Gaussian approximation of the (2+1)-dimensional Gross-Neveu model

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The (2+1)-dimensional Gross-Neveu model is analyzed by the Gaussian approximation in the functional Schrödinger picture. It is shown that the Gaussian effective potential implies the existence of two phases, and one of them is inconsistent. In the phase where the bare coupling constant approaches positive infinitesimal, the effective potential shows the existence of dynamical symmetry breaking when the renormalized coupling constant is negative.

I. INTRODUCTION

The Schrödinger-picture Gaussian variational method is quite useful for the study of quantum structures of field theories. This method has been proved to be well suited for the study of boson field theories.[1] On the other hand, the functional variational method has not yet been well established for the study of fermion field theories. In the case of fermion field theories, the form of the wave functional is strongly dependent on the way the fermion field operators ψ and $\psi^\dagger$ are realized. As far as we know, there are two realization prescriptions for ψ and $\psi^\dagger$ so that the trial wave function may take the Gaussian form. One of these prescriptions has been suggested by Floreanini and Jackiw[2] and the other by Duncan, Meyer-Ortmanns, and Roskies.[3] In the Floreanini-Jackiw prescription the fermion field operators are realized as

$$\begin{aligned}\psi(x)&=\frac{1}{\sqrt{2}}\left[u(x)+\frac{\delta}{\delta u^\dagger(x)}\right],\\ \psi^\dagger(x)&=\frac{1}{\sqrt{2}}\left[u^\dagger(x)+\frac{\delta}{\delta u(x)}\right],\end{aligned}\tag{1.1}$$

where u and $u^\dagger$ are anticommuting Grassman variables and ψ and $\psi^\dagger$ satisfy the equal-time anticommutation relation $\{\psi_i(x,t),\psi_j^\dagger(x',t)\}=\delta_{ij}\delta(x-x')$. For the variational approximation, one can take the trial wave functional in the Gaussian form

$$|\Psi\rangle\to|G\rangle=\frac{1}{(\det G)^{1/4}}\exp\left[\int_{x,y}u^\dagger(x)G(x,y)y(y)\right]\tag{1.2a}$$

and its dual

$$\langle\Psi|\to\langle G|=\frac{1}{(\det G)^{1/4}}\exp\left[\int_{x,y}u^\dagger(x)\overline{G}(x,y)u(y)\right],\tag{1.2b}$$

where $\overline{G}=(G^\dagger)^{-1}$.

It is quite recent that the variational method based on the functional representation (1.1) has been applied for the study of fermion field theories. In our previous work,[4] we used the prescription (1.1) for the variational calculation of the effective potential of the Gross-Neveu (GN) model. Our result agrees with other Gaussian approximations[5] and it reproduces the well-known *GN* result in the large-N limit.[6]

The main aim of this work is to test further the Schrödinger-picture Gaussian variational method for the study of fermion field theory. For this purpose, we will calculate the effective potential of the Gross-Neveu (GN) model in 2+1 dimensions, using the Gaussian variational method. The (2+1)-dimensional GN model has been already studied and shown to be renormalizable in the $1/N$ expansion by several authors.[7] The existence of the two phases of the quantum-field-theoretic systems, implied by the Gaussian effective potential, has been controversial recently,[8] and our calculations can provide some further insight into this issue.

In Sec. II we introduce our notation and explain the overall features for obtaining the Gaussian effective potential by the functional variational method. In Sec. III we perform the renormalization of the effective potential, and in the last section we summarize our results and discuss their implications.

II. THE GAUSSIAN EFFECTIVE POTENTIAL OF THE (2+1)-DIMENSIONAL GN MODEL

The GN model is defined by the Lagrangian density

$$\mathcal{L}=\bar{\psi}^a(i\partial\!\!\!/)\psi^a+\tfrac{1}{2}g^2(\bar{\psi}^a\psi^a)^2,\quad a=1,\ldots,N\,.\tag{2.1}$$

In 2+1 dimensions, the γ algebra of the theory is defined by

$$\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu},\quad g_{\mu\nu}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix},\quad \mu,\nu=0,1,2\ . \tag{2.2}$$

The γ matrices can be represented by the 2×2 Pauli matrices.

By using the expressions (1.2a) and (1.2b) for the trial wave functional and its dual, we can obtain the expectation value of the Hamiltonian of the system in the form

$$\langle G|H|G\rangle\equiv\langle H\rangle=\tfrac{1}{2}\int d^2x\, d^2y\{\mathrm{tr}[-i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}_{(x,y)}\Omega(\mathbf{y},\mathbf{x})]\}$$
$$-\frac{g^2}{8}\int d^2x\{\mathrm{tr}[\gamma^0\Omega(\mathbf{x},\mathbf{x})]\mathrm{tr}[\gamma^0\Omega(\mathbf{x},\mathbf{x})]+\mathrm{tr}[2\Omega(\mathbf{x},\mathbf{x})\delta(\mathbf{x},\mathbf{x})-\gamma^0\Omega(\mathbf{x},\mathbf{x})\gamma^0\Omega(\mathbf{x},\mathbf{x})]\}\ , \tag{2.3}$$

where the boldfaced vector quantities stand for the two-dimensional Euclidean vectors, e.g.,

$$\mathbf{x}=(x^1,x^2)\ .$$

The symbol $\boldsymbol{\nabla}_{(x,y)}$ operates on the arbitrary function $f(\mathbf{x},\mathbf{y})$ as

$$\int d^2y\ \boldsymbol{\nabla}_{(x,y)}f(y,x)=\int d^2y\ \delta(\mathbf{x}-\mathbf{y})[\boldsymbol{\nabla}_y f(y,x)]\ .$$

