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Zero-point field in a circular-motion frame

S. K. Kim
Department of Physics, Ehwa Women's University, Seoul 120, Korea

K. S. Soh
Department of Physics Education, Seoul National University, Seoul 151, Korea

J. H. Yee
Department of Physics, Yonsei University, Seoul 120, Korea
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The energy spectrum of zero-point fields of a massless scalar field observed by a detector in circular motion is studied by analyzing the Wightman function. It is shown to be quite different from the Planck spectrum which would have been expected from the result of a uniformly accelerated detector. In a nonrelativistic limit zero-point fields with frequencies only up to the first harmonics of the circular-motion frequency contribute dominantly. In an extremely relativistic case the energy spectrum is dominated by a particular pole in the complex proper-time plane.

I. INTRODUCTION

In the last decade a uniformly accelerated particle detector[1] has attracted much attention in connection with Hawking's black-hole radiation effects.[2] It is remarkable that a uniformly accelerated detector in Minkowski space-time would see a Planckian form of particle distributions in the Minkowski vacuum; i.e., the response function of a uniformly accelerated particle detector is identical to that of the same detector at rest in a thermal bath. Interpretations on the origin of the Planck spectrum may vary depending upon one's viewpoint. One may argue that the accelerated detector sees a thermal state,[1,3] or that there appears a distortion of zero-point fields without implying creation of particles.[4,5]

We follow the zero-point field viewpoint of Hacyan *et al.*[5] and extend their study to a more realistic problem: the response function of a particle detector in a uniform circular motion. Circular motion is much different from uniform linear acceleration in that it has no event horizon whose existence was considered to be closely related to the Planck spectrum.[1,3] The question we ask is whether there is still any distortion of zero-point fields in a motion without an event horizon such as circular motion. We will show that there is indeed a distortion of a zero-point field energy density but its spectrum is not thermal. This is in agreement with Sciama,[4] who maintains that the spectrum of vacuum fluctuations observed by a detector not moving along a geodesic is different from that of an inertial observer.

The spectrum in rotating coordinates has been extensively studied by Letaw and Pfautsch.[6,7] We confirm their results and extend their work by applying different methods. We study the problem using different parameters, v (the circular-motion speed) and ω_0 (the circular motion frequency), instead of their parameters (the acceleration or torsion and the speed). By performing the integration as a power series of v we show that harmonics of the circular-motion frequency appear, which should be expected from physical intuitions but cannot be clearly exhibited by a numerical analysis. For the extremely relativistic case, the power-series method is not effective because the integration has a singularity at $v=c$ when ω_0 is fixed. We study this case by examining the pole contributions in the complex proper-time plane, which provides another valuable piece of information on the structure of the spectrum.

We restricted our attention to massless scalar fields in four-dimensional space-time. In Sec. II a brief review of an inertial and uniformly accelerated detector is given for the purpose of comparison. The Wightman function for circular motion is introduced and the zero-point field energy spectrum is computed as a power series of the linear velocity of circular motion. It shows that the zero-point fields whose frequencies are lower than the circular-motion frequency contributes dominantly because higher-frequency fields average out when $v \ll c$. The series is, however, divergent as v approaches the speed of light, which makes it difficult to study the spectrum. This problem is dealt with in Sec. III by computing pole contributions in the complex proper-time contour integral. The divergent contribution at $v=c$ is concentrated only at a pole while all other poles give a finite amount at $v=c$, which renders it possible to evaluate the dominant contribution as v goes to c. In Sec. IV a summary is given along with a further example of motion which exhibits a nonthermal spectrum. From now on we will take $\hbar=c=k_B=1$.

II. NONRELATIVISTIC LIMIT

When a particle detector moves along a world line, its response in the vacuum is represented by the two-point Wightman function evaluated at two points $\chi^\mu=\chi^\mu(\tau+\frac{1}{2}\sigma)$ and $\chi'^\mu=\chi'^\mu(\tau-\frac{1}{2}\sigma)$ on the world line:

$$D^{\pm}(\tau+\tfrac{1}{2}\tau,\tau-\tfrac{1}{2}\sigma) = \langle 0|\phi(X^{\mu}(\tau\pm\tfrac{1}{2}\sigma))\phi(X^{\mu}(\tau\mp\tfrac{1}{2}\sigma))|0\rangle , \quad (1)$$

where $\phi(X)$ is a massless scalar field, τ is the proper time of the world line, and $|0\rangle$ is the vacuum in Minkowski space-time. Fourier transforms of the Wightman functions are defined as

