Gaussian approximation of the Gross-Neveu model in the functional Schrödinger picture

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The Gross-Neveu model is analyzed by the Gaussian approximation in the functional Schrödinger picture. It is shown that in the large-N limit the Gaussian approximation exactly reproduces the Gross-Neveu results, but for finite N it contains more information than the large-N approximation. There are two nontrivial phases of the theory depending upon the sign of the infinitesimal bare coupling constant. Dynamical symmetry breaking occurs in one of the phases. We also apply our analysis to the chiral Gross-Neveu model.

I. INTRODUCTION

It is believed that the Schrödinger-picture Gaussian variational method is quite promising for the study of quantum structures of field theories.[1] This method can be easily applied for boson field theories. In the case of fermion field theories, on the other hand, it is not easy to take the wave functional in Gaussian form.

Recently, Floreanini and Jackiw[2] proposed that the fermion field operator ψ and its conjugate momentum $i\psi^\dagger$ be realized as

$$\psi(x)=\frac{1}{\sqrt{2}}\left[u(x)+\frac{\delta}{\delta u^\dagger(x)}\right],$$
$$i\psi^\dagger(x)=\frac{i}{\sqrt{2}}\left[u^\dagger(x)+\frac{\delta}{\delta u(x)}\right], \tag{1.1}$$

to satisfy the equal-time anticommutation relation $\{\psi_i(x,t),\psi_j^\dagger(x',t)\}=\delta_{ij}\delta(x-x')$, where u and $u^\dagger$ are anticommuting Grassmann variables. The time-independent functional Schrödinger equation is then given by

$$H\left[u^\dagger+\frac{\delta}{\delta u},u+\frac{\delta}{\delta u^\dagger}\right]|\Psi\rangle=E|\Psi\rangle \tag{1.2}$$

and we can take the trial wave functional, for the variational approximation, in the Gaussian form

$$|\Psi\rangle\rightarrow|G\rangle=\frac{1}{(\det G)^{1/4}}\exp\left[\int_{x,y}u^\dagger(x)G(x,y)u(y)\right] \tag{1.3a}$$

and its dual

$$\langle\Psi|\rightarrow\langle G|=\frac{1}{(\det\bar{G})^{1/4}}\exp\left[\int_{x,y}u^\dagger(x)\bar{G}(x,y)u(y)\right], \tag{1.3b}$$

where $\bar{G}=(G^\dagger)^{-1}$.

The purpose of this paper is to use the realization (1.1) to establish the Gaussian variational method for fermion theories. We believe that the Gross-Neveu (GN) model[3] is most suitable for this purpose since the model is solvable in the large-N limit. We note that a variational analysis of the GN model already has been carried out by Latorre and Soto.[4] Our work is different from theirs in two features. Their wave functional is a delta functional, while ours is a Gaussian one.[2] They introduced constant background spinor fields and expressed the effective potential (V_{eff}) in terms of a mass parameter which is constructed from constant spinor fields. Following Gross and Neveu who used the vacuum expectation value of the compositive operator $\langle g\bar{\psi}\psi\rangle\equiv-\sigma$ for V_{eff}, we expressed V_{eff} with an effective scalar field α which is analogous to the σ variable so that our results can be compared with the GN model. Despite the differences in the choice of the trial wave functionals and the variational parameters the results turn out to be equivalent. Still another work by Lou and Ni,[5] who take a coherent-state wave functional, give similar results. The equivalence of the results suggest that the Gaussian variational approximations of fermion fields share the same essential core albeit apparently different formulations.

As Latorre and Soto[4] pointed out there is a strong parallel between $\lambda\phi^4$ in four-dimensional spacetime and the GN model. In scalar theory there exist two phases: the precarious and the autonomous phases.[6] In the precarious phase the bare coupling constant is negative infinitesimal, while in the autonomous phase it is positive infinitesimal. In our approach to the GN model there are indeed two phases exactly analogous to scalar theory.

