{1}{2}\sigma)$, where τ is the proper time of the detector. The energy density measured by the detector with four-velocity $u^\alpha[d\chi^\alpha(\tau)/d\tau]$ is

$$e\equiv u^\alpha u^\beta T_{\alpha\beta}\ , \tag{6}$$

where $T_{\alpha\beta}$ is the energy-momentum tensor of the field.

In Sec. II we will evaluate the energy-density spectrum of spinor fields in the extremely relativistic and nonrelativistic limit of the circular motion. In Sec. III the case of vector fields will be treated in a similar manner, and in Sec. IV summary of the results will be given.

II. SPIN $\frac{1}{2}$

The energy-momentum tensor of a massless fermion field is

$$T_{\alpha\beta}=\frac{i}{8}[\sigma_{\alpha}^{A\dot{A}}\langle(\partial_\beta\phi_A)\phi_{\dot{A}}-\phi_A\partial_\beta\phi_{\dot{A}}\rangle+\sigma_{\beta}^{A\dot{A}}\langle(\partial_\alpha\phi_A)\phi_{\dot{A}}-\phi_A\partial_\alpha\phi_{\dot{A}}\rangle] \tag{7}$$

and the energy density of the field as detected by an observer with a world line $\chi^\alpha=\chi^\alpha(\tau)$ and four-velocity u^α is

$$e\equiv u^\alpha u^\beta T_{\alpha\beta}=\frac{i}{4}u^{A\dot{A}}\left\langle\left[\frac{d}{d\tau}\phi_A\right]\phi_{\dot{A}}-\phi_A\left[\frac{d}{d\tau}\phi_{\dot{A}}\right]\right\rangle, \tag{8}$$

where all the quantities are evaluated at the proper time τ. Following the work of Hacyan,[2] we obtain

$$\frac{de}{d\omega}=\frac{-2i}{4\pi}\int_{-\infty}^{\infty}d\sigma\, e^{i\omega\sigma}\omega u^\alpha\times\frac{\partial}{\partial\chi^\alpha}[D^+(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma)-D^-(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma)], \tag{9}$$

where only positive values of ω are involved, and $-2i$ is the normalization constant.

In the case of uniform linear acceleration, the world line is given by

$$\chi^\alpha(\tau)=a^{-1}(\cosh a\tau,\sinh a\tau,0,0),\quad u^\alpha(\tau)=a^{-1}(\sinh a\tau,\cosh a\tau,0,0), \tag{10}$$

where a is acceleration. From this we can show that

$$\chi^\alpha(\tau+\tfrac{1}{2}\sigma)-\chi^\alpha(\tau-\tfrac{1}{2}\sigma)=\frac{2}{a}\sinh\left[\frac{a\sigma}{2}\right]u^\alpha(\tau) \tag{11}$$

and

$$u^\alpha\frac{\partial}{\partial\chi^\alpha}D^\pm\left[\tau+\frac{\sigma}{2},\tau-\frac{\sigma}{2}\right]=\frac{-2}{4\pi^2}\frac{1}{\left[\frac{2}{a}\sinh\left[\frac{2}{a}\sigma\mp i\epsilon\right]\right]^3}. \tag{12}$$

The energy density is

$$\frac{de}{d\omega}=\frac{4i}{16\pi^3}\omega\int_{-\infty}^{\infty}d\sigma\, e^{i\omega\sigma}\left[\frac{1}{\left[\frac{2}{a}\sinh\left[\frac{a}{2}\sigma-i\epsilon\right]\right]^3}+\frac{1}{\left[\frac{2}{a}\sinh\left[\frac{a}{2}\sigma+i\epsilon\right]\right]^3}\right]=\frac{1}{2\pi^2}\omega\left[\omega^2+\left[\frac{a}{2}\right]^2\right]\left[-\frac{1}{2}+\frac{1}{e^{2\pi\omega/a}+1}\right]. \tag{13}$$

The last factor in the energy density shows the thermal characteristics of the spectrum, and the other factor represents the number density, which can be seen by calculating the fermion number density $dn/d\omega$ (Ref. 2).

