

Emergent geometry from quantized spacetime

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We examine the picture of emergent geometry arising from a mass-deformed matrix model. Because of the mass deformation, a vacuum geometry turns out to be a constant curvature spacetime such as d -dimensional sphere and (anti-)de Sitter spaces. We show that the mass-deformed matrix model giving rise to the constant curvature spacetime can be derived from the d -dimensional Snyder algebra. The emergent geometry beautifully confirms all the rationale inferred from the algebraic point of view that the d -dimensional Snyder algebra is equivalent to the Lorentz algebra in $(d + 1)$ -dimensional flat spacetime. For example, a vacuum geometry of the mass-deformed matrix model is completely described by a G -invariant metric of coset manifolds G/H defined by the Snyder algebra. We also discuss a nonlinear deformation of the Snyder algebra.

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I. INTRODUCTION

The wave-particle duality in quantum mechanics is a remarkable consequence of particle dynamics in quantum phase space defined by $[x^i, p_k] = i\hbar\delta_k^i$. In a classical world with $\hbar = 0$, the wave and the particle are completely independent with exclusive properties. But, when $\hbar \neq 0$, the particle phase space becomes noncommutative (NC). As a result, the particle dynamics in the NC phase space reveals a novel duality such that the wave and the particle are no longer exclusive entities but complementary aspects of the same physical reality. That is, they are unified into a single entity with a dual nature in the quantum world.

A NC spacetime arises from endowing spacetime with a symplectic structure $B = \frac{1}{2}B_{ab}dy^a \wedge dy^b$ and then quantizing the spacetime with its Poisson structure $\theta^{ab} \equiv (B^{-1})^{ab}$, treating it as a quantum phase space described by

$$[y^a, y^b]_\star = i\theta^{ab}. \quad (1.1)$$

Just as the wave-particle duality emerges in the NC phase space (quantum mechanics) which has never been observed in classical physics, the NC spacetime (1.1) may also introduce a new kind of duality between physical or mathematical entities. So an interesting question is what kind of duality arises from the quantization of spacetime triggered by the θ -deformation (1.1). We will see that it is the gauge/gravity duality as recently demonstrated in [1–4].

The gauge/gravity duality in NC spacetime is realized in the context of emergent gravity where spacetime geometry emerges as a collective phenomenon of underlying microscopic degrees of freedom defined by NC gauge fields. Remarkably the emergent gravity reveals a noble picture

about the origin of spacetime, dubbed as emergent spacetime, which is radically different from any previous physical theory all of which describe what happens in a given spacetime. The emergent gravity has been addressed, according to their methodology, from two facets of quantum field theories: NC field theories [2–8] and large N matrix models [9–14]. But it turns out [3,4] that the two approaches are intrinsically related to each other. In particular, the AdS/CFT correspondence [12] has been known as a typical example of the emergent gravity based on a large N matrix model (or gauge theory) which has been extensively studied for a decade. Furthermore, the emergent gravity has also been suggested to resolve the cosmological constant problem and dark energy [15,16]. Nevertheless, there has been little understanding about why and when the gravity in higher dimensions can emerge from some kind of lower dimensional quantum field theory and what the first (dynamical) principle is for the emergent spacetime.

The issues for the emergent gravity seem to be more accessible from the approach based on NC geometry. See a recent review, Ref. [17], for various issues on emergent gravity. In usual commutative spacetime, a gauge theory such as the electromagnetism is very different from the gravity described by general relativity since the former is based on an internal symmetry while the latter is formulated with the spacetime symmetry. A remarkable property in the NC spacetime (1.1) is that the internal symmetry in gauge theory turns into the spacetime symmetry. This can be seen from the fact that translations in NC directions are an inner automorphism of NC \star -algebra \mathcal{A}_θ , i.e., $e^{ik \cdot y} \star \hat{f}(y) \star e^{-ik \cdot y} = \hat{f}(y + \theta \cdot k)$ for any $\hat{f}(y) \in \mathcal{A}_\theta$ or, in its infinitesimal form,

$$-i[B_{ab}y^b, \hat{f}(y)]_\star = \partial_a \hat{f}(y). \quad (1.2)$$

To be specific, let us consider a $U(1)$ bundle supported on a symplectic manifold (M, B) . Because the symplectic