In Sec. II we introduce our notation and the overall schemes. It is well known in the scalar theory that the Gaussian effective potential contains the leading $1/N$ result as its formal $N \to \infty$ limit.[6-9] We will show explicitly in Sec. III that the large-N limit is a special approximation to the Gaussian method. In this section we will also present how the variational equation can be solved to obtain the wave functional, which demonstrates essentials of mathematical techniques without getting involved in the complexity of the full equation. In Sec. IV we obtain the effective potential of the GN model and the properties of two phases are investigated. In the last section we summarize our results and briefly present the Gaussian analysis of the chiral Gross-Neveu model.[10,11]

II. SCHRÖDINGER PICTURE GAUSSIAN APPROXIMATION

The GN model is defined by the Lagrangian density

$$\mathcal{L}_\psi = \bar{\psi}^a (i\partial\!\!\!/)\psi^a + \tfrac{1}{2}g^2(\bar{\psi}^a\psi^a)^2, \quad a=1,\ldots,N , \tag{2.1}$$

in two-dimensional space-time. Note that we take the plus sign for g^2 as Gross and Neveu did. The γ algebra of the theory is defined by

$$\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}, \quad g_{\mu\nu}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} .$$

By employing the Gaussian trial wave functional $|G\rangle$ and its dual defined in Eqs. (1.3a) and (1.3b), we calculate the expectation value of the Hamiltonian of the system in the form

$$\begin{aligned}\langle G|H|G\rangle &\equiv \langle H\rangle \\ &= \tfrac{1}{2}\int dx\, dy\{\mathrm{tr}[-i\gamma^0\gamma^1\partial(x,y)\Omega(y,x)]\} - \frac{g^2}{8}\int dx\{\mathrm{tr}[\gamma^0\Omega(x,x)]\mathrm{tr}[\gamma^0\Omega(x,x)] \\ &\qquad + \mathrm{tr}[2\Omega(x,x)\delta(x,x)-\gamma^0\Omega(x,x)\gamma^0\Omega(x,x)]\} ,\end{aligned} \tag{2.2}$$

where tr denotes the trace taken over Dirac spinor and color indices. The matrix $\Omega(x,y)$ is defined as

$$\begin{aligned}\Omega(x,y) &= 2\langle G|\psi(x)\psi^\dagger(y)|G\rangle \\ &= \langle x|(I+G)S^{-1}(I+\bar{G})|y\rangle ,\end{aligned} \tag{2.3}$$

where $S \equiv (G+\bar{G})$ and I denotes the identity matrix. The divergent function $\delta(x,x)$ can be defined as, in momentum space,

$$\delta(x,x) \equiv \int \frac{dp}{2\pi} . \tag{2.4}$$

If we are interested only in obtaining the ground-state energy of the system, we take variations on $\langle H\rangle$ directly with respect to G and $\bar{G}$:

$$\delta_{G,\bar{G}}\langle H\rangle = 0 . \tag{2.5}$$

These conditions yield the equations

$$(I-G)h_\Omega(I+G)=0 , \tag{2.6a}$$

$$(I+\bar{G})h_\Omega(I-\bar{G})=0 , \tag{2.6b}$$

where

$$\begin{aligned}h_\Omega(x,y) &\equiv -i\gamma^0\gamma^1\partial(x,y) \\ &\quad -\frac{g^2}{2}I(x,y)\{\gamma^0\mathrm{tr}[\gamma^0\Omega(x,x)] \\ &\qquad + I(x,x) - \mathrm{tr}[\gamma^0\Omega(x,x)\gamma^0]\} .\end{aligned} \tag{2.7}$$

Equation (2.6a) can be solved exactly for G, to obtain the ground-state energy of the system. The result will be presented later. One can show that the condition (2.6b) is equivalent to Eq. (2.6a).