The matrix $\Omega(\mathbf{x},\mathbf{y})$ is defined as

$$\Omega(\mathbf{x},\mathbf{y})=2\langle G|\psi(\mathbf{x})\psi^\dagger(\mathbf{y})|G\rangle$$
$$=\langle\mathbf{x}|(I+G)S^{-1}(I+\bar{G})|\mathbf{y}\rangle\ , \tag{2.4}$$

where $S\equiv(G+\bar{G})$ and I denotes the identity matrix. The divergent function $\delta(\mathbf{x},\mathbf{x})$ is defined in momentum space by

$$\delta(\mathbf{x},\mathbf{x})\equiv\int\frac{d^2p}{(2\pi)^2}\ . \tag{2.5}$$

If we are interested only in obtaining the ground-state energy of the system we may take variations on $\langle H\rangle$ directly with respect to G and $\bar{G}$. We are, however, interested in obtaining the effective potential to see if dynamical symmetry breaking occurs in the system:

$$H_{\mathrm{eff}}=H_\sigma+\int d^2x\ \alpha\left[\sigma+\frac{g}{2}\mathrm{tr}[\gamma^0\Omega(\mathbf{x},\mathbf{x})]\right]$$
$$\equiv\int d^2x\ V_{\mathrm{eff}}\ , \tag{2.6}$$

where H_σ is defined by replacing $\frac{1}{2}g\ \mathrm{tr}[\gamma^0\Omega(\mathbf{x},\mathbf{x})]\to-\sigma$ in $\langle H\rangle$ expressed in Eq. (2.3). The α field introduced in H_{eff} is the Langrange multiplier auxiliary field.[4] Solving the variation equations

$$\delta_\sigma H_{\mathrm{eff}}=0,\quad \delta_{\bar{G}}H_{\mathrm{eff}}=0,\quad \delta_G H_{\mathrm{eff}}=0\ ,$$

we can determine G, $\bar{G}$, and σ as functions of α: i.e.,

$$G=G(\alpha),\quad \bar{G}=\bar{G}(\alpha),\quad \sigma=\sigma(\alpha)\ .$$

Using these results we can determine H_{eff} and thus V_{eff} as a function of α:

$$H_{\mathrm{eff}}=H_{\mathrm{eff}}(\alpha)\equiv\int d^2x\ V_{\mathrm{eff}}(\alpha)\ . \tag{2.7}$$

The procedure obtaining V_{eff} in $2+1$ dimensions is exactly the same as in the case of $1+1$ dimensions. This procedure is described in detail in Ref. 4. We present here only the result

$$V_{\mathrm{eff}}=\tfrac{1}{2}\alpha^2-\frac{N}{g^2}(m-g\alpha)^2-N\int\frac{d^2p}{(2\pi)^2}\sqrt{p^2+m^2}\ , \tag{2.8}$$

where the effective mass m is related to the α field through the equation

$$m=\alpha g-\frac{g^2}{2}m\int\frac{d^2p}{(2\pi)^2}\frac{1}{\sqrt{p^2+m^2}}\ . \tag{2.9}$$

Since the $(2+1)$-dimensional GN theory is renormalizable in the $1/N$ expansion,[7] we expect that the divergences in V_{eff} can be absorbed by a suitable renormalization procedure, which will be done in the next section.

III. RENORMALIZATION

By adding to and subtracting from V_{eff} certain divergent terms, we can write V_{eff} in the form

$$V_{\mathrm{eff}}=\tfrac{1}{2}\alpha^2-\frac{N}{g^2}(m-g\alpha)^2$$
$$-N\int\frac{d^2p}{(2\pi)^2}\left[\sqrt{p^2+m^2}-\sqrt{p^2}-\frac{m^2}{2\sqrt{p^2}}\right]$$
$$-N\frac{m^2}{2}\int\frac{d^2p}{(2\pi)^2}\frac{1}{\sqrt{p^2}}-N\int\frac{d^2p}{(2\pi)^2}\sqrt{p^2}\ . \tag{3.1}$$

The last term in (3.1) is the zero-point energy: i.e.,

$$E_0=-N\int\frac{d^2p}{(2\pi)^2}\sqrt{p^2}=-\sum\tfrac{1}{2}\hbar\omega\ .$$

We will neglect this term from Eq. (3.1). The integral in the second line in Eq. (3.1) is convergent. This integral can be easily calculated:

$$\int\frac{d^2p}{(2\pi)^2}\left[\sqrt{p^2+m^2}-\sqrt{p^2}-\frac{m^2}{2\sqrt{p^2}}\right]=-\frac{1}{6\pi}(m^2)^{3/2}\ . \tag{3.2}$$