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structure $B: TM \rightarrow T^*M$ is nondegenerate at any point $y \in M$, we can invert this map to obtain the map $\theta \equiv B^{-1}: T^*M \rightarrow TM$. This cosymplectic structure $\theta \in \Lambda^2 TM$ is called the Poisson structure of M which defines a Poisson bracket $\{\cdot, \cdot\}_\theta: C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$. The NC spacetime (1.1) is then obtained by quantizing the symplectic manifold (M, B) with the Poisson structure $\theta = B^{-1}$. An important point is that the gauge symmetry acting on $U(1)$ gauge fields as $A \rightarrow A + d\phi$ is a diffeomorphism symmetry generated by a vector field X satisfying $\mathcal{L}_X B = 0$, which is known as the symplectomorphism in symplectic geometry. In other words, $U(1)$ gauge transformations are generated by the Hamiltonian vector field X_ϕ satisfying $\iota_{X_\phi} B + d\phi = 0$ and the action of X_ϕ on a smooth function $f(y) \in C^\infty(M)$ is given by

$$\delta f(y) \equiv X_\phi(f)(y) = \{f, \phi\}_\theta(y). \quad (1.3)$$

Therefore, the gauge symmetry (1.3) on the symplectic manifold (M, B) should be regarded as a spacetime symmetry rather than an internal symmetry [2].

The above reasoning implies that $U(1)$ gauge fields in NC spacetime can be realized as a spacetime geometry like as the gravity in general relativity [2–4]. In general relativity the equivalence principle beautifully explains why the gravitational force has to manifest itself as a spacetime geometry. If the gauge/gravity duality is realized in NC spacetime, a natural question is what is the corresponding equivalence principle for the geometrization of the electromagnetic force. Because the geometrical framework of NC spacetime is apparently based on the symplectic geometry in sharp contrast to the Riemannian geometry, the question should be addressed in the context of the symplectic geometry rather than the Riemannian geometry. Remarkably it turns out that NC spacetime admits a novel form of the equivalence principle such that there “always” exists a coordinate transformation to locally eliminate the electromagnetic force [4]. This geometrization of the electromagnetism is inherent as an intrinsic property in the symplectic geometry known as the Darboux theorem or the Moser lemma [18]. As a consequence, the electromagnetism in NC spacetime can be realized as a geometrical property of spacetime like gravity.

This noble form of the equivalence principle can be understood as follows [4]. The presence of fluctuating gauge fields on a symplectic manifold (M, B) appears as a deformation of the symplectic manifold (M, B) such that the resulting symplectic structure is given by $\omega_1 \equiv B + F$ where $F = dA$. Because the original symplectic structure $\omega_0 = B$ is a nondegenerate and closed two-form, the associated map $B^b: TM \rightarrow T^*M$ is a vector bundle isomorphism. Therefore, there exists a natural pairing $\Gamma(TM) \rightarrow \Gamma(T^*M): X \mapsto B^b(X) = \iota_X B$ between C^∞ -sections of tangent and cotangent bundles. Because the $U(1)$ gauge field A on M only appears as the combination $\omega_1 = B + dA$, one may identify the connection A with an element in $\Gamma(T^*M)$ such that

$$\iota_X B + A = 0. \quad (1.4)$$

The identification (1.4) is defined up to symplectomorphisms or equivalently $U(1)$ gauge transformations, that is, $X \sim X + X_\phi \Leftrightarrow A \sim A + d\phi$ where $\iota_{X_\phi} B = d\phi$. Using the Cartan’s magic formula $\mathcal{L}_X = d\iota_X + \iota_X d$ and so $[\mathcal{L}_X, d] = 0$, it is easy to see that $\omega_1 = B + dA = B - \mathcal{L}_X B$ and $d\omega_1 = 0$ because of $dB = 0$. This means that a smooth family $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ of symplectic structures joining ω_0 to ω_1 is all deformation-equivalent and there exists a map $\phi: M \times \mathbf{R} \rightarrow M$ as a flow—a one-parameter family of diffeomorphisms—generated by the vector field X_t satisfying $\iota_{X_t} \omega_t + A = 0$ such that $\phi_t^*(\omega_t) = \omega_0$ for all $0 \leq t \leq 1$.