We are, however, interested in obtaining the effective potential to see if dynamical symmetry breaking occurs in the system. For this purpose, we define the effective Hamiltonian

$$H_{\mathrm{eff}} = H_\sigma + \int dx\, \alpha \left[\sigma + \frac{g}{2}\mathrm{tr}[\gamma^0\Omega(x,x)]\right] , \tag{2.8}$$

where H_σ is obtained by replacing $\frac{1}{2}g\,\mathrm{tr}[\gamma^0\Omega(x,x)] \to -\sigma$ in $\langle H\rangle$ by Eq. (2.2). More explicitly, we write it as

$$\begin{aligned}H_\sigma &\equiv \tfrac{1}{2}\int dx\, dy\, \mathrm{tr}[-i\gamma^0\gamma^1\partial(x,y)\Omega(y,x)] - \tfrac{1}{2}\int dx\, \sigma^2 \\ &\quad -\frac{g^2}{8}\int dx\, \mathrm{tr}[2\Omega(x,x)\delta(x,x) \\ &\qquad -\gamma^0\Omega(x,x)\gamma^0\Omega(x,x)] .\end{aligned} \tag{2.9}$$

The α field introduced in H_{eff} is the Lagrange multiplier auxiliary field. If we take a variation on H_{eff} with respect to α we obtain

$$\begin{aligned}\sigma &= -\frac{g}{2}\mathrm{tr}[\gamma^0\Omega(x,x)] = -g\langle\bar{\psi}(x)\psi(x)\rangle , \\ \langle H\rangle &= H_\sigma = H_{\mathrm{eff}} .\end{aligned} \tag{2.10}$$

Other variational equations are

$$\delta_\sigma H_{\mathrm{eff}} = 0 , \tag{2.11a}$$

$$\delta_{\bar{G}} H_{\mathrm{eff}} = 0 , \tag{2.11b}$$

$$\delta_G H_{\mathrm{eff}} = 0 . \tag{2.11c}$$

In obtaining the extremum value of H_{eff}, Eqs. (2.10)–(2.11) are equivalent to Eqs. (2.6a) and (2.6b). We now explain our strategy in detail.

We first solve Eqs. (2.11a)–(2.11c) to determine G, $\bar{G}$, and σ as functions of α: i.e.,

$$G=G(\alpha),\quad \bar{G}=\bar{G}(\alpha),\quad \sigma=\sigma(\alpha)\ .$$

Using these results we determine $H_{\rm eff}$ and thus $V_{\rm eff}$ as a function of α:

$$H_{\rm eff}=H_{\rm eff}(\alpha)\equiv\int dx\ V_{\rm eff}(\alpha)\ . \tag{2.12}$$

The extremum value of $H_{\rm eff}$ is then determined from the condition

$$\frac{d}{d\alpha}H_{\rm eff}(\alpha)=0\ . \tag{2.13}$$

Note that we never use Eq. (2.10) in computing $V_{\rm eff}$ from Eq. (2.12). The information of Eq. (2.10) is determined, in this case, by Eq. (2.13). In the next section we solve Eqs. (2.11a)–(2.11c) in the large-N limit and obtain the effective potential $V_{\rm eff}$.

III. EFFECTIVE POTENTIAL IN THE LARGE-N LIMIT

In the large-N limit, $H_{\rm eff}$ can be written in the form

$$H_{N_{\rm eff}}=\frac{N}{2}\int dx\,dy\ {\rm tr}'[h_N(x,y)\Omega(y,x)] +\int dx(\alpha\sigma-\tfrac{1}{2}\sigma^2)\ , \tag{3.1}$$

where

$$h_N(x,y)=-i\gamma^0\gamma^1\partial(x,y)+g\alpha\delta(x-y)\gamma^0 \tag{3.2}$$

and the tr′ denotes the trace taken over Dirac spinor indices only. The variation of $H_{\rm eff}$ with respect to σ, $\bar{G}$, and G yields, respectively,

$$\sigma=\alpha\ , \tag{3.3a}$$

$$(I-G)h_N(I+G)=0\ , \tag{3.3b}$$

$$(I+\bar{G})h_N(I-\bar{G})=0\ . \tag{3.3c}$$

It turns out that the condition (3.3c) is equivalent to the condition (3.3b).

