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Equivalence between the Weyl, Coulomb, and unitary gauges in the functional Schrödinger picture

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Equivalence between the Weyl (temporal), Coulomb, and unitary gauges of scalar electrodynamics is shown in the context of the functional Schrödinger picture.

I. INTRODUCTION

The functional Schrödinger-picture formulation of quantum field theories[1] has recently attracted increasing attention. It has been found especially convenient for nonperturbative variational studies of quantum field theories in flat[1] and curved space-time,[2] and of the dynamical-symmetry-breaking phenomena of fermion field theories.[3] The formalism has also been found in studying gauge field theories in the Weyl (temporal, or timelike axial) and Coulomb gauges.[4]

The noncovariant gauge formulations, especially the Weyl-gauge formulation[5] of gauge field theories, have long attracted considerable attention because of the apparent absence of ghost fields and because dynamical and gauge degrees of freedom seem easily separable. Upon closer inspection of the quantization procedure of the Weyl-gauge formulation, however, it has been found that there arise several serious difficulties. The difficulties encountered in the Weyl-gauge quantization may be traced back to the gauge-fixing condition, $A^0(x)=0$, which does not fix the gauge completely.[5]

The purpose of this paper is to understand the relations between the Weyl- and other gauge formulations and to see how the gauge degrees of freedom are eliminated. This can give some further insight into the nature of the difficulties encountered in the noncovariant gauge formulations, and may give a clue to the correct way for computing the physical quantities in such gauges. The functional Schrödinger-picture formulation is especially suited for understanding the relations between the quantization procedures with different gauge-fixing conditions. For simplicity we will consider the Abelian Higgs model in the functional Schrödinger picture, and study the relations between the Weyl-, Coulomb-, and unitary-gauge formulations of the theory. We find that the well-known fact of the gauge equivalence can be shown almost trivially in the functional Schrödinger-picture formulation.

In the next section we consider the relation between the Weyl- and Coulomb-gauge formulations and show the equivalence. In Sec. III we show the equivalence between the Weyl and unitary gauges, and in the last section we discuss some related problems.

II. EQUIVALENCE BETWEEN THE WEYL AND COULOMB GAUGES

The Lagrangian density of the scalar electrodynamics of two real scalar fields ϕ_1 and ϕ_2 is

$$L=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}(\partial_\mu\phi_1-eA_\mu\phi_2)^2 +\tfrac{1}{2}(\partial_\mu\phi_2+eA_\mu\phi_1)^2-V(\phi_a\phi_a)\ , \tag{2.1}$$

where the potential V is a function of $(\phi_1^2+\phi_2^2)$. The Hamiltonian density is

$$\begin{aligned}H=&\tfrac{1}{2}(\pi_k\pi_k+B_kB_k)\\&+\tfrac{1}{2}(\partial_k\phi_a+eA_k\epsilon_{ab}\phi_b)(\partial_k\phi_a+eA_k\epsilon_{ab}\phi_b)\\&+\tfrac{1}{2}P_aP_a-A^0(\partial_k\pi_k+e\phi_a\epsilon_{ab}P_b)+V(\phi_a\phi_a)\ ,\end{aligned} \tag{2.2}$$

where $\pi_k=\delta L/\delta\dot{A}_k=\partial_k A^0+\partial_t A_k$, $P_a=\partial L/\partial\dot{\phi}_a=\partial_t\phi_a-eA_0\epsilon_{ab}\phi_b$, and ϵ_{ab} is the usual antisymmetric real matrix. We use i,j,k to denote spatial indices 1,2,3, and a,b,c to denote the scalar field components $\hat{1}$ and $\hat{2}$.

In the Weyl gauge we simply put

$$A^0=0\ , \tag{2.3}$$

so that the Weyl Hamiltonian density is

$$H_w\equiv H|_{A^0=0}\ . \tag{2.4}$$

We need the Gauss-law constraint

$$\nabla\cdot\mathbf{E}+j^0=0\ ,$$

which takes care of the spatial gauge degrees of freedom that are left unfixed by the gauge condition (2.3). In the functional Schrödinger picture the Gauss-law constraint is implemented as

$$\left[\partial_k\left[i\frac{\delta}{\delta A_k}\right]+\frac{e}{i}\left[\phi_1\frac{\delta}{\delta\phi_2}-\phi_2\frac{\delta}{\delta\phi_1}\right]\right]\psi_w(A_j,\phi_a)=0\ , \tag{2.5}$$

where $\psi_w(A_j,\phi_a)$ is the wave functional for the Weyl gauge. By introducing polar coordinates for the scalar fields as

$$\begin{aligned}\phi_1&=\rho\cos\theta\ ,\\ \phi_2&=\rho\sin\theta\ ,\end{aligned} \tag{2.6}$$

we can rewrite the Gauss-law constraint (2.5) as

$$\left[\partial_k\frac{\delta}{\delta A_k}-e\frac{\delta}{\delta\theta}\right]\psi_w(A_j,\rho,\theta)=0\ . \tag{2.7}$$