This can be explicitly checked by considering a local Darboux chart $(U; y^1, \dots, y^{2n})$ centered at $p \in M$ and valid on the neighborhood U such that $\omega_0|_U = \frac{1}{2} B_{ab} dy^a \wedge dy^b$ where B_{ab} is a constant symplectic matrix of rank $2n$. Now consider a flow $\phi_t: U \times [0, 1] \rightarrow M$ generated by the vector field X_t satisfying (1.4). Under the action of ϕ_ϵ with an infinitesimal ϵ , one finds that a point $p \in U$ whose coordinate is y^a is mapped to $\phi_\epsilon(y) \equiv x^a(y) = y^a + \epsilon X^a(y)$. Using the inverse map $\phi_\epsilon^{-1}: x^a \mapsto y^a(x) = x^a - \epsilon X^a(x)$, the symplectic structure $\omega_0|_U = \frac{1}{2} B_{ab}(y) dy^a \wedge dy^b$ can be expressed as

$$\begin{aligned} (\phi_\epsilon^{-1})^*(\omega_0|_y) &= \frac{1}{2} B_{ab}(x - \epsilon X) d(x^a - \epsilon X^a) \wedge d(x^b - \epsilon X^b) \\ &\approx \frac{1}{2} [B_{ab} - \epsilon X^\mu (\partial_\mu B_{ab} + \partial_b B_{\mu a} + \partial_a B_{b\mu}) + \epsilon (\partial_a (B_{b\mu} X^\mu) - \partial_b (B_{a\mu} X^\mu))] dx^a \wedge dx^b \equiv B + \epsilon F, \end{aligned} \quad (1.5)$$

where $A_a(x) = B_{a\mu}(x) X^\mu(x)$ or $\iota_X B + A = 0$ and $dB = 0$ was used for the vanishing of the second term. Equation (1.5) can be rewritten as $\phi_\epsilon^*(B + \epsilon F) = B$, which means that the electromagnetic force $F = dA$ can always be eliminated by a local coordinate transformation generated by the vector field X satisfying Eq. (1.4).

Surprisingly, it is easy to understand how the Darboux theorem in symplectic geometry manifests itself as a novel form of the equivalence principle such that the electromagnetism in NC spacetime can be regarded as a theory of gravity [2–4]. It is well known that, for a given Poisson algebra $(C^\infty(M), \{\cdot, \cdot\}_\theta)$, there exists a natural map

$C^\infty(M) \rightarrow TM: f \mapsto X_f$ between smooth functions in $C^\infty(M)$ and vector fields in TM such that

$$X_f(g) = \{g, f\}_\theta \quad (1.6)$$

for any $g \in C^\infty(M)$. Indeed the assignment (1.6) between a Hamiltonian function f and the corresponding Hamiltonian vector field X_f is the Lie algebra homomorphism in the sense

$$X_{\{f,g\}_\theta} = -[X_f, X_g] \quad (1.7)$$

where the right-hand side represents the Lie bracket between the Hamiltonian vector fields.

The correspondence (1.6) between the Poisson algebra $(C^\infty(M), \{\cdot, \cdot\}_\theta)$ and vector fields in $\Gamma(TM)$ can be generalized to the NC \star algebra $(\mathcal{A}_\theta, [\cdot, \cdot]_\star)$ by considering an adjoint operation of NC gauge fields $\hat{D}_a(y) \in \mathcal{A}_\theta$ as follows:

$$\begin{aligned} ad_{\hat{D}_a}[\hat{f}](y) &\equiv -i[\hat{D}_a(y), \hat{f}(y)]_\star \\ &= -\theta^{\mu\nu} \frac{\partial D_a(y)}{\partial y^\nu} \frac{\partial f(y)}{\partial y^\mu} + \dots \\ &\equiv V_a[f](y) + \mathcal{O}(\theta^3). \end{aligned} \quad (1.8)$$

The leading term in Eq. (1.8) exactly recovers the vector fields in Eq. (1.6) and the vector field $V_a(y) = V_a^\mu(y) \times \frac{\partial}{\partial y^\mu} \in \Gamma(TM_y)$ takes values in the Lie algebra of volume-preserving diffeomorphisms since $\partial_\mu V_a^\mu = 0$ by definition. But it can be shown [4] that the vector fields $V_a \in \Gamma(TM)$ are related to the orthonormal frames (vielbeins) E_a by $V_a = \lambda E_a$ where $\lambda^2 = \det V_a^\mu$. Therefore, we see that the Darboux theorem in symplectic geometry implements a deep principle to realize a Riemannian manifold as an emergent geometry from NC gauge fields through the correspondence (1.8) whose metric is given by [3,4]

$$ds^2 = g_{ab} E^a \otimes E^b = \lambda^2 g_{ab} V_a^\mu V_b^\nu dy^\mu \otimes dy^\nu, \quad (1.9)$$

where $E^a = \lambda V^a \in \Gamma(T^*M)$ are dual oneforms.