We can then write V_{eff} in the form

$$V_{\mathrm{eff}}=\tfrac{1}{2}\alpha^2-\frac{N}{g}(m-g\alpha)^2+\frac{N}{6\pi}(m^2)^{3/2}-\frac{N}{2}m^2I_\Lambda\ , \tag{3.3}$$

where

$$I_\Lambda \equiv \int^\Lambda \frac{d^2p}{(2\pi)^2}\frac{1}{\sqrt{p^2}} .$$

Λ is the ultraviolet cutoff. The effective mass m is calculated from Eq. (2.9) as

$$m = g\alpha + \frac{g^2}{4\pi}m(m^2)^{1/2} - \frac{g^2}{2}mI_\Lambda . \tag{3.4}$$

Imposing the renormalization condition

$$\left.\frac{d^2V_{\text{eff}}}{dm^2}\right|_{m=m_0} = \frac{1}{g_R^2} , \tag{3.5}$$

we find two possibilities for obtaining the finite effective potential. They are

$$g^2I_\Lambda = -2 \quad\text{or}\quad g^2I_\Lambda = \frac{2}{2N-1} . \tag{3.6}$$

This implies that there exist two ways to renormalize the coupling constant, which might lead to two different phases of the theory.

For $g^2I_\Lambda = 2/2N-1$, the Eq. (3.5) can be written in the form

$$\frac{1}{g_R^2} = \frac{1}{g^2} - \frac{2N-1}{2}I_\Lambda . \tag{3.7}$$

In this case the bare coupling constant becomes positive infinitesimal as $\Lambda \to \infty$. From Eqs. (3.4) and (3.7), we can define the effective mass m as the function of renormalized quantities:

$$m = \frac{2N-1}{2N}g_R\alpha_R , \tag{3.8}$$

where we have imposed the wave-function renormalization condition

$$\alpha g = \alpha_R g_R . \tag{3.9}$$

Substituting Eqs. (3.7) through (3.9) into Eq. (3.3), we obtain V_{eff}:

$$V_{\text{eff}} = \frac{m^2}{2g_R^2} + \frac{N}{6\pi}(m^2)^{3/2} . \tag{3.10}$$

The effective potential has a minimum at $m = -2\pi/g_R^2N$. Therefore symmetry breaking occurs if $g_R^2 < 0$ (Ref. 9). This has been already observed by Rosenstein, Warr, and Park.[7]

For the case with $g^2I_\Lambda = -2$, the renormalization conditions are

$$\frac{1}{g_R^2} = \frac{1}{g^2} + \tfrac{1}{2}I_\Lambda , \tag{3.11a}$$

$$\frac{\alpha}{g} = \frac{\alpha_R}{g_R} . \tag{3.11b}$$

With these conditions we can obtain the finite effective potential. We, however, encounter an inconsistency in the wave-function renormalization, since the left-hand side of (3.11b) contains an infinite term due to the relation

$$\sigma = \alpha = -\frac{g}{2}\text{tr}[\gamma^0\Omega(\mathbf{x},\mathbf{x})] .$$

This shows that only one of the two phases implied by the Gaussian effective potential of the GN model is acceptable.

IV. DISCUSSIONS

We have shown that the $(2+1)$-dimensional Gross-Neveu model is also renormalizable in the Gaussian approximation in the functional Schrödinger picture. The Gaussian effective potential can be renormalized by two different prescriptions, thus implying the existence of two phases of the theory. One of these two phases (where the bare coupling approaches negative infinitesimal as the cutoff goes to infinity), however, is shown to be invalid due to the inconsistency in the wave-function renormalization condition. The other phase (where the bare coupling constant approaches positive infinitesimal) is consistent and shows that dynamical symmetry breaking occurs when the renormalized coupling constant becomes negative.[9] This is in agreement with the result of the large-N approximation obtained by Rosenstein, Warr, and Park.[7]

It has been well known that the Gaussian approximation of the $(3+1)$-dimensional ϕ^4 theory also implies the existence of the two phases in the theory. Although both renormalization prescriptions seem to be consistent in $(3+1)$-dimensional ϕ^4 theory, one of the phases contains many questionable features, as pointed out by several authors.[8] We have shown that, in both $(1+1)$-dimensional[4] and $(2+1)$-dimensional Gross-Neveu models, one of the two phases contains an inconsistency and thus is not valid. This strongly suggests that one of the phases (where the bare coupling constant approaches positive infintesimal for $\lambda\phi^4$ theory and negative infinitesimal for the GN model) implied by the Gaussian approximation is not real but a fictitious effect of the computational method.

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[9]Since γ_5 in odd space-time dimensions is proportional to the identity matrix, the symmetry breaking is not that of the chiral symmetry as in the case of the $(1+1)$-dimensional GN model. Since the nontrivial minimum of the effective potential (3.10) implies the spontaneous generation of the fermion mass and the fermion mass term in $2+1$ dimensions violates P and T invariance [see, for example, R. Jackiw and S. Templeton, Phys. Rev. D **23**, 2291 (1981)], the symmetry breaking is that of the parity and the time-reversal invariance.