Splitting the vector potential into the transverse part $\mathbf{A}^T$ and the longitudinal part $\mathbf{A}^L$ as

$$A_k=A_k^T+A_k^L\equiv\left[A_k-\partial_k\frac{1}{\nabla^2}\nabla\cdot\mathbf{A}\right]+\partial_k\left[\frac{1}{\nabla^2}\nabla\cdot\mathbf{A}\right]\ , \tag{2.8}$$

we notice that the Gauss-law constraint (2.7) requires the wave functional to be of the form

$$\psi_w\left[\mathbf{A}^T,\rho,\ \mathbf{A}^L-\frac{1}{e}\nabla\theta\right]\ . \tag{2.9}$$

We can see more clearly the Gauss-law constraint by introducing variable transformations

$$s=\nabla\cdot\mathbf{A}^L-\nabla^2(\theta/e)\ , \tag{2.10}$$

$$t=a\nabla\cdot\mathbf{A}^L-b\nabla^2(\theta/e)\ , \tag{2.11}$$

where (a,b) are free real parameters such that $a-b\neq0$. With these variables we see that Eq. (2.7) can be written as

$$\frac{\partial}{\partial t}\psi_w(\mathbf{A}^T,\rho,s,t)=0 \tag{2.12}$$

and we are free to take any fixed value of t, say $t=0$, in order to remove either $\nabla\cdot\mathbf{A}^L$ or $\nabla^2\theta$, which will be used to derive the wave functionals in other gauges.

Turning to the Coulomb gauge condition[4]

$$\nabla\cdot\mathbf{A}=0\ , \tag{2.13}$$

we have the Hamiltonian density

$$\begin{aligned}H_c=&\tfrac{1}{2}(\pi_k^T\pi_k^T+B_kB_k)\\&+\tfrac{1}{2}(\partial_k\phi_a+eA_k^T\epsilon_{ab}\phi_b)(\partial_k\phi_a+eA_k^T\epsilon_{ab}\phi_b)\\&+\frac{1}{8\pi}\int d^3y\,j^0(x)\frac{1}{|\mathbf{x}-\mathbf{y}|}j^0(y)+\tfrac{1}{2}P_aP_a+V(\phi_a\phi_a)\ ,\end{aligned} \tag{2.14}$$

and $\mathbf{A}^L$-independent wave functional $\psi_c(\mathbf{A}^T,\rho,\theta)$. In order to show the equivalence between the Weyl and the Coulomb gauges we will derive H_c and ψ_c from H_w and ψ_w by removing $\mathbf{A}^L$ from the Weyl-gauge case, using the above-mentioned condition $t=0$, or

$$\nabla\cdot\mathbf{A}^L=-\frac{b}{a}\nabla^2(\theta/e)\ . \tag{2.15}$$

First we define the Coulomb wave functional as

$$\psi_c(\mathbf{A}^T,\rho,\theta)\equiv\psi_w\left[\mathbf{A}^T,\rho,\ \mathbf{A}^L-\frac{\nabla}{e}\theta\right]\Bigg|_{\mathbf{A}^L=(b/a)\nabla(\theta/e)}\ . \tag{2.16}$$

For the Hamiltonian density the Coulomb term in the H_c comes from the longitudinal contribution of the electric field energy in the H_w of (2.4) as

$$\tfrac{1}{2}\int d^3x(\pi_k^T\pi_k^T+\pi_k^L\pi_k^L)=\tfrac{1}{2}\int d^3x\,\pi_k^T\pi_k^T+\frac{1}{8\pi}\int d^3x\,d^3y\,j^0(\mathbf{x})\frac{1}{|\mathbf{x}-\mathbf{y}|}j^0(\mathbf{y})\ , \tag{2.17}$$