Multiplying Eq. (3.3b) by h_N from the left yields

$$h_N^2-K_N^2+[h_N,K_N]=0\ , \tag{3.4}$$

where

$$K_N\equiv h_NG\ .$$

One trivial solution of Eq. (3.4) is $K_N=\pm h_N$, which leads to $H_{\rm eff}$ with vanishing quantum corrections. To obtain nontrivial solutions of Eq. (3.4), we decompose the 2×2 matrix $K_N(x,y)$ in terms of four Dirac matrices. We denote them collectively by Γ^a:

$$\Gamma^0=I,\quad \Gamma^1=i\gamma^1,\quad \Gamma^2=\gamma^0\gamma^1,\quad \Gamma^3=\gamma^0\ . \tag{3.5}$$

Γ^a's are taken to be Hermitian and satisfy the commutation relations

$$\{\Gamma^i,\Gamma^j\}=2\delta_{ij},\quad i=1,2,3\ ,$$
$$[\Gamma^i,\Gamma^j]=2i\epsilon_{ijk}\Gamma^k\ , \tag{3.6}$$
$$[\Gamma^0,\Gamma^i]=0\ .$$

$K_N(x,y)$ is then decomposed in the form

$$K_N(x,y)=\sum_{a=0}^{3}\Gamma^a\int\frac{dp}{2\pi}e^{-ip(x-y)}K_{Na}(p)\ . \tag{3.7}$$

Other functions relevant to Eq. (3.4) are written in the form

$$h_N(x,y)=\int\frac{dp}{2\pi}(-p\Gamma^2+g\alpha\Gamma^3)e^{-ip(x-y)}\ , \tag{3.8}$$

$$h_N^2(x,y)\equiv\int dz\ h_N(x,z)h_N(z,y) =\int\frac{dp}{2\pi}(p^2+g^2\alpha^2)e^{-ip(x-y)}\ , \tag{3.9}$$

$$[h_N,K_N](x,y)=2i\int\frac{dp}{2\pi}\{\Gamma^1[-pK_{N3}(p)-g\alpha K_{N2}(p)]+\Gamma^2g\alpha K_{N1}(p)+\Gamma^3pK_{N1}(p)\}e^{-ip(x-y)}\ , \tag{3.10}$$

$$K_N^2(x,y)=\int\frac{dp}{2\pi}\left[\left[K_{N0}^2(p)+\sum_{i=1}^{3}K_{Ni}^2(p)\right]+2\sum_{i=1}^{3}\Gamma^iK_{N0}(p)K_{Ni}(p)\right]e^{-ip(x-y)}\ . \tag{3.11}$$

Substituting Eqs. (3.7)–(3.11) into (3.4) yields

$$K_{N0}^2+\sum_{i=1}^{3}K_{Ni}^2=p^2+g^2\alpha^2\ , \tag{3.12a}$$

$$K_{N0}K_{N1}=-i(pK_{N3}+g\alpha K_{N2})\ , \tag{3.12b}$$

$$K_{N0}K_{N2}=ig\alpha K_{N1}\ , \tag{3.12c}$$

$$K_{N0}K_{N3}=ipK_{N1}\ . \tag{3.12d}$$

From the equations we find

$$K_{N0}^2=(p^2+g^2\alpha^2),\quad K_{N1}=K_{N2}=K_{N3}=0\ . \tag{3.13}$$

We can thus determine the matrices $K_N(x,y)$ and $G_N(x,y)$, the solution of Eq. (3.3b), uniquely except for the sign

$$K_N(x,y)=\pm I\int\frac{dp}{2\pi}\sqrt{p^2+g^2\alpha^2}\,e^{-ip(x-y)}\ ,$$
$$G_N(x,y)\equiv(h_N^{-1}K_N)(x,y) =\pm\int\frac{dp}{2\pi}\frac{1}{\sqrt{p^2+g^2\alpha^2}}(-p\Gamma^2+g\alpha\Gamma^3)e^{-ip(x-y)}\ . \tag{3.14}$$

We note the matrices K_N and G_N have properties

$$G_N=G_N^\dagger=G_N^{-1}\ ,$$
$$K_N=K_N^\dagger,\quad h_N=h_N^\dagger\ ,\tag{3.15}$$
$$[h_N,K]=[h_N,G_N]=0\ .$$