Consider now the case of circular motion, which is given by

$$\chi^\alpha(\tau)=(\gamma\tau,\rho\sin(\omega_0\gamma\tau),\rho\cos(\omega_0\gamma\tau),0),\quad u^\alpha(\tau)=(\gamma,\gamma\sigma\cos(\omega_0\gamma\tau),-\gamma\sigma\sin(\omega_0\gamma\tau),0), \tag{14}$$

where ρ, ω_0, v are the radius, angular frequency, and speed of the circular motion, respectively, and $\gamma=1/\sqrt{1-v^2}$. From this we easily obtain

$$\chi^\alpha\left[\tau+\frac{\sigma}{2}\right]-\chi^\alpha\left[\tau-\frac{\sigma}{2}\right]=\left[\gamma\sigma,2\rho\sin\left[\frac{\omega_0\gamma\sigma}{2}\right]\cos(\omega_0\gamma\tau),-2\rho\sin\left[\frac{\omega_0\gamma\sigma}{2}\right]\sin(\omega_0\gamma\tau),0\right] \tag{15}$$

and

$$D^\pm\left[\tau+\frac{\sigma}{2},\tau-\frac{\sigma}{2}\right]=\frac{-1}{4\pi^2}\frac{1}{(\gamma\sigma\mp i\epsilon)^2-4\rho^2\sin^2\left[\frac{\omega_0\gamma\sigma}{2}\right]}, \tag{16}$$

$$u^\alpha\frac{\partial}{\partial\chi^\alpha}D^\pm=\frac{2}{4\pi^2}\frac{\gamma\left[\gamma\sigma-2\rho v\sin\left[\frac{\omega_0\gamma\sigma}{2}\right]\right]}{\left[(\gamma\sigma\mp i\epsilon)^2-4\rho^2\sin^2\left[\frac{\omega_0\gamma\sigma}{2}\right]\right]^2}. \tag{17}$$

The energy density is

$$\frac{de}{d\omega}=\frac{n}{4\pi}\omega\int_{-\infty}^{\infty}d\sigma\, e^{i\omega\sigma}u^{\alpha}\partial_{\alpha}\left[D^{+}\left[\tau+\frac{\sigma}{2},\tau-\frac{\sigma}{2}\right]+D^{-}\left[\tau+\frac{\sigma}{2},\tau-\frac{\sigma}{2}\right]\right]$$

$$=\frac{n}{4\pi^3}\left[\frac{\omega_0}{2}\right]^3\left[\frac{\omega}{\omega_0}\right]\int_{-\infty}^{\infty}dz\, e^{ziWz}\left[\frac{z-v^2\sin z}{[(z-i\epsilon)^2-v^2\sin^2 z]^2}+\frac{z-v^2\sin z}{[(z+i\epsilon)^2-v^2\sin^2 z]^2}\right]$$

$$\equiv\frac{n}{4\pi^3}\left[\frac{\omega_0}{2}\right]^3\left[\frac{\omega}{\omega_0}\right][J(-i\epsilon)+J(+i\epsilon)]\ , \tag{18}$$

where $W=\omega/\gamma\omega_0$ and $z=(\gamma\omega_0/2)\sigma$.

