Some exact solutions of the semilocal Popov equations

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A B S T R A C T

We study the semilocal version of Popov's vortex equations on S^2. Though they are not integrable, we construct two families of exact solutions which are expressed in terms of rational functions on S^2. One family is a trivial embedding of Liouville-type solutions of the Popov equations obtained by Manton, where the vortex number is an even integer. The other family of solutions is constructed through a field redefinition which relates the semilocal Popov equation to the original Popov equation but with the ratio of radii $\sqrt{3/2}$, which is not integrable. These solutions have vortex number $N = 3n - 2$ where n is a positive integer, and hence $N = 1$ solutions belong to this family. In particular, we show that the $N = 1$ solution with reflection symmetry is the well-known CP^1 lump configuration with unit size where the scalars lie on S^3 with radius $\sqrt{3/2}$. It generates the uniform magnetic field of a Dirac monopole with unit magnetic charge on S^2.

Recently, Popov [1] obtained a set of vortex-type equations on a 2-sphere by dimensional reduction of SU(1, 1) Yang–Mills instanton equations on the four-manifold $S^2 \times H^2$ where H^2 is a hyperbolic plane. It was shown that they are integrable when the scalar curvature of the manifold vanishes. Subsequently, Manton [2] constructed explicit solutions with even vortex numbers from rational functions on the sphere. They have a geometric interpretation in terms of conformal rescalings of the 2-sphere metric.

The Popov equations involve a complex scalar field and a U(1) gauge potential. Except a flipped sign, they are the same as the well-known Bogomolny equations [3] for abelian Higgs vortices [4,5] on S^2. In this paper, we would like to consider the semilocal [6,7] version of the Popov equations, which consist of two scalar fields instead of one. The equations have an additional global SU(2) symmetry with respect to the rotation of the scalars as well as the local U(1) symmetry. We will show that they appear in $2 + 1$ dimensional Chern–Simons systems with nonrelativistic matter on S^2. Such systems on the plane have been extensively studied to understand the quantum Hall effect and other related phenomena [8–10]. Then we construct two families of exact solutions of the semilocal Popov equations. One family of solutions is trivially obtained by a simple ansatz that the two scalars are proportional to each other, with which the equations reduce to the original Popov equations. For the other family of solutions, we will relate the equations to the semilocal version of the Liouville equations considered in [11,12]. In addition to Liouville solutions, they admit

another family of exact solutions [12] which involves an arbitrary rational function on S^2. We will construct solutions of the semilocal Popov equations from them.

It turns out that semilocal Popov equations have another connection to the original Popov equations with a single scalar. As mentioned above, it is integrable only when the scalar curvature of the underlying four-manifold $S^2 \times H^2$ vanishes [1], which happens for equal radii $R_1 = R_2$, where R_1, R_2 are the radii of S^2 and H^2, respectively. Here we will show that the semilocal Popov equations with equal radii can be transformed to the Popov equations with different radii $R_1/R_2 = \sqrt{3/2}$. The aforementioned solutions of the semilocal equation correspond to the constant solution of the latter.

The Liouville solutions have only even vortex numbers [2]. However the vortex number of the other family of solutions is $N = 3n - 2$, where n is a positive integer, so that odd vortex numbers are possible. In particular, the solutions with unit vorticity $N = 1$ belong to this family. We will show that the $N = 1$ solution with reflection symmetry in the equator of S^2 is precisely given by the CP^1 lump configuration with unit size. The S^3 where the scalar fields lie has radius $\sqrt{3/2}$ which is the ratio R_1/R_2 above. The magnetic field is that of a Dirac monopole with unit magnetic charge on S^2.

Let us begin with writing the Popov equations on S^2. For convenience, the radius of S^2 is fixed to be $\sqrt{2}$. The metric of S^2 is given by $ds^2 = \Omega \, dz d\bar{z}$ with

$$\Omega = \frac{8}{(1 + |z|^2)^2}. \tag{1}$$

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The Popov equations are[1]

$$D_{\bar{z}}\phi \equiv \partial_{\bar{z}}\phi - ia_{\bar{z}}\phi = 0, \tag{2}$$