Using Eqs. (2.3) and (3.15) we find

$$\Omega=\Omega_N=(I+G_N)S_N^{-1}(I+\overline{G}_N)=(I+G_N)\ .\tag{3.16}$$

We can thus determine $H_{\rm eff}$ in the large-N limit:

$$\begin{aligned}H_{N\rm eff}&=\frac{N}{2}\int dx\ {\rm tr}'K(x,x)+\tfrac{1}{2}\int dx\ \sigma^2(x)\\&=\int dx\left[\tfrac{1}{2}\sigma^2\pm N\int\frac{dp}{2\pi}\sqrt{p^2+g^2\sigma^2}\right]\\&\equiv\int dx\ V_{N\rm eff}\ .\end{aligned}\tag{3.17}$$

We take the minus sign from Eq. (3.14) in order to have the correct zero-point energy E_0:

$$E_0=-\sum\tfrac{1}{2}\hbar\omega=-N\int\frac{dp}{2\pi}\sqrt{p^2}\ ,\tag{3.18}$$

in the free fermion field theory limit. If we take the plus sign from Eq. (3.14), the system will be unstable.

The effective potential is then given by

$$\begin{aligned}V_{\rm eff}&=\tfrac{1}{2}\alpha^2-N\int\frac{dp}{2\pi}\sqrt{p^2+g^2\alpha^2}\\&=\tfrac{1}{2}\alpha^2-N\left[I_1+\frac{g^2\alpha^2}{2}I_0(M)\right.\\&\qquad\left.-\frac{g^2\alpha^2}{4\pi}\left[\ln\frac{g^2\alpha^2}{M^2}-1\right]\right]\ ,\end{aligned}\tag{3.19}$$

where

$$I_1=\int\frac{dp}{2\pi}|p|\ ,\tag{3.20}$$

$$I_0(M)=\int\frac{dp}{2\pi}\frac{1}{\sqrt{p^2+M^2}}\ .\tag{3.21}$$

Here M is an arbitrary constant. The effective potential (3.19) is the same as that of Gross and Neveu.[3] After renormalization we obtain the renormalized effective potential

$$V(\alpha)=\tfrac{1}{2}\alpha^2+\frac{Ng^2}{4\pi}\alpha^2\left[\ln\left[\frac{\alpha^2}{\alpha_0^2}\right]-3\right]\ .\tag{3.22}$$

It is straightforward to show that this potential exhibits spontaneous symmetry breaking as is done in Ref. 3. Accordingly the Gaussian approximation in the Schrödinger picture is equivalent to the Gross-Neveu approach in the large-N limit.

IV. EFFECTIVE POTENTIAL BEYOND THE LARGE-N LIMIT

We first write $H_{\rm eff}$ defined in Eq. (2.8) in the form, taken trace over color indices,

$$\begin{aligned}H_{\rm eff}&=\int dx(\alpha\sigma-\tfrac{1}{2}\sigma^2)\\&\quad+\frac{N}{2}\int dx\,dy\ {\rm tr}'[h(x,y)\Omega(y,x)]\\&\quad-\frac{g^2N}{8}\int dx\ {\rm tr}'[\gamma^0\Omega(x,x)\gamma^0\Omega(x,x)]\ ,\end{aligned}\tag{4.1}$$

where

$$\begin{aligned}h(x,y)&\\&=-i\gamma^0\gamma^1\partial(x,y)\\&\quad+\gamma^0\delta(x-y)\left[\alpha g+\frac{g^2}{2}[\gamma^0\Omega(x,x)\gamma^0-\delta(x,x)]\right]\ .\end{aligned}\tag{4.2}$$

Taking variations on $H_{\rm eff}$ with respect to σ, $\overline{G}$, and G yields the same equation as Eqs. (3.3a)–(3.3c) with h_N replaced by h: i.e.,

$$\sigma=\alpha\ ,\tag{4.3a}$$

$$(I-G)h(I+G)=0\ ,\tag{4.3b}$$

$$(I+\overline{G})h(I-\overline{G})=0\ .\tag{4.3c}$$

It again turns out that Eq. (4.3c) is equivalent to Eq. (4.3b). Multiplying Eq. (4.3b) by h from the left yields

$$h^2-K^2+[h,K]=0\ ,\tag{4.4}$$

where $K\equiv hG$. Nontrivial solution of Eq. (4.4) for K is uniquely determined in exactly the same way as the solution of Eq. (3.4) for K_N. The results are