In the nonrelativistic limit ($v\simeq 0$) we can evaluate the energy density as a power series in v. We illustrate the procedure by computing the first two terms explicitly:

$$J(-i\epsilon)\simeq\int_{-\infty}^{\infty}dz\, e^{2izW}\frac{1}{(z-i\epsilon)^3}\left[1+v^2\left[\frac{\sin^2 z}{z^2}-\frac{\sin z}{z}\right]+\cdots\right]$$

$$=\frac{\pi}{i}\left[4W^2+\frac{v^2}{3}[(W+1)^4-2W^4+(W-1)^4\theta(W-1)]-\frac{4v^2}{3}[(W+\tfrac{1}{2})^3-(W-\tfrac{1}{2})^3\theta(W-\tfrac{1}{2})]+\cdots\right]. \tag{19}$$

For the calculation of $J(+i\epsilon)$ we make use of the relation

$$J(-i\epsilon)-J(+i\epsilon)=\frac{-2\pi i}{1-v^2}\left[2W^2+\frac{v^2}{2(1-v^2)}\right]$$

$$=-4\pi i\left[\frac{\omega^2}{{\omega_0}^2}+\frac{1}{4}\frac{v^2}{(1-v^2)^2}\right]=\frac{-4\pi i}{{\omega_0}^2}\left[\omega^2+\left[\frac{a}{2}\right]^2\right], \tag{20}$$

where $a=\omega_0 v/(1-v^2)$ is the circular acceleration. Now the energy density is

$$\frac{de}{d\omega}=\frac{n}{4\pi^3}\left[\frac{\omega_0}{2}\right]^3\left[\frac{\omega}{\omega_0}\right]\left[(-4\pi i)\left[\frac{\omega^2}{{\omega_0}^2}+\frac{1}{4}\frac{v^2}{(1-v^2)^2}\right]+\frac{2\pi i}{3}v^2[(W-1)^4\theta(1-W)+4(W-\tfrac{1}{2})^3\theta(\tfrac{1}{2}-W)]\right]$$

$$=\frac{1}{2\pi^2}\omega\left[\omega^2+\left[\frac{a}{2}\right]^2\right]\left[-\frac{1}{2}\right]+\frac{\omega}{2\pi^2}\frac{{\omega_0}^2v^2}{12}[(W-1)^4\theta(1-W)+4(W-\tfrac{1}{2})^3\theta(\tfrac{1}{2}-W)]\ . \tag{21}$$

We have chosen the normalization factor $n=-2i$ such that we obtain the (negative) zero-point energy density of the neutrino field for $v=0$. Notice that the form of the first term agrees completely with the result of Hacyan in the uniform acceleration case, namely, Eqs. (13). A new feature we found in this energy density is the appearance of the factor which involves harmonics of one-half of the circular-motion frequency. In the scalar case only integer multiples of the circular-motion frequency were involved. The origin of the half-integer harmonics is the $u^{\alpha}\chi_{\alpha}$ term which was produced by the $u^{\alpha}\partial_{\alpha}$ operation on $D^{\pm}$ functions. A kind of frequency mixing of $\chi^{\alpha}(\tau)$ and $u^{\alpha}(\tau)$ occurs.

In the extremely relativistic limit ($v\to 1$), the integral is dominated by the pole contribution at $z=iR$ ($R=v\sinh R$). We note that this pole is a double pole in the spin-$\frac{1}{2}$ case while it was a simple one in the scalar case. After some calculations we obtain

$$J(+i\epsilon)=\oint dz\frac{e^{2iWz}(z-v^2\sin z)}{(z^2-v^2\sin^2 z)^2}$$

$$\simeq 2\pi i\frac{\exp(-2WR)(1-v)}{[2R(v\cosh R-1)]^2}\left[2WR+\frac{R^2}{v\cosh R-1}+\frac{Rv(\cosh R-1)}{1-v}\right]. \tag{22}$$

The energy density coming from this term is

$$\frac{de}{d\omega}\simeq\frac{\omega^3}{\pi^2}\frac{e^{-2WR}}{32W^2R^2}(WR+\tfrac{3}{2}),\quad R\to 0\ , \tag{23}$$

where we have retained only the highest order in $(1-v)$. This is different from the thermal spectrum.[1,2]