If a coordinate transformation is generated by a Hamiltonian vector field X_ϕ satisfying $\iota_{X_\phi} B = d\phi$ or $X_\phi^\mu = \theta^{\mu\nu} \partial_\nu \phi$, the symplectic structure remains intact as can easily be checked from Eq. (1.5). It should be the case since the symplectomorphism generated by the Hamiltonian vector field is equal to the $U(1)$ gauge transformation. So let us look at a response of the metric (1.9) under the coordinate transformation in the symplectomorphism or the $U(1)$ gauge transformation. Using the definition of the vector fields in Eq. (1.8), one can rewrite the inverse metric of Eq. (1.9) as follows:

$$\begin{aligned} \left(\frac{\partial}{\partial s}\right)^2 &= g^{ab} E_a \otimes E_b = \lambda^{-2} g^{ab} V_a^\mu V_b^\nu \partial_\mu \otimes \partial_\nu \\ &\equiv \mathbb{G}^{\mu\nu} \partial_\mu \otimes \partial_\nu = \theta^{\mu a} \theta^{\nu b} G_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^a} \frac{\partial x^\beta}{\partial y^b} \partial_\mu \otimes \partial_\nu \\ &= \theta^{\mu a} \theta^{\nu b} (G_{ab} + \mathcal{L}_{X_\phi} G_{ab}) \partial_\mu \otimes \partial_\nu, \end{aligned} \quad (1.10)$$

where $G_{ab} = -\lambda^{-2} B_{ac} g^{cd} B_{db}$ and $x^\alpha(y) = y^\alpha + X_\phi^\alpha(y)$. For consistency the metric (1.10) should remain intact under the $U(1)$ gauge transformation or the symplectomorphism since it does not change the symplectic structure. It is easy to see that this consistency condition is equivalent to require $\mathcal{L}_{X_\phi} G_{ab} = 0$ since $V_a^\mu = \delta_a^\mu$ in this case and so $\lambda^2 = \det V_a^\mu = 1$. Therefore, we get a consistent result that the $U(1)$ gauge transformation or the symplectomorphism corresponds to a Killing symmetry and the emergent metric (1.9) does not change, i.e., $\mathbb{G}_{\mu\nu} = g_{\mu\nu}$.

As emphasized by Elvang and Polchinski [19], the emergence of gravity requires the emergence of spacetime itself. That is, spacetime is not given *a priori* but defined by “spacetime atoms,” NC gauge fields in our case, in quantum gravity theory. It should be required for consistency that the entire spacetime including a flat spacetime has to be emergent from NC gauge fields. In other words, the emergent gravity should necessarily be background independent where any spacetime structure is not *a priori* assumed but defined from the theory. Let us elucidate using the relation between a matrix model and a NC gauge theory [11,20,21] how the emergent gravity based on the NC geometry achieves the background independence [3,4].

Consider the zero-dimensional IKKT matrix model [10] whose action is given by

$$S_{\text{IKKT}} = -\frac{1}{4} \text{Tr}([X_a, X_b][X^a, X^b]). \quad (1.11)$$

Because the action (1.11) is zero-dimensional, it does not assume the prior existence of any spacetime structure. There are only a bunch of $N \times N$ Hermitian matrices $X^a (a = 1, \dots, 2n)$ which are subject to a couple of algebraic relations given by

$$[X_a, [X^a, X^b]] = 0, \quad (1.12)$$

$$[X^a, [X^b, X^c]] + [X^b, [X^c, X^a]] + [X^c, [X^a, X^b]] = 0. \quad (1.13)$$

In order to consider fluctuations around a vacuum of the matrix theory (1.11), first one has to specify the vacuum of the theory where all fluctuations are supported. Of course, the vacuum solution itself should also satisfy the Eqs. (1.12) and (1.13). Suppose that the vacuum solution is given by $X_{\text{vac}}^a = y^a$. In the limit $N \rightarrow \infty$, the Moyal NC space defined by Eq. (1.1) where θ^{ab} is a constant matrix of rank $2n$ definitely satisfies the equations of motion (1.12) as well as the Jacobi identity (1.13). Furthermore, in this case, the matrix algebra $(M_N, [\cdot, \cdot])$ defining the action