$$\tilde{D}^{\pm}(\omega,\tau)=\int_{-\infty}^{\infty} d\sigma\, e^{i\omega\sigma}D^{\pm}(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma) , \quad (2)$$

where ω is the frequency of the zero-point fields. According to the viewpoint of Ref. 5, the "particle number density of the vacuum seen by the moving detector" is given by

$$f(\omega,\tau)=\frac{1}{(2\pi)^2\omega}[\tilde{D}^{+}(\omega,\tau)-\tilde{D}^{-}(\omega,\tau)] , \quad (3)$$

and the energy density per mode is

$$de=\frac{\omega^2}{\pi}[\tilde{D}^{+}(\omega,\tau)+\tilde{D}^{-}(\omega,\tau)]d\omega . \quad (4)$$

For the massless scalar field the Wightman functions are

$$D^{\pm}(X^{\mu},X'^{\mu})=\frac{-1}{4\pi^2}\frac{1}{(t-t'\mp i\epsilon)^2-(X-X')^2} . \quad (5)$$

Before we present the circular-motion case we review the results of an inertial and uniformly accelerating motion for the purpose of comparison. An inertial trajectory with velocity v is

$$t=\tau/(1-v^2)^{1/2},\quad X=X_0+v\tau/(1-v^2)^{1/2} , \quad (6)$$

the Wightman functions are

$$D^{\pm}(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma)=\frac{-1}{4\pi^2}\frac{1}{(\sigma\mp i\epsilon)^2} , \quad (7)$$

and their Fourier transforms are

$$\begin{aligned}&\tilde{D}^{+}(\omega,\tau)-\tilde{D}^{-}(\omega,\tau)=\frac{\omega}{2\pi},\quad \omega>0\\ &\tilde{D}^{-}(\omega,\tau)=0,\omega>0 .\end{aligned} \quad (8)$$

The world line of a uniformly accelerated detector is given by

$$t=\alpha^{-1}\sinh(\alpha\tau),\quad x=\alpha^{-1}\cosh(\alpha\tau) , \quad (9)$$

where α is the magnitude of the acceleration. The Wightman functions are

$$D^{\pm}(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma)=\frac{-\alpha^2}{16\pi^2}\frac{1}{\sinh^2[\tfrac{1}{2}\alpha(\sigma\mp i\epsilon)]} , \quad (10)$$

and, for positive ω,

$$\begin{aligned}&\tilde{D}^{+}(\omega,\tau)-\tilde{D}(\omega,\tau)=\frac{\omega}{2\pi},\\ &\tilde{D}^{-}(\omega,\tau)=\frac{\omega}{2\pi}\frac{1}{\exp(2\pi\omega/\alpha)-1} .\end{aligned} \quad (11)$$

Comparing these two cases we note that the particle number densities are identical, but the energy densities are quite different: a Planck term appears in the uniformly accelerated case. This term is exactly what one would observe at a thermal bath with temperature $T=\alpha/2\pi$. In many articles[1,3] the Planck spectrum is implicitly considered to be closely related to the existence of an event horizon in Rindler coordinates[8] of the uniformly accelerated detector. However, there may still exist a distortion of the zero-point energy density even when there is no event horizon, and its spectrum may not be thermal. In order to study this problem, we chose a uniform circular motion which is a relatively simple motion without an event horizon, and its predictions could possibly be tested by accelerator experiments. We also contrived another less realistic example which is discussed in the last section.

The trajectory of a uniform circular motion is given by

$$\begin{aligned}&t=\tau/(1-v^2)^{1/2} ,\\ &x=\rho\sin(\omega_0 t),\quad y=\rho\cos(\omega_0 t) ,\end{aligned} \quad (12)$$

where ρ is the radius of the circle, ω_0 is the angular frequency, and $v=\rho\omega_0$. The magnitude of acceleration in the detector frame is

$$\alpha=\frac{v\omega_0}{1-v^2} . \quad (13)$$

The Wightman functions are

$$D^{\pm}(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma)=\frac{-1}{4\pi^2}\frac{1}{(\gamma\sigma\mp i\epsilon)^2-4\rho^2\sin^2(\gamma\omega_0\sigma/2)} , \quad (14)$$

where $\gamma=1/(1-v^2)^{1/2}$, and their Fourier transforms are

$$\tilde{D}^{\pm}(\omega,\tau)=\frac{-1}{4\pi^2}\frac{\omega_0}{2\gamma}\int_{-\infty}^{\infty}ds\frac{\exp(2iWs)}{(s\mp i\epsilon)^2-v^2\sin^2(s)} , \quad (15)$$

where $s\equiv\gamma\omega_0\sigma/2$, $W\equiv\omega/\gamma\omega_0$.