$$F_{z\bar{z}} = -\frac{2i}{(1+|z|^2)^2}(C^2 - |\phi|^2), \tag{3}$$

where $C = R_1/R_2 = \sqrt{2}/R_2$ is the ratio of radii as described above. ϕ is a complex scalar field, a is a U(1) gauge potential and $F_{z\bar{z}} = \partial_z a_{\bar{z}} - \partial_{\bar{z}} a_z$ is the field strength which is imaginary. If the right hand side of the second equation has opposite sign, these would be the same as the Bogomolny equations for abelian Higgs vortices on S^2. As mentioned above, the equations are integrable only for $C = 1$ [1]. From (2) the gauge potential $a_{\bar{z}}$ may be expressed as

$$a_{\bar{z}} = -i\partial_{\bar{z}} \ln \phi, \tag{4}$$

away from zeros of ϕ. Since

$$F_{z\bar{z}} = -i\partial_z \partial_{\bar{z}} \ln |\phi|^2, \tag{5}$$

we can eliminate the gauge potential and are left with a single equation

$$\partial_z \partial_{\bar{z}} \ln |\phi|^2 = \frac{2}{(1+|z|^2)^2}(C^2 - |\phi|^2), \tag{6}$$

which is valid away from zeros of ϕ.

Eqs. (2) and (3) may be obtained from an energy function [1,2] which comes from a dimensional reduction of the Yang–Mills action,

$$
\begin{aligned}
E &= \frac{1}{2}\int_{S^2}\left[\frac{4}{\Omega}|F_{z\bar{z}}|^2 - 2(|D_z\phi|^2 + |D_{\bar{z}}\phi|^2)\right.\\
&\quad \left. + \frac{\Omega}{4}(C^2 - |\phi|^2)^2\right]\frac{i}{2}dz \wedge d\bar{z}\\
&= \frac{1}{2}\int_{S^2}\left\{-\frac{4}{\Omega}\left[F_{z\bar{z}} + i\frac{\Omega}{4}(C^2 - |\phi|^2)\right]^2\right.\\
&\quad \left. - 4|D_{\bar{z}}\phi|^2\right\}\frac{i}{2}dz \wedge d\bar{z} - \pi C^2 N,
\end{aligned}
\tag{7}
$$

where N is the first Chern number

$$N = \frac{1}{2\pi}\int_{S^2} F_{z\bar{z}} dz \wedge d\bar{z}, \tag{8}$$

which is an integer and is the same as the vortex number which counts the number of isolated zeros of ϕ. Therefore, for fields satisfying the Popov equations, the energy is stationary and has value $-\pi C^2 N$. It is however not minimal because of the negative sign in the second term of (7).

The Popov equation (6) can also arise in a completely different physics system. Let us consider a $2+1$ dimensional Chern–Simons gauge theory with a nonrelativistic matter field on S^2 of which the action is

$$
\begin{aligned}
S &= \int dt \int_{S^2}\left[\frac{\kappa}{2}\epsilon^{\mu\nu\lambda}a_\mu \partial_\nu a_\lambda + \Omega(i\phi^* D_t\phi - V)\right.\\
&\quad \left. - (|\tilde{D}_z\phi|^2 + |\tilde{D}_{\bar{z}}\phi|^2)\right]\frac{i}{2}dz \wedge d\bar{z},
\end{aligned}
\tag{9}
$$

where κ is the Chern–Simons coefficient and

$$D_t\phi = (\partial_t - ia_t)\phi$$

$$\tilde{D}_z\phi = (\partial_z - a_z - iA_z^{ex})\phi. \tag{10}$$

Note that we applied an external U(1) gauge potential A^{ex} given by

$$A_{\bar{z}}^{ex} = \frac{i}{2}\frac{gz}{1+|z|^2}, \tag{11}$$

which generates uniform magnetic field with magnetic charge g on S^2. The potential V has the form

$$V = -\frac{g}{8}|\phi|^2 + \frac{1}{2\kappa}|\phi|^4. \tag{12}$$

This action has been extensively studied on the plane in the context of anyon physics to understand the quantum Hall effect and other related phenomena [9,10].