III. SPIN 1

The energy-momentum tensor of the electromagnetic field f_{AB} is given by

$$T_{\alpha\beta}=\frac{1}{4\pi}\langle f_{AB}f_{\dot{A}\dot{B}}\rangle \tag{24}$$

and using the same procedure as in the previous section the energy density can be obtained as

$$\begin{aligned}\frac{de}{d\omega}&=\frac{n_1}{8\pi^2}\frac{-1}{16\pi^2}\frac{\omega_0^3}{\gamma}\int_{-\infty}^{\infty}dz\,e^{2izW}\left[\frac{4\gamma^2(z-v^2\sin z)^2-(z^2-v^2\sin^2 z)}{[(z-i\epsilon)^2-v^2\sin^2 z]^3}\right]+(i\epsilon\to-i\epsilon)\\&\equiv\frac{n_1}{8\pi^2}\frac{-1}{16\pi^2}\frac{\omega_0^3}{\gamma}[K(-i\epsilon)+K(+i\epsilon)]\ ,\end{aligned}\tag{25}$$

where $W=\omega/\gamma\omega_0$ and $z=(\gamma\omega_0/2)\sigma$ as in the previous section.

In the nonrelativistic case ($v\simeq 0$) the evaluation of the energy density can be done as a power series in v^2. In order to have a concrete idea we give only the first two terms:

$$\begin{aligned}K(+i\epsilon)&=\int_{-\infty}^{\infty}dz\,e^{2izW}\frac{1}{(z+i\epsilon)^4}\left[\frac{4\left[1-\frac{v^2\sin z}{z}\right]^2-\left[1-\frac{v^2\sin^2 z}{z^2}\right]}{\left[1-\frac{v^2\sin^2 z}{z^2}\right]^3}\right]\\&\simeq\frac{-4\pi v^2}{3}[4(W-\tfrac{1}{2})^4\theta(\tfrac{1}{2}-W)+(W-1)^5\theta(1-W)]\ .\end{aligned}\tag{26}$$

Here we note again that the half-frequency harmonics appear in the vector fields in the same way as the spinor case. The energy density is obtained using (26) and the relations

$$K(-i\epsilon)-K(+i\epsilon)=\frac{8\pi\gamma}{\omega_0^3}(\omega^3+a^2\omega)\ , \tag{27}$$

where a is the circular-motion acceleration. The factor $(\omega^3+a^2\omega)$ agrees with the one in the uniform linear acceleration motion found by Candelas and Deutsch,[3] Boyer,[6] and Hacyan.[2] The energy density is normalized by taking $n_1=8\pi$ in order to recover, in the limit $v\to 0$, the well-known spectrum $(\omega^3/\pi^2)\frac{1}{2}$. The spectrum up to the order of v^2 is

$$\frac{de}{d\omega}=\frac{\omega^3}{\pi^2}\left[1+\left[\frac{a}{\omega}\right]^2\right]\frac{1}{2}+\frac{\omega_0^3}{\pi^2}\frac{v^2}{6}[4(W-\tfrac{1}{2})^4\theta(\tfrac{1}{2}-W)+(W-1)^5\theta(1-W)]\ . \tag{28}$$

In the extremely relativistic limit the integral is again dominated by the pole contribution at $z=iR$, but the pole is a triple one which makes the evaluation very tedious. The result is

$$\begin{aligned}K(+i\epsilon)&\simeq\frac{2\pi\exp(-2WR)(1-v)}{[2R(v\cosh R-1)]^3}[4(WR)^2+30WR+36]\ ,\\\frac{de}{d\omega}&\simeq\frac{\omega^3}{\pi^2}\frac{\exp(-2WR)}{(2WR)^3}\left[\frac{1}{2}(WR)^2+\frac{15WR}{4}+\frac{9}{2}\right],\quad R\to 0\ ,\end{aligned}\tag{29}$$

where only the highest term in $(1-v)$ is retained. This spectrum is somewhat different from the scalar and the spinor case.

Hacyan and Sarmiento[8] have computed the total vacuum energy density of the electromagnetic field in a circular-motion frame. By equating the result with the blackbody energy density, they obtained the effective temperature of the system. This effective temperature might conveniently be used as a rough measure of the zero-point field energy of the system.