It is straightforward to show that

$$\tilde{D}^{+}(\omega)-\tilde{D}^{-}(\omega)=\omega/2\pi , \quad (16)$$

which means that the particle number density is identical to the previous ones. The integrand of $\tilde{D}^{-}(\omega)$ can be expanded as a power series of v:

$$\tilde{D}^{-}(\omega)=-\frac{1}{4\pi^2}\frac{\omega_0}{2\gamma}\int_{-\infty}^{\infty}ds\, e^{2iWs}\sum_{n=0}^{\infty}\frac{v^{2n}\sin^{2n}(s)}{(s+i\epsilon)^{2n+2}} , \quad (17)$$

which can be integrated term by term. The result is

$$\tilde{D}^{-}(\omega)=\frac{\omega_0}{2\pi\gamma}\sum_{n=0}^{\infty}\frac{v^{2n}}{2n+1}\sum_{k=0}^{n}(-1)^k\frac{(n-k-W)^{2n+1}}{k!(2n-k)!}\times\Theta(n-k-W) , \quad (18)$$

where the Θ function is the usual step function. For explicitness we give the first few terms ($n=1,2$):

$$\tilde{D}^{-}=\frac{\omega_0}{4\pi\gamma}\left[\frac{v^2}{3}(1-W)^3\Theta(1-W)-\frac{v^4}{15}(1-W)^5\Theta(1-W)+\frac{v^4}{60}(2-W)^5\Theta(2-W)+\cdots\right] . \quad (19)$$

The arguments of the Θ functions indicate zero-point field modes whose frequencies are harmonics of the circular-motion frequency ($W=n$ means $\omega=n\omega_0\gamma$). For a very small velocity only the first term is important, in which case the contribution of the zero-point fields with frequencies $\omega>\omega_0$ is averaged out due to faster oscillations than the circular motion. This is why the Θ function appears.

It is instructive to compare the energy densities of the uniformly accelerated detector and the circular-motion case:

$$\left[\frac{de}{d\omega}\right]_a=\frac{\omega^3}{\pi^2}\left[\frac{1}{2}+\frac{1}{\exp(2\pi\omega/\alpha)-1}\right], \tag{20}$$

$$\left[\frac{de}{d\omega}\right]_c=\frac{\omega^3}{\pi^2}\left[\frac{1}{2}+\frac{\omega_0}{2\gamma\omega}\sum_{n=0}^{\infty}v^{2n}f_n(W)\right], \tag{21}$$

where $f_n(W)$ can be read from (18), and the subscripts a and c refer the uniformly accelerated and circular case, respectively. From the results of the uniformly accelerated detector, one might conjecture that the energy spectrum in other motions is also determined by the acceleration parameter. We can see that this does not hold by comparing (20) and (21). If it were true (20) and (21) should be identical if we substitute the parameter α in (20) by the circular-motion acceleration $\alpha_c=\gamma^2\omega_0 v$.

The function

$$\frac{1}{\exp[2\pi W(1-v^2)^{1/2}v]-1} \tag{22}$$

has an essential singularity at $v=0$ which forbids an expansion such as Eq. (21). This means that the spectrum of the circular motion is not determined by the acceleration alone. Indeed, the spectrum also depends on the speed v, and the numerical results are explicitly displayed in Ref. 6 for the range of v from 0.05 to 0.95, which clearly shows that the spectrum is quite different from the thermal distribution.

III. EXTREMELY RELATIVISTIC CASE

As the speed v approaches the speed of light, the integral (15) develops a singularity at $v=1$ due to relativistic kinematics, and the series (18) is indeed divergent at $v=1$. Since in many realistic situations v is very close to 1, it is necessary to evaluate the integral in a different way.