Variation of a_t gives the Gauss constraint

$$F_{z\bar{z}} = -i\frac{\Omega}{2\kappa}|\phi|^2. \tag{13}$$

The energy function is

$$E = \int_{S^2}(|\tilde{D}_z\phi|^2 + |\tilde{D}_{\bar{z}}\phi|^2 + \Omega V)\frac{i}{2}dz \wedge d\bar{z}, \tag{14}$$

which has no explicit contribution from the Chern–Simons term. It can be rewritten by the usual Bogomolny rearrangement

$$
\begin{aligned}
|\tilde{D}_z\phi|^2 &= |\tilde{D}_{\bar{z}}\phi|^2 - i(F_{z\bar{z}} + F_{z\bar{z}}^{ex})|\phi|^2\\
&= |\tilde{D}_{\bar{z}}\phi|^2 - \frac{\Omega}{2\kappa}|\phi|^4 + \frac{g}{8}\Omega|\phi|^2,
\end{aligned}
\tag{15}
$$

up to a total derivative term, where in the second line we have used (13) and $F_{z\bar{z}}^{ex} = \frac{ig}{8}\Omega$. The last two terms in (15) are cancelled by the potential (12) and the energy becomes

$$E = 2\int_{S^2}|\tilde{D}_{\bar{z}}\phi|^2 \frac{i}{2}dz \wedge d\bar{z}, \tag{16}$$

which is positive definite. Therefore the energy vanishes if

$$\tilde{D}_{\bar{z}}\phi = 0. \tag{17}$$

Combining this equation with the Gauss constraint (13), we get

$$\partial_z \partial_{\bar{z}} \ln |\phi|^2 = -\frac{\Omega}{2\kappa}\left(\frac{\kappa g}{4} - |\phi|^2\right), \tag{18}$$

away from zeros of ϕ. With $\kappa = -2$ and $g = -2C^2$ this becomes the Popov equation (6).

Now we introduce the semilocal Popov equations which involve two scalar fields ϕ_i ($i = 1, 2$). We will only consider the case of equal radii, i.e., $C = 1$. The semilocal Popov equations read

$$D_{\bar{z}}\phi_i \equiv \partial_{\bar{z}}\phi_i - ia_{\bar{z}}\phi_i = 0, \quad (i = 1, 2) \tag{19}$$

$$F_{z\bar{z}} = -\frac{2i}{(1+|z|^2)^2}(1 - |\phi_1|^2 - |\phi_2|^2), \tag{20}$$

which have an obvious global SU(2) symmetry. These equations can again be obtained from the energy function generalizing (7) by introducing two scalars

$$
\begin{aligned}
E &= \frac{1}{2}\int_{S^2}\left\{\frac{(1+|z|^2)^2}{2}|F_{z\bar{z}}|^2 - 2\sum_{i=1}^2(|D_z\phi_i|^2 + |D_{\bar{z}}\phi_i|^2)\right.\\
&\quad \left. + \frac{2}{(1+|z|^2)^2}(1 - |\phi_1|^2 - |\phi_2|^2)^2\right\}\frac{i}{2}dz \wedge d\bar{z}.
\end{aligned}
\tag{21}
$$

Another way is to consider the action

$$S = \int dt \int_{S^2} \left\{ \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \Omega(i\bar{\phi}_1 D_t \phi_1 + i\bar{\phi}_2 D_t \phi_2 - V) \right.$$
$$\left. - \sum_{i=1}^{2} \left(|\tilde{D}_z \phi_i|^2 + |\tilde{D}_{\bar{z}} \phi_i|^2 \right) \right\} \frac{i}{2} dz \wedge d\bar{z}, \tag{22}$$

with

$$V = -\frac{g}{8} \left(|\phi_1|^2 + |\phi_2|^2 \right) + \frac{1}{2\kappa} \left(|\phi_1|^2 + |\phi_2|^2 \right)^2. \tag{23}$$

This class of theories was considered on the plane to study double-layer electron systems [10]. Now following the same procedure as above, it is straightforward to get the semilocal Popov equations.

A Bradlow-type constraint [13] on N can be obtained by integrating (20) over the sphere of radius $\sqrt{2}$,

$$\int_{S^2} \Omega \left(|\phi_1|^2 + |\phi_2|^2 \right) \frac{i}{2} dz \wedge d\bar{z} = 8\pi + 4\pi N, \tag{24}$$

which implies $N \geq -2$.