$$K(p)=-\left[\left[\sum_{i=1}^{3}h_i^2(p)\right]^{1/2}+\frac{h_0(p)h_i(p)}{\left[\sum_{i=1}^{3}h_i^2(p)\right]^{1/2}}\Gamma_i\right]\ ,\tag{4.5}$$

$$G(p)=-\frac{h_i(p)}{\left[\sum_{i=1}^{3}h_i^2(p)\right]^{1/2}}\Gamma^i\equiv G_i\Gamma^i\ ,\tag{4.6}$$

where

$$h=h_0+\sum_{i=1}^{3}h_i\Gamma^i\ .$$

We again choose the minus sign in Eqs. (4.5) and (4.6) to be consistent with the zero-point energy of the system as shown in Eq. (3.18). We note the Γ^0 component of G is zero. The results given in Eqs. (3.15) and (3.16) are also reproduced if we drop the index N, i.e.,

$$G=G^\dagger=G^{-1}\ ,\tag{4.7a}$$

$$K=K^\dagger,\quad h=h^\dagger\ ,\tag{4.7b}$$

$$[h,K]=[h,G]=0\ ,\tag{4.7c}$$

$$\Omega=I+G\ .\tag{4.7d}$$

Using the results (4.6) and (4.7d), we can write $h(x,y)$ in the form

$$h(x,y)=\int\frac{dp}{2\pi}\left[-\frac{g^2}{2}G_1(0)\Gamma^1-\left[\frac{g^2}{2}G_2(0)+p\right]\Gamma^2+\left[\frac{g^2}{2}G_3(0)+\alpha g\right]\Gamma^3\right]e^{-ip(x-y)}, \tag{4.8}$$

where $G_i(0)$'s are defined in momentum space:

$$G_i(0)=\int\frac{dp}{2\pi}G_i(p). \tag{4.9}$$

From Eqs. (4.6), (4.8), and (4.9) we obtain the consistency condition

$$G_3(0)=-\left[\frac{g^2}{2}G_3(0)+\alpha g\right]\int\frac{dp}{2\pi}\left[p^2+\left[\frac{g^2}{2}G_3(0)+\alpha g\right]^2\right]^{-1/2} \tag{4.10}$$

and $G_1(0)=G_2(0)=0$. Defining the effective mass m as

$$m=\alpha g+\frac{g^2}{2}G_3(0), \tag{4.11}$$

the condition becomes

$$\alpha g=m-\frac{g^2m}{2}\left[\frac{1}{2\pi}\ln\left[\frac{m}{M}\right]^2-I_0(M)\right], \tag{4.12}$$

where $I_0(M)$ is given by (3.21).

We are now ready to evaluate $V_{\rm eff}$ as a function of α and m. It is

$$\begin{aligned}V_{\rm eff}&=\tfrac{1}{2}\alpha^2-N\frac{g^2}{4}G_3(0)^2+\frac{N}{2}{\rm tr}'K(x,x)\\&=\tfrac{1}{2}\alpha^2-N\left[\frac{m}{g}-\alpha\right]^2+\frac{Nm^2}{2}\left[\frac{1}{2\pi}\left[\ln\frac{m^2}{M^2}-1\right]-I_0(M)\right],\end{aligned} \tag{4.13}$$

where we have neglected irrelevant infinite constants. We notice that the condition (4.12) may be understood as $\partial V/\partial m=0$, which reminds us of the large-N approximation of O(N)-symmetric scalar theory.[7] The ground state of the theory is determined by the stationary points of V, i.e., by the equations $\partial V/\partial m=0$ and $\partial V/\partial\alpha=0$.

Since the GN model is renormalizable the divergences in (4.12) and (4.13) can be absorbed by a suitable renormalization of coupling constant and wave function. We can do this by defining $(d^2V/dm^2)|_{m=m_0}=1/g_R^2$ or equivalently by adding counterterms to the original Lagrangian. If we define

$$\left.\frac{d^2V_{\rm eff}}{dm^2}\right|_{m=m_0}=\frac{1}{g_R^2}$$

and require that the renormalized coupling constant g_R be finite, we find two cases,

$$g^2I_0(M)=\frac{2}{2N-2}\ \text{ or } -2,$$

for which the effective potential can be made finite. We consider each of these two cases separately.