(1.11) can be mapped to the NC \star algebra $(\mathcal{A}_\theta, [\cdot, \cdot]_\star)$ defined by the NC space (1.1) [11]. To be explicit, let us expand the large N matrices $X^a \equiv \theta^{ab} \hat{D}_b$ around the Moyal vacuum (1.1) as follows:

$$\hat{D}_a(y) = B_{ab}y^b + \hat{A}_a(y). \quad (1.14)$$

Note that

$$\begin{aligned} -i[\hat{D}_a(y), \hat{D}_b(y)]_\star &= \partial_a \hat{A}_b(y) - \partial_b \hat{A}_a(y) \\ &\quad - i[\hat{A}_a(y), \hat{A}_b(y)]_\star - B_{ab} \\ &= \hat{F}_{ab}(y) - B_{ab}. \end{aligned} \quad (1.15)$$

Then the IKKT matrix model (1.11) becomes the NC $U(1)$ gauge theory in $2n$ dimensions [11,21]

$$\hat{S}_{\text{NC}} = \frac{1}{4g_{\text{YM}}^2} \int d^{2n}y G^{ac} G^{bd} (\hat{F} - B)_{ab} \star (\hat{F} - B)_{cd}, \quad (1.16)$$

where $G^{ab} = \theta^{ac} \theta^{bc}$ and $\text{Tr} \rightarrow \int \frac{d^{2n}y}{(2\pi)^n |\text{Pf} \theta|}$ and we have recovered a $2n$ -dimensional gauge coupling constant g_{YM}^2 [20].

According to the correspondence (1.8), the NC gauge fields $\hat{D}_a(y) \in \mathcal{A}_\theta$ in Eq. (1.14) are mapped to (generalized) vector fields $\hat{V}_a(y) \equiv ad_{\hat{D}_a}(y)$ as an inner derivation in \mathcal{A}_θ [2–4]. In particular, we have the property

$$[ad_{\hat{D}_a}, ad_{\hat{D}_b}]_\star = ad_{\hat{F}_{ab}} = [\hat{V}_a, \hat{V}_b]_\star, \quad (1.17)$$

where $[\hat{V}_a, \hat{V}_b]_\star = [V_a, V_b] + \mathcal{O}(\theta^3)$ is a generalization of the Lie bracket to the generalized vector fields in Eq. (1.8). Using the maps in Eqs. (1.8) and (1.17), one can further deduce that

$$[ad_{\hat{D}_a}, [ad_{\hat{D}_b}, ad_{\hat{D}_c}]_\star]_\star = ad_{\hat{D}_a \hat{F}_{bc}} = [\hat{V}_a, [\hat{V}_b, \hat{V}_c]_\star]_\star. \quad (1.18)$$

Using the relation (1.18), one can easily show that the equations of motion for NC gauge fields derived from the action (1.16) are mapped to the geometric equations for (generalized) vector fields defined by Eq. (1.8) [4]:

$$\hat{D}_{[a} \hat{F}_{bc]} = 0 \Leftrightarrow [\hat{V}_{[a}, [\hat{V}_b, \hat{V}_c]_\star]_\star = 0, \quad (1.19)$$

$$\hat{D}^a \hat{F}_{ab} = 0 \Leftrightarrow [\hat{V}^a, [\hat{V}_a, \hat{V}_b]_\star]_\star = 0. \quad (1.20)$$

To be specific, if one confines to the leading order in Eq. (1.8) where one recovers usual vector fields, the Jacobi identity (1.13) [or the Bianchi identity (1.19) for NC gauge fields] is equivalent to the first Bianchi identity for Riemann tensors, i.e., $R_{[abc]d} = 0$, and the equations of motion (1.12) for $N \times N$ matrices or (1.20) for NC gauge fields are mapped to the Einstein equations, $R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T_{ab}$, for the emergent metric (1.9) [4].

Though the emergence of Einstein gravity from NC gauge fields is shown after some nontrivial technical com-

putations [4], it can easily be verified for the self-dual sector without any further computation. First notice the following equality directly derived from Eq. (1.17)

$$\hat{F}_{ab} = \pm \frac{1}{2} \varepsilon_{ab}{}^{cd} \hat{F}_{cd} \quad \cong \quad [\hat{V}_a, \hat{V}_b]_\star = \pm \frac{1}{2} \varepsilon_{ab}{}^{cd} [\hat{V}_c, \hat{V}_d]_\star. \quad (1.21)$$

Because $[\hat{V}_a, \hat{V}_b]_\star = [V_a, V_b] + \mathcal{O}(\theta^3)$, the right-hand side of Eq. (1.21) in a commutative, i.e. $\mathcal{O}(\theta)$, limit describes self-dual and Ricci-flat four-manifolds as was rigorously proved in [1,4,22]. In other words, the self-dual Einstein gravity arises from the leading order of self-dual NC gauge fields [7].