IV. SUMMARY AND DISCUSSION

We summarize the results of the previous and present papers and also include those of the uniformly accelerating detectors for the sake of completeness and comparison.

(1) Uniform acceleration.

$$\frac{de}{d\omega}=\begin{cases}\dfrac{\omega^3}{\pi^2}\left[\dfrac{1}{2}+\dfrac{1}{\exp(2\pi\omega/a)-1}\right], & s=0\ ,\\[2ex] \dfrac{\omega^3}{2\pi^2}\left[1+\left[\dfrac{a}{2\omega}\right]^2\right]\left[-\dfrac{1}{2}+\dfrac{1}{\exp(2\pi\omega/a)+1}\right], & s=\tfrac{1}{2}\ ,\\[2ex] \dfrac{\omega^3}{\pi^2}\left[1+\left[\dfrac{a}{\omega}\right]^2\right]\left[\dfrac{1}{2}+\dfrac{1}{\exp(2\pi\omega/a)-1}\right], & s=1\ .\end{cases} \tag{30}$$

(2) Circular motion $v \ll 1$.

$$\frac{de}{d\omega}=\begin{cases}\dfrac{\omega^3}{\pi^2}\left[\dfrac{1}{2}+\dfrac{1}{6}\dfrac{\omega_0}{\omega}v^2(1-W)^3\theta(1-W)+\cdots\right], & s=0\ ,\\[2ex] \dfrac{\omega^3}{2\pi^2}\left\{\left[1+\left[\dfrac{a}{2\omega}\right]^2\right]\left[\dfrac{-1}{2}+\dfrac{1}{12}\left[\dfrac{\omega_0}{\omega}\right]^2 v^2[(W-1)^4\theta(1-W)+4(W-\tfrac{1}{2})^3\theta(\tfrac{1}{2}-W)]+\cdots\right]\right\}, & s=\tfrac{1}{2}\ ,\\[2ex] \dfrac{\omega^3}{\pi^2}\left\{\left[1+\left[\dfrac{a}{\omega}\right]^2\right]\left[\dfrac{1}{2}\right]+\dfrac{1}{6}\left[\dfrac{\omega_0}{\omega}\right]^2 v^2[(W-1)^5\theta(1-W)+4(W-\tfrac{1}{2})^4\theta(\tfrac{1}{2}-W)]+\cdots\right\}, & s=1\ .\end{cases} \tag{31}$$

(3) Circular motion $v \to 1$ $(R \to 0)$.

$$\frac{de}{d\omega}=\begin{cases}\dfrac{\omega^3}{\pi^2}\dfrac{\exp(-2WR)}{4WR}, & s=0\ ,\\[2ex] \dfrac{\omega^3}{2\pi^2}\dfrac{\exp(-2WR)}{(2WR)^2}(\tfrac{1}{2}WR+\tfrac{3}{4}), & s=\tfrac{1}{2}\ ,\\[2ex] \dfrac{\omega^3}{\pi^2}\dfrac{\exp(-2WR)}{(2WR)^3}[\tfrac{1}{2}(WR)^2+\tfrac{15}{4}(WR)+\tfrac{9}{2}], & s=1\ .\end{cases} \tag{32}$$

The thermal nature of the energy spectrum holds only in the uniform acceleration. The appearance of harmonics of circular motion in the low-velocity limit is common to the fields of any spin, but the spinor and vector fields have extra harmonics of half-frequency. The extreme relativistic limit is somewhat similar in that they have the same exponential factor, but they have spin-dependent polynomial function of frequency.

One common feature in the uniform linear acceleration and the circular motion is the multiplicative factor, i.e., the density-of-states factor $(\omega^2+s^2a^2)$, s being the spin 0, $\frac{1}{2}$, or 1. This may suggest that this factor may be somewhat universal, independent of the details of motion.

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