(i) The case with $g^2I_0=2/(2N-1)$: The renormalization conditions are

$$\frac{1}{g_R^2}=\frac{1}{g^2}-(N-\tfrac{1}{2})I_0(M), \tag{4.14a}$$

$$g_R\alpha_R=g\alpha, \tag{4.14b}$$

for which the consistency condition becomes

$$g_R\alpha_R=\left[\frac{2N}{2N-1}\right]m, \tag{4.15}$$

and the effective potential becomes

$$\begin{aligned}V_{\rm eff}&=\tfrac{1}{2}\alpha_R^2-N\left[\frac{m}{g_R}-\alpha_R\right]^2+\frac{Nm^2}{4\pi}\left[\ln\left[\frac{m}{M}\right]^2-1\right]\\&=\frac{1}{2}\left[\frac{2N-1}{2N}\right]^2\alpha_R^2+\frac{Ng_R^2\alpha_R^2}{4\pi}\left[\ln\left[\frac{\alpha R}{\alpha_0}\right]^2-3\right]\left[\frac{2N-1}{2N}\right]^2,\end{aligned} \tag{4.16}$$

where we have set $M=[(2N-1)/2N]g_R\alpha_0e$. In this case the bare coupling constant becomes positive infinitesimal as the cutoff goes to infinity. The behavior of $V_{\rm eff}$ is qualitatively similar to the GN result, and in the large-N limit it reproduces the GN result. It has a minimum at

$$\alpha_R=\alpha_0\exp\left[1-\frac{2}{(2N-1)g_R}\right] \tag{4.17}$$

and therefore the symmetry breaking occurs for any finite N. The behavior of the effective potential resembles that of the so-called autonomous phase of $\lambda\phi^4$ theory[6] and our result is qualitatively equivalent to the result of Lou and Ni[5] who used a coherent wave functional which is somewhat different from ours.

(ii) The case with $g^2I_0=-2$: In this case the renormalization conditions are

$$\frac{1}{g_R^2}=\frac{1}{g^2}+\tfrac{1}{2}I_0(M), \tag{4.17a}$$

$$\frac{\alpha_R}{g_R}=\frac{\alpha}{g}. \tag{4.17b}$$

Although the renormalization conditions (4.17) can make the effective potential (4.13) finite and this phase seems similar to a phase in the case of the scale $\lambda\phi^4$ theory, one has to question the existence of this phase since the right-hand side of (4.17b) contains an infinite term due to Eq. (2.10). We think that this fact may be related to the point raised by the authors of Ref. 11 and warrants further investigation.

V. DISCUSSIONS

For the Gaussian effective potential evaluation of the GN model, so far there have been developed three different formulations for the wave functionals of fermion fields: namely, the delta wave functional,[4] the coherent wave functional,[5] and Floreanini-Jackiw wave functional which was used in our work. We have found that these apparently different approaches lead to qualitatively equivalent results, and furthermore the results are quite similar to the $\lambda\phi^4$ theory case: there are two phases, i.e., the precarious phase in which the bare coupling constant g^2 is negative infinitesimal and the autonomous phase where g^2 is positive infinitesimal, and symmetry breaking occurs for any finite N in the latter phase.

Both in $\lambda\phi^4$ theory and in the GN model the Gaussian variational approach contains the leading $1/N$ result as N goes to infinity.[6,8,9] However, in the chiral Gross-Neveu (CGN) model this does not hold. We will briefly present our analysis on this model showing that the Gaussian approximation is equivalent to the $1/N$ approximation.