where the Gauss-law constraint (2.5) is used. We notice that the left-hand side of Eq. (2.17) applies only to ψ_w, but the right-hand side both to ψ_w and ψ_c. There is one more term to be identified: The covariant derivative term in H_w of (2.4) has $\mathbf{A}^L$ components in it as

$$\begin{aligned}\tfrac{1}{2}(\partial_k\phi_a+eA_k\epsilon_{ab}\phi_b)(\partial_k\phi_a+eA_k\epsilon_{ab}\phi_b)&=\tfrac{1}{2}\nabla\rho\cdot\nabla\rho+\tfrac{1}{2}\rho^2(\nabla\theta\cdot\nabla\theta+e^2\mathbf{A}^T\cdot\mathbf{A}^T+2e\,\mathbf{A}^T\cdot\nabla\theta\\&\quad+2e^2\mathbf{A}^T\cdot\mathbf{A}^L+e^2\mathbf{A}^L\cdot\mathbf{A}^L\\&\quad+2e\,\mathbf{A}^L\cdot\nabla\theta)\ ,\end{aligned} \tag{2.18}$$

where the similar term in H_c lacks $\mathbf{A}^L$ components. There are two possible ways to get the Coulomb-gauge Hamiltonian: One is to set simply $b/a=0$ which reproduces H_c exactly; another one is to set $b/a=2$ which gives the charge-conjugated Hamiltonian $(e\to-e)$ of H_c.

We have constructed the Coulomb-gauge wave functional ψ_c and the Hamiltonian density H_c from the Weyl ones as

$$\psi_c(\mathbf{A}^T,\rho,\theta)\equiv\psi_w[\mathbf{A}^T,\rho,\ \mathbf{A}^L-\nabla(\theta/e)]|_{\mathbf{A}^L=0}\ , \tag{2.19a}$$

$$H_c\equiv H_w|_{\nabla\cdot\pi=j^0,\ \mathbf{A}^L=0}\ . \tag{2.19b}$$

We finally note that the longitudinal component of the electric field, $\mathbf{E}^L$, in the Weyl gauge is determined by the longitudinal component of the vector potential, $\mathbf{A}^L$, by the Gauss-law constraint (2.5), while $\mathbf{E}^L$ in the Coulomb gauge is determined by A^0. Thus the Gauss-law constraint (2.5) becomes

$$(-\nabla^2A^0+j^0)\psi_c=0 \tag{2.20}$$

in the Coulomb gauge, which determines A^0 in terms of j^0, where A^0 is defined by the field equation

$$\nabla\cdot\mathbf{E}=-\nabla^2 A^0 . \tag{2.21}$$

We have thus established the equivalence between the Weyl- and Coulomb-gauge formulations. Using the above relations we can obtain the Coulomb-gauge expression of any quantity from the Weyl-gauge expression, or vice versa. For example, we can write the energy expectation value as

$$\begin{aligned} E&=\int d^3x\int (D\phi_a)(DA_k)\delta(\nabla\cdot\mathbf{A})\psi_w^*\left[\mathbf{A}^T,\rho,\,\mathbf{A}^L-\frac{1}{e}\nabla\theta\right]H_w\psi_w\left[\mathbf{A}^T,\rho,\,\mathbf{A}^L-\frac{1}{e}\nabla\theta\right]\\ &=\int d^3x\int (D\phi_a)(DA_k^T)\psi_c^*(\mathbf{A}^T,\rho,\theta)H_c\psi_c(\mathbf{A}^T,\rho,\theta) . \end{aligned}\tag{2.22}$$

III. EQUIVALENCE BETWEEN THE WEYL AND UNITARY GAUGES

By a unitary gauge we mean a kind of coordinate transformation from A_μ,ϕ_a to a_μ,ρ such that only gauge-invariant variables appear in the Hamiltonian and the wave functionals. The desired transformation is

$$\begin{aligned} A_\mu&=a_\mu-\frac{1}{e}\partial_\mu\theta ,\\ \phi_1&=\rho\cos\theta ,\\ \phi_2&=\rho\sin\theta , \end{aligned}\tag{3.1}$$

with which the Lagrangian density becomes

$$\begin{aligned} L=&-\tfrac{1}{4}(\partial_\mu a_\nu-\partial_\nu a_\mu)(\partial^\mu a^\nu-\partial^\nu a^\mu)\\ &+\tfrac{1}{2}e^2\rho^2 a_\mu a^\mu+\tfrac{1}{2}(\partial_\mu\rho)(\partial^\mu\rho)-V(\rho) , \end{aligned}\tag{3.2}$$