Let us now consider the first Eq. (19) which represents the semilocal nature of the equations. As before it can be written as

$$a_{\bar{z}} = -i \partial_{\bar{z}} \ln \phi_i \tag{25}$$

for both ϕ_1 and ϕ_2. Then

$$\partial_{\bar{z}} \ln \left(\frac{\phi_2}{\phi_1} \right) = 0, \tag{26}$$

so that the ratio

$$w(z) = \frac{\phi_2}{\phi_1} \tag{27}$$

is locally holomorphic. Moreover, (19) implies that ϕ_i's have zeros at discrete points [14]. Therefore $w(z)$ should be a rational function of z. We can again eliminate $a_{\bar{z}}$ and (20) becomes

$$\partial_z \partial_{\bar{z}} \ln |\phi_1|^2 = \partial_z \partial_{\bar{z}} \ln |\phi_2|^2 = \frac{2}{(1+|z|^2)^2} \left(1 - |\phi_1|^2 - |\phi_2|^2 \right), \tag{28}$$

away from zeros of ϕ_i. Eq. (28) is the same as the semilocal abelian Higgs vortex equations [6,7] on S^2 except the flipped sign, as it should be. It is illuminating to introduce

$$e^{u_i} = \frac{|\phi_i|^2}{(1+|z|^2)^2}, \tag{29}$$

and rewrite (28) as a Toda-type equation,

$$\partial_z \partial_{\bar{z}} u_i = -K_{ij} e^{u_j}, \qquad K = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}. \tag{30}$$

It is known to be integrable if the matrix K corresponds to the Cartan matrix of a Lie algebra [16,17], which is not the case for (30). It may be considered as the semilocal version of the Liouville equation. Although it is not integrable, some exact solutions have been obtained in the context of nonrelativistic self-dual Chern–Simons matter systems where semilocal solitons satisfy the same equation as (30) [11,12].

In the present context, we proceed as follows. Using (27), we eliminate $u_2(z)$,

$$\partial_z \partial_{\bar{z}} u_1 = -2 \left(1 + |w(z)|^2 \right) e^{u_1}, \tag{31}$$

where $w(z)$ is an arbitrary rational function of z. If $w(z) = c$ is a constant, which means ϕ_2 is proportional to ϕ_1, (31) reduces to the Popov equation of a single scalar. That is, (30) becomes the Liouville equation of which the exact solutions are well-known [15]. They are given by [2]

$$|\phi_1|^2 = \frac{(1+|z|^2)^2 |R'(z)|^2}{(1+|c|^2)(1+|R(z)|^2)^2}, \tag{32}$$

where $R(z)$ is a rational function of z. With an appropriate local gauge choice, ϕ_i and $a_{\bar{z}}$ are then

$$\phi_1 = \frac{(1+|z|^2) R'(z)}{\sqrt{1+|c|^2} (1+|R(z)|^2)},$$
$$\phi_2 = c\phi_1,$$
$$a_{\bar{z}} = i \left[\frac{R(z) \overline{R'(z)}}{1+|R(z)|^2} - \frac{z}{1+|z|^2} \right]. \tag{33}$$

If $R(z)$ is a ratio of polynomials of degree n, the vortex number is $N = 2n - 2$ which is even. Note that the vortex points of ϕ_1 and ϕ_2 are the same.

For later use, we express these Liouville solutions in a different form. Let us write $R(z)$ as a ratio of two polynomials $P(z)$ and $Q(z)$ which are generically of order n and have no common zeros,

$$R(z) = \frac{Q(z)}{P(z)}. \tag{34}$$

Then (32) becomes

$$|\phi_1|^2 = \frac{(1+|z|^2)^2 |P(z)Q'(z) - Q(z)P'(z)|^2}{(1+|c|^2)(|P(z)|^2 + |Q(z)|^2)^2}, \tag{35}$$

and a natural gauge choice would be

$$\phi_1 = \frac{(1+|z|^2)(P(z)Q'(z) - Q'(z)P(z))}{\sqrt{1+|c|^2} (|P(z)|^2 + |Q(z)|^2)},$$
$$\phi_2 = c\phi_1,$$
$$a_{\bar{z}} = i \left[\frac{P(z) \overline{P'(z)} + Q(z) \overline{Q'(z)}}{|P(z)|^2 + |Q(z)|^2} - \frac{z}{1+|z|^2} \right]. \tag{36}$$

This form may be directly obtained from (33) by multiplying $P^2/|P|^2$ to ϕ_i and adding $i\partial_{\bar{z}} \ln \bar{P}$ to $a_{\bar{z}}$. Of course this is a gauge transformation which removes singular phases at the vortex points at finite z.