Consider the integral

$$\tilde{D}^{\pm}(\omega)=\frac{-1}{4\pi^2}\frac{\omega_0}{2\gamma}\int_{-\infty}^{\infty}ds\frac{\exp(2iWs)}{(s\mp i\epsilon)^2-v^2\sin^2(s)},\quad W=\frac{\omega}{\gamma\omega_0}, \tag{23}$$

as a contour integral in the complex s plane, and study the poles of the integrand. When v is near 1 the poles can be grouped into three classes. The pole at $s=\pm i\epsilon$ is a double pole which is a sort of universal pole in the sense that it is common in all three motions we have considered and it gives rise to the result $\tilde{D}^{+}(\omega)-\tilde{D}^{-}(\omega)=\omega/2\pi$. There is another pole near the origin at $s=iR$, where R is the positive real root of the equation

$$R=v\sinh R\ . \tag{24}$$

There is one and only one solution when $v<1$, and it approaches zero as $v\to 1$. The contribution from this pole is very large and is to be identified as the divergent part of (23) at $v\simeq 1$. All other poles are located outside the unit circle, the nearest one to the origin at $s=\pm 2.3\pm 1.7i$ when $v\simeq 1$. These poles contribute only finite amounts even at $v=1$.

The integral of (23) can be written as the sum of two contour integrals:

$$\int_{-\infty}^{\infty}ds=\int_{c_1}ds+\oint_{c_2}ds\ ,$$

where the contours c_1 and c_2 are shown in the Fig. 1. The semicircle in the figure is of unit radius with its center at the origin. It is straightforward to show that the integration along the contour c_1 remains finite as $v\to 1$. Since the ϵ-pole contribution is constant and independent of v, the divergent part at $v=1$ is only due to the pole at $s=iR$, whose contribution can be evaluated by the residue theorem.

When v is extremely close to one the integral (23) is dominated by the pole at $s=iR$:

$$\tilde{D}(\omega)\simeq\frac{\omega_0}{4\pi\gamma}\frac{\exp(-2WR)}{2R(v\cosh R-1)}\ . \tag{25}$$

Notice that $\gamma^2(v\cosh R-1)$ approaches 1 as $v\to 1$. The energy density is approximately

$$\frac{de}{d\omega}\simeq\frac{\omega^3}{\pi^2}\left[\frac{1}{2}+\frac{1}{4WR}\exp(-2WR)\right]. \tag{26}$$

The frequency range of appreciable energy density change is $\omega\lesssim\omega_0\gamma^2/2\sqrt{3}$, which is much smaller than a synchrotron radiation frequency range ($\omega\lesssim 3\omega_0\gamma^3$).

Let us compare this result with a Planck spectrum. From (20), by identifying $\alpha=\gamma^2 v\omega_0=\sqrt{3}\gamma\omega_0/R$, we obtain

$$\left[\frac{de}{d\omega}\right]_a=\frac{\omega^3}{\pi^2}\left[\frac{1}{2}+\frac{1}{\exp(2\pi WR/\sqrt{3})-1}\right]$$
$$\simeq\frac{\omega^3}{\pi^2}\frac{\exp(-2\pi WR/\sqrt{3})}{2\pi WR/\sqrt{3}}\ ,\quad R\to 0 \tag{27}$$

for a suitable range of ω. On the other hand, from (26) we get, as $R\to 0$,

$$\left[\frac{de}{d\omega}\right]_c\simeq\frac{\omega^3}{\pi^2}\frac{\exp(-2WR)}{4WR},\quad R\to 0\ . \tag{28}$$

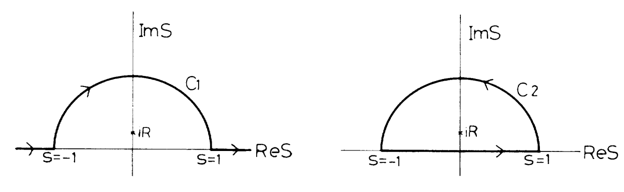

FIG. 1. The integral of (23) can be written as $\int_{-\infty}^{\infty}ds=\int_{c_1}ds+\oint_{c_2}ds$, when $v\sim 1$.