where the θ variable disappears completely.[6] The wave functional ψ_u is a functional of a_k and ρ only, and the Hamiltonian density becomes

$$\begin{aligned} H_u=&\tfrac{1}{2}[\boldsymbol{\pi}\cdot\boldsymbol{\pi}+(\nabla\times\mathbf{a})\cdot(\nabla\times\mathbf{a})]+\tfrac{1}{2}e^2\rho^2\mathbf{a}\cdot\mathbf{a}+\frac{(\nabla\cdot\boldsymbol{\pi})^2}{2e^2\rho^2}\\ &+\tfrac{1}{2}(\nabla\rho)\cdot(\nabla\rho)+\frac{1}{2}\left[\frac{1}{i}\frac{\partial}{\partial\rho}\right]^2+V(\rho) , \end{aligned}\tag{3.3}$$

where we have eliminated the Lagrange multiplier field a_0 by minimization procedures.

We can rewrite the Weyl Hamiltonian H_w, Eq. (2.4), in terms of ρ and θ as

$$\begin{aligned} H_w=&\tfrac{1}{2}[\boldsymbol{\pi}\cdot\boldsymbol{\pi}+(\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{A})]\\ &+\tfrac{1}{2}e^2\rho^2\left[\mathbf{A}+\frac{1}{e}\nabla\theta\right]\cdot\left[\mathbf{A}+\frac{1}{e}\nabla\theta\right]\\ &+\frac{1}{2}\left[\left[\frac{1}{i}\frac{\partial}{\partial\rho}\right]^2+\frac{1}{\rho^2}\left[\frac{1}{i}\frac{\partial}{\partial\theta}\right]\right]+\tfrac{1}{2}\nabla\rho\cdot\nabla\rho+V(\rho) . \end{aligned}\tag{3.4}$$

By defining the unitary-gauge wave functional as

$$\psi_u(\mathbf{a},\rho)\equiv\psi_w\left[\mathbf{A}^T,\rho,\,\mathbf{A}^L-\frac{1}{e}\nabla\theta\right]\Bigg|_{\substack{\theta=0\\ \mathbf{A}=\mathbf{a}}} , \tag{3.5}$$

we can obtain the unitary Hamiltonian H_u from the H_w using the following relations:

$$-\frac{1}{2}\left[\frac{\delta}{\delta A_k}\frac{\delta}{\delta A_k}\right]\psi_w(\mathbf{A},\rho,\theta) =-\frac{1}{2}\left[\frac{\delta}{\delta a_k}\frac{\delta}{\delta a_k}\right]\psi_u(\mathbf{a},\rho) , \tag{3.6a}$$

$$\tfrac{1}{2}\rho^2\left[\mathbf{A}+\frac{1}{e}\nabla\theta\right]\cdot\left[\mathbf{A}+\frac{1}{e}\nabla\theta\right]=\tfrac{1}{2}\rho^2\mathbf{a}\cdot\mathbf{a} , \tag{3.6b}$$

$$\int d^3x\,\frac{1}{2}\frac{1}{\rho^2}\left[\frac{\partial}{\partial\theta}\right]^2\psi_w(\mathbf{A},\rho,\theta) =\int d^3x\,\frac{1}{2}\frac{1}{e^2\rho^2}\left[\partial_k\frac{\delta}{\delta a_k}\right]^2\psi_u(\mathbf{a},\rho) , \tag{3.6c}$$

where the last equality comes from the Gauss-law constraint. The energy expectation value is

$$\begin{aligned} E&=\int d^3x\int D\phi_a D\,\mathbf{A}\delta(\theta)\psi_w^* H_w\psi_w\left[\mathbf{A}^T,\rho,\,\mathbf{A}^L-\frac{1}{e}\nabla\theta\right]\\ &=\int d^3x\int(\rho D\rho)D\mathbf{a}\psi_u^* H_u\psi_u(\mathbf{a},\rho) . \end{aligned}\tag{3.7}$$

We can also obtain ψ_w and H_w from the ψ_u and H_u by the simple replacement

$$\mathbf{a}^L\rightarrow\mathbf{a}^L-\frac{1}{e}\nabla\theta , \tag{3.8}$$

thereby showing the equivalence between the Weyl- and unitary-gauge formulations of the scalar QED.