For an arbitrary $w(z)$ which is not constant, it turns out to be useful to introduce

$$e^{u_1} = \frac{|w'|^2}{(1+|w|^2)^3} e^v. \tag{37}$$

Then (31) becomes an equation for v,

$$\partial_z \partial_{\bar{z}} v = \frac{2|w'|^2}{(1+|w|^2)^2} \left(\frac{3}{2} - e^v \right). \tag{38}$$

Changing the differentiation variable from z to w, we get

$$\partial_w \partial_{\bar{w}} v = \frac{2}{(1+|w|^2)^2} \left(\frac{3}{2} - e^v \right). \tag{39}$$

Note that this equation is identical to the Popov equation (6) with the ratio of radii $C = \sqrt{3/2}$. In other words, the semilocal Popov equation with equal radii reduces to the Popov equation with $C = \sqrt{3/2}$.

As shown in [1], (39) is not integrable and general solutions are not known. Nevertheless there is one known exact solution,

namely $e^v = 3/2$. Though it is a trivial solution by itself, it provides nontrivial solutions to the semilocal Popov equation under consideration. From (29) and (37), we see that

$$|\phi_1|^2 = \frac{3}{2} \frac{(1 + |z|^2)^2 |w'|^2}{(1 + |w|^2)^3} \tag{40}$$

solves the semilocal Popov equation (28). Given $|\phi_1|^2$, we can construct other fields. At this time $w(z)$ plays the role of $R(z)$ of (32) and is written as a ratio of two polynomials $P(z)$ and $Q(z)$ with no common zeros,

$$w(z) = \frac{Q(z)}{P(z)}. \tag{41}$$

Then we find

$$\begin{pmatrix} |\phi_1|^2 \\ |\phi_2|^2 \end{pmatrix} = \frac{3}{2} \frac{(1 + |z|^2)^2 |P(z)Q'(z) - Q(z)P'(z)|^2}{(|P(z)|^2 + |Q(z)|^2)^3} \begin{pmatrix} |P(z)|^2 \\ |Q(z)|^2 \end{pmatrix}. \tag{42}$$

With a local gauge choice as in (36), the solutions are then given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sqrt{\frac{3}{2}} \frac{(1 + |z|^2)(P(z)Q'(z) - Q(z)P'(z))}{(|P(z)|^2 + |Q(z)|^2)^{3/2}} \begin{pmatrix} P(z) \\ Q(z) \end{pmatrix},$$
$$a_{\bar{z}} = i \left[\frac{3}{2} \frac{P(z)\overline{P'(z)} + Q(z)\overline{Q'(z)}}{|P(z)|^2 + |Q(z)|^2} - \frac{z}{1 + |z|^2} \right]. \tag{43}$$

Note that ϕ_1 vanishes at the zeros of P and $PQ' - QP'$, while ϕ_2 vanishes at the zeros of Q and $PQ' - QP'$. Thus they share only part of the vortex points and this is genuinely different family of solutions from the Liouville solutions (36).

Let us count the vortex number of the solution (43) for a generic rational function w of degree n with P and Q being polynomials of order n. Since $PQ' - QP'$ is a polynomial of order $2n - 2$, ϕ_1 and ϕ_2 have $3n - 2$ zeros respectively. In addition, we have to consider the behavior at $z = \infty$ but it is easy to see that the scalar fields remain finite at $z = \infty$. Thus the vortex number is $N = 3n - 2$. We can also confirm the result by investigating the singularities of the solution. With the local gauge choice in (43), ϕ_i's are regular everywhere except at $z = \infty$. In particular, around a zero z_0 of ϕ_i, ϕ_i behave as $\phi_i \sim c(z - z_0)$. The expression (43) however is singular at $z = \infty$ since

$$\phi_i \sim c \frac{z^{3n-2}}{|z|^{3n-2}}. \tag{44}$$

This can be removed by a gauge transformation of winding number $3n - 2$ defined on an annulus on S^2 enclosing $z = \infty$. The winding number is then the first Chern number N, which is the same as the vortex number. If the order of P or Q is not n but less than n, there would be vortices sitting at $z = \infty$ but it is easy to see that the vortex number is still $N = 3n - 2$. Note that solutions with odd vortex numbers are possible in contrast to the Liouville solutions (36) which have even vortex numbers $N = 2n - 2$. Moreover the solutions with unit vorticity belong to this family.