The CGN model is defined by the Lagrangian density

$$L=\bar{\psi}^a i\partial\!\!\!/\psi^a+\tfrac{1}{2}g^2[(\bar{\psi}\psi)^2-(\bar{\psi}\gamma_5\psi)^2],\quad a=1,2,\ldots,N\ . \tag{5.1}$$

The expectation value of the Hamiltonian with respect to a trial wave functional is

$$\langle H\rangle=\tfrac{1}{2}\int dx\,dy\,\mathrm{tr}[-i\gamma^0\gamma^1\partial(x,y)\Omega(y,x)]$$
$$+\frac{g^2}{8}\int dx(\mathrm{tr}\{[\gamma^0\Omega(x,x)]^2-[\gamma^0\gamma^5\Omega(x,x)]^2\}-[\mathrm{tr}\gamma^0\Omega(x,x)]^2+[\mathrm{tr}\gamma^0\gamma^5\Omega(x,x)]^2-4\,\mathrm{tr}[\delta(x,x)\Omega(x,x)])\ . \tag{5.2}$$

We introduce auxiliary fields σ and π as

$$\sigma\equiv-\langle g\bar{\psi}\psi\rangle=\frac{-1}{2}g\,\mathrm{tr}[\gamma^0\Omega(x,x)]\ , \tag{5.3a}$$

$$i\pi\equiv-\langle g\bar{\psi}\gamma_5\psi\rangle=\tfrac{1}{2}g\,\mathrm{tr}[\gamma^0\gamma^5\Omega(x,x)]\ . \tag{5.3b}$$

In order to study the effective potential in terms of σ and π we use an effective Hamiltonian

$$H_{\mathrm{eff}}=H_{\sigma,\pi}+\int dx\,\alpha\left[\sigma+\frac{g}{2}\mathrm{tr}[\gamma^0\Omega(x,x)]\right]+\beta\left[\pi-\frac{ig}{2}\mathrm{tr}[\gamma^0\gamma^5\Omega(x,x)]\right]\ , \tag{5.4}$$

where α and β are Lagrange multiplier fields and

$$H_{\sigma,\pi}=\tfrac{1}{2}\int dx\,dy\,\mathrm{tr}[-i\gamma^0\gamma^1\partial(x,y)\Omega(y,x)]-\tfrac{1}{2}\int dx(\sigma^2+\pi^2)$$
$$-\frac{g^2}{8}\int dx\,\mathrm{tr}\{2\Omega(x,x)\delta(x,x)-[\gamma^0\Omega(x,x)]^2+[\gamma^0\gamma^5\Omega(x,x)]^2\}\ . \tag{5.5}$$

After performing similar calculations to the previous sections we obtain the effective potential

$$V_{\mathrm{eff}}=\tfrac{1}{2}(\alpha^2+\beta^2)-N\int\frac{dp}{2\pi}[p^2+g^2(\alpha^2+\beta^2)]^{1/2}-\frac{Ng^2}{2}I(0)I(0)$$
$$=\tfrac{1}{2}\phi^2+\frac{Ng^2\phi^2}{2}\left\{\frac{1}{2\pi}\left[\ln\left[\frac{g^2\phi^2}{M^2}\right]-1\right]-I_0(M)\right\}, \tag{5.6}$$

where $\phi^2=\alpha^2+\beta^2$, and irrelevant infinite constants are neglected. This is indeed of the same form as the $1/N$ effective potential (3.19), and after renormalization one obtains the finite effective potential of the CGN model:[10]

$$V_{\mathrm{eff}}=\tfrac{1}{2}\phi^2+\frac{g^2N}{4\pi}\phi^2\left[\ln\frac{\phi^2}{\phi_0^3}-3\right]\ . \tag{5.7}$$

Thus the Gaussian variational approximation is equivalent to the large-N approximation in the case of the CGN model, while the former contains the latter as a special limit in the $\lambda\phi^4$ scalar theory and the GN model.

The effective potential (5.7) has a local maximum at $\phi^2=0$, and the absolute minimum at $\phi^2=\phi_0^2e^{2-2/Ng}$. This implies that the chiral symmetry is spontaneously broken and the fermions acquire masses. But in two space-time dimensions, it is well known that spontaneous breaking of a continuous symmetry is not possible. As shown in the case of the large-N approximation by Witten,[12] therefore, the appearance of the nontrivial minimum in the Gaussian effective potential (5.7) must be interpreted not as an indication of the symmetry breaking of the CGN model, but as an indication of a phase transition which gives rise to mass generation of the physical fermions.

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