From (27) and (28) we see that, in the extremely relativistic case, the energy spectrum looks similar to the Planck form but with a different numerical factor; i.e., the circular-motion spectrum falls more slowly by a factor $\pi/\sqrt{3}$ as ω increases. This result agrees with Letaw's extreme realistic case.[7]

An experimental test of (26) by polarization measurements was proposed by Bell and Leinaas.[9] Their formula is in agreement with (26) for fixed $\alpha=\gamma^2 v\omega_0$ in the limit $\gamma\to\infty$, and can be regarded as a special case of ours whose range of ω is limited to the region $\omega\gtrsim\gamma v\omega_0$. In accelerator situations the change of the energy density is very small. For an illustration let us take a circular motion with a radius $\rho=1$ km and $\gamma=E/m=10^3$. Then the change is

$$\Delta e=\int_0^\infty d\omega\frac{\omega^3}{2\pi^2}\frac{\exp(-2WR)}{2WR}=\frac{\omega_0^4\gamma^8}{24\pi^2R^4}\simeq 1\times 10^{-4}\ \mathrm{eV/cm}^3\ . \tag{29}$$

But this value increases as γ^8 for fixed radius; hence, for $\gamma=10^8$, Δe becomes extremely large,

$$\Delta e\simeq 1.0\times 10^3\ \mathrm{GeV}/(\mathrm{\AA})^3\ ,$$

which is about 10 times denser than ordinary solids. This implies that acceleration of particles in the circular accelerators is limited up to $\gamma\simeq 10^7-10^8$ for a fixed radius $\rho=1$ km even when energy supply conditions are met.

We would like to point out that singularities of $\tilde{D}^{\pm}(\omega)$ are branch cuts (Θ functions) and those of $D^{\pm}(s)$ are poles determined by the equation

$$j_0(s)=(\sin s)/s=1/v\ .$$

The branch points reflect the harmonics of the circular-motion frequency, but we could not find a simple physical meaning of the poles. It is also worth noticing that all of the poles in the uniform acceleration case are located at integer points on the imaginary axis, while none of those in circular motion are on the imaginary axis except the ϵ poles and the pole at $s=iR$. As has been shown in this section, the difference in the location of poles leads to the distinct spectra.

IV. DISCUSSION

By comparing three kinds of motion (inertial, uniformly accelerating, and circular motion) we found that the mode number densities in the vacuum are all the same, but the energy densities vary depending upon the details of motion. Circular motion does not have the Planck spectrum, which one might have naively expected from the result of uniformly accelerating cases. This signifies that the similarity between a thermal spectrum and the zero-point field spectrum may be only coincidental.

It is natural to expect that other kinds of motion will also exhibit nonthermal zero-point energy, but it is not so easy to find a case analytically integrable to obtain $\tilde{D}^{\pm}(\omega)$ in a closed form. We were able to contrive an example which is not realistic but illustrative:

$$\alpha t=\tan(\alpha\tau)\ ,$$
$$\alpha x=\frac{[1+\cos^2(\alpha\tau)]^{1/2}}{\cos(\alpha\tau)}-\ln\{\cos(\alpha\tau)+[1+\cos^2(\alpha\tau)]^{1/2}\}\ , \tag{30}$$

where $-\pi/2\le\alpha\tau\le\pi/2$. This motion has a turning point at $t=\tau=0$, and the acceleration is not uniform. The proper time is finite but coordinate time is infinite, which may remind one of the motion of a particle which is shot up from the outside of a black-hole event horizon to the turning point and then falls back to the black hole. At the turning point ($\tau=0$) the Wightman function is

$$D^{\pm}(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma)\,|_{\tau=0}=\frac{-\alpha^2}{4\pi^2}\frac{\cos^4(\alpha\sigma/2)}{\sin^2(\alpha\sigma\mp i\epsilon)}\ , \tag{31}$$

and their Fourier transforms satisfy

$$\tilde{D}^{+}(\omega)-\tilde{D}^{-}(\omega)=\frac{\omega}{2\pi}\ ,$$
$$\tilde{D}^{+}(\omega)+\tilde{D}^{-}(\omega)=\frac{\omega}{2\pi}+\frac{\alpha}{2\pi^2}\sin\frac{\pi\omega}{\alpha}\times\left[1+\frac{\alpha}{2\omega}-\beta(\omega/\alpha)\frac{2\omega}{\alpha}\right]\ , \tag{32}$$

where

$$\beta(x)=\int_0^\infty\frac{e^{-xt}}{1+e^{-t}}dt\ .$$

Evidently, the spectrum is quite different from the uniform acceleration case. Especially for large ω, the difference is prominent, i.e.,

$$\tilde{D}^{+}(\omega)+\tilde{D}^{-}(\omega)\to\frac{\omega}{2\pi}+\frac{\alpha^2}{2\pi^2}\frac{1}{\omega}\sin(\pi\omega/\alpha)+O(1/\omega^2)\ , \tag{33}$$

which decreases only as $1/\omega$ while the Planck spectrum diminishes exponentially.

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