As an example of the solution, choose $P(z) = c^n$ and $Q(z) = z^n$ with $c > 0$ for simplicity. Then

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sqrt{\frac{3}{2}} \frac{nc^n(1 + |z|^2)z^{n-1}}{(c^{2n} + |z|^{2n})^{3/2}} \begin{pmatrix} c^n \\ z^n \end{pmatrix}. \tag{45}$$

This is a circular symmetric solution with vortices at $z = 0$ with multiplicities $n - 1$ and $2n - 1$ for ϕ_1 and ϕ_2, respectively. The constant c may be considered as a parameter representing the size of the vortices. In terms of the coordinate $\xi = 1/z$, this becomes

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sqrt{\frac{3}{2}} \left(\frac{|\xi|}{\xi} \right)^{3n-2} \frac{nc^{-n}(1 + |\xi|^2)\xi^{n-1}}{(c^{-2n} + |\xi|^{2n})^{3/2}} \begin{pmatrix} \xi^n \\ c^{-n} \end{pmatrix}, \tag{46}$$

where $(|\xi|/\xi)^{3n-2}$ is the phase factor mentioned above and should be removed by a gauge transformation. The exponent $3n - 2$ is the first Chern number which should be the total vortex number. Indeed there are vortices at $z = \infty$ with multiplicities $2n - 1$ and $n - 1$ for ϕ_1 and ϕ_2, respectively, as seen in (46). The size of the vortices at $z = \infty$ is $1/c$. For $c = 1$, the solution has the reflection symmetry in the equator $|z| = 1$ if ϕ_1 and ϕ_2 are exchanged.

The solution with unit vorticity $N = 1$ and $c = 1$ is worth mentioning separately. With $P(z) = 1$ and $Q(z) = z$, the conformal factor $1 + |z|^2$ in the numerator is cancelled by the denominator in (45) and we get

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sqrt{\frac{3}{2}} \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 \\ z \end{pmatrix}. \tag{47}$$

This is precisely the well-known CP^1 lump configuration with unit size [18]. The scalars satisfy

$$|\phi_1|^2 + |\phi_2|^2 = \frac{3}{2}, \tag{48}$$

which defines S^3 fibered as a circle bundle over CP^1. Note that the radius of S^3 is not one but $\sqrt{3/2}$. The origin of this radius may be traced back to the connection to the Popov equation with ratio of radii $C = \sqrt{3/2}$ as seen in (39). From (43) the gauge potential of the solution is calculated as

$$a_{\bar{z}} = \frac{i}{2} \frac{z}{1 + |z|^2}, \tag{49}$$

and the corresponding field strength is

$$F_{z\bar{z}} = \frac{i}{(1 + |z|^2)^2}, \tag{50}$$

which can also be directly obtained from (20). The magnetic charge is uniform on S^2 and is that of the Dirac monopole with unit magnetic charge. Considering the form of (20), it is clear that the relation (48) is crucial to obtain the magnetic field with unit magnetic charge.

In conclusion, we have considered the semilocal version of Popov's vortex equations on S^2. Though they are not integrable, we were able to construct two families of exact solutions both of which involve arbitrary rational functions on S^2. One family of solutions is just a trivial embedding of the solutions of the Popov equations and they have even vortex numbers $N = 2n - 2$, where n is the degree of the rational function. The other family of solutions have vortex number $N = 3n - 2$ and they correspond to the constant solution with the Popov equation with the ratio of radii $C = \sqrt{3/2}$. The $N = 1$ solution with the reflection symmetry is given by the CP^1 lump configuration with unit size. Obviously there should be other solutions. This is evident if we recall that the solutions found here are just obtained from the constant solution of (39). A related issue is that we have even vortex numbers for even n. Thus for the vortex numbers $N = 6k - 2$ ($k \in \mathbb{Z}$), we have two distinct families of solutions: Liouville-type solutions are generated by rational functions of degree $3k$ while the other solutions, by those of degree $2k$. A natural question is then whether the solutions are smoothly connected to each other in the solution space through solutions not found here. To answer these questions it may be helpful to do the zero-mode analysis similar to that in [19]. In this paper, we have shown that the semilocal Popov equations are obtained as Bogomolny equations in the nonrelativistic Chern–Simons matter systems on S^2 in the presence of a constant

external magnetic field. The Popov equations however are originally discussed as a reduction of Yang–Mills instanton equations with noncompact gauge group. It would be interesting to find such an interpretation for the semilocal equations for example as an embedding in nonabelian Popov equations.

Acknowledgements

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