IV. DISCUSSIONS

The three gauges, Weyl, Coulomb, and unitary gauges, are represented by linear combinations of the two degrees of freedom $\mathbf{A}^L$ and θ. The wave functionals are, respectively,

$$\psi_W=\psi\left[\mathbf{A}^T,\rho,\,\mathbf{A}^L-\frac{1}{e}\nabla\theta\right] , \tag{4.1}$$

$$\psi_C=\psi(\mathbf{A}^T,\rho,\theta) , \tag{4.2}$$

$$\psi_u=\psi(\mathbf{A}^T,\rho,\mathbf{A}^L) , \tag{4.3}$$

where $\mathbf{A}^T$, and ρ are gauge-invariant variables. If we write ψ in general as

$$\psi\left[\mathbf{A}^T,\rho,a\,\mathbf{A}^L+b\frac{1}{e}\nabla\theta\right],$$

then the three gauges are represented as points in the (a,b) parameter space, namely, $(1,1)=$Weyl, $(0,1)=$Coulomb, and $(1,0)=$unitary. It would be possible to define a one-parameter family of gauge (a,b) (modulo normalization) as a generalization of the known gauge choices.

The equivalence between these gauges is not only interesting pedagogically but also could have practical significance. For example, in the Gaussian approximation of the ground-state wave functional and the energy expectation value the three gauges are quite distinct in the explicit evaluations: In the Weyl gauge[7] it is difficult to construct a nontrivial Gaussian wave functional satisfying the spatial gauge condition (2.5). In the unitary gauge the functional integral in the ρ variable is not a simple Gaussian, and therefore troublesome. In the Coulomb gauge the Coulomb interaction term in the Hamiltonian (2.11) makes computations very difficult.[4] This suggests that a change of gauge might be useful for computational purposes at a certain stage of problem solving.

We give an explicit example of the wave functional in order to show how it transforms with the gauge choices. Let ψ_c be a Gaussian wave functional in the Coulomb gauge[4]

$$\psi_c(\phi,A)=Ne^{(\Phi-v)G(\Phi-v)/4}, \tag{4.4}$$

where $\Phi_\alpha=(\phi_a,A_k^T)$, $v=(\hat{\phi}_a,0,0,0)$, G is a 5×5 matrix function, and N is a normalization factor. In the exponent of Eq. (4.4) the integration convention is used for the three-dimensional spatial coordinates. $\mathbf{A}$ has only transverse components and v is chosen to be a fixed function for possible symmetry breaking.

In order to obtain the Weyl-gauge wave functional we use polar coordinates (2.6) for ϕ_a, and take the variable transformation as

$$\phi\rightarrow\theta+e\frac{1}{\nabla^2}\nabla\cdot\mathbf{A}^L\,. \tag{4.5}$$

Thus the Weyl-gauge wave functional is obtained if we substitute the variable

$$\Phi_\alpha\rightarrow\left[\rho\cos\left[\theta+e\frac{1}{\nabla^2}\nabla\cdot\mathbf{A}^L\right],\rho\sin\left[\theta+e\frac{1}{\nabla^2}\nabla\cdot\mathbf{A}^L\right],A_k^T\right] \tag{4.6}$$

where the longitudinal component $\mathbf{A}^L$ appears only through the polar variable of ϕ_a.

The unitary-gauge wave functional is obtained simply by a replacement

$$\theta+e\frac{1}{\nabla^2}\nabla\cdot\mathbf{A}^L\rightarrow e\frac{1}{\nabla^2}\nabla\cdot\mathbf{a}^L$$

so that the field variable Φ_α becomes

$$\Phi_\alpha\rightarrow\left[\rho\cos e\frac{1}{\nabla^2}\nabla\cdot\mathbf{a}^L,\rho\sin e\frac{1}{\nabla^2}\nabla\cdot\mathbf{a}^L,\mathbf{a}_k^T\right],$$

where $\mathbf{a}^T=\mathbf{A}^T$ in this case.

We finally note that, in an attempt to obtain the consistent Feynman rules in the Weyl gauge, some authors use the relaxed Gauss-law constraint[5]

$$\langle\psi|\nabla\cdot\mathbf{E}-j^0|\psi\rangle=0 \tag{4.7}$$

rather than Eq. (2.5). As Kerman and Vautherin[7] have noted, the Gaussian wave functional (4.4) with $\Phi=(\phi_a,A_k)$ satisfies the relaxed condition (4.7), but it is not the correct Weyl-gauge vacuum state. Our analysis implies that the correct Weyl-gauge states must satisfy the Gauss-law constraint (2.5), rather than the relaxed one (4.7).

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