One can trace the emergent metric (1.9) back to see where the flat spacetime comes from. It turns out [16] that the flat spacetime is emergent from the uniform condensation of gauge fields giving rise to the NC spacetime (1.1). This is a tangible difference from Einstein gravity where the flat spacetime is a completely empty space. Furthermore, since gravity emerges from NC gauge fields, the parameters, g_{YM}^2 and $|\theta|$, defining a NC gauge theory should be related to the Newton constant G in emergent gravity. A simple dimensional analysis shows that $\frac{G\hbar^2}{c^2} \sim g_{\text{YM}}^2 |\theta|$. In four dimensions, this relation immediately leads to the fact that the energy density of the vacuum (1.1) is $\rho_{\text{vac}} \sim |B_{ab}|^2 \sim M_P^4$ where $M_P = (8\pi G)^{-1/2} \sim 10^{18}$ GeV is the Planck mass. Therefore, the emergent gravity reveals a remarkable picture that the huge Planck energy M_P is actually used to generate a flat spacetime. It is very surprising but should be expected from the background independence of the emergent gravity that a flat spacetime is not free gratis but a result of Planck energy condensation in a vacuum. Hence the vacuum energy does not gravitate unlike Einstein gravity. It was argued in [4,16] that this emergent spacetime picture will be essential to resolving the cosmological constant problem, to understanding the nature of dark energy, and to explaining why gravity is so weak compared to other forces.

In this paper we will generalize the picture of emergent geometry to the case with a nontrivial vacuum geometry, especially, a constant curvature spacetime. This kind of emergent geometry will arise from a mass-deformed matrix model. The subsequent parts of this paper will be organized as follows.

In Sec. II, we will consider the matrix model of $\text{SO}(3-p, p)$ Lie algebra with $p = 0, 1, 2$ which is the matrix version of Maxwell-Chern-Simons theory or massive Chern-Simons theory [23]. We show that either compact or noncompact (fuzzy) Riemann surfaces such as a two-dimensional sphere and (anti-)de Sitter spaces are emergent from the matrix model. A well-known example of quantized compact Riemann surfaces is a fuzzy sphere [24]. We discuss how a nonlinear deformation of the underlying Lie algebra can trigger a topology change of the Riemann surfaces [25].

In Sec. III, we will generalize the matrix model of two-dimensional Riemann surfaces to higher dimensions. The emergent geometry in higher dimensions is deduced from a mass-deformed IKKT matrix model [26]. Because of the mass deformation, a vacuum geometry is no longer flat but a constant curvature spacetime such as a d -dimensional sphere and (anti-)de Sitter spaces. We show that the mass-deformed matrix model giving rise to the constant curvature spacetime can be derived from the d -dimensional Snyder algebra [27]. The emergent gravity beautifully confirms all the rationale inferred from the algebraic point of view that the d -dimensional Snyder algebra is equivalent to the Lorentz algebra in $(d + 1)$ -dimensional *flat* spacetime. We also discuss a nonlinear deformation of the Snyder algebra.

In Sec. IV, we show that a vacuum geometry of the mass-deformed matrix model is completely described by a G -invariant metric of coset manifolds G/H [28] defined by the Snyder algebra. We thus advocate the picture that the geometrical aspects of emergent gravity for the mass-deformed matrix model can be nicely captured by the equivalence between the d -dimensional Snyder algebra and the $(d + 1)$ -dimensional Lorentz algebra. Finally, we conclude with several remarks about the significance of emergent geometry based on the results we have obtained.

In the Appendix, it is shown that the two-dimensional Snyder algebra is precisely equal to the three-dimensional $SO(3 - p, p)$ Lie algebra in Sec. II.

II. TWO-DIMENSIONAL MANIFOLDS FROM MATRIX MODEL

Consider the following master matrix action:

$$S_M = \text{Tr} \left(\frac{g_{\text{YM}}^2}{2} P_A P^A - \lambda P_A X^A + \frac{i\kappa}{3!} \varepsilon_{ABC} X^A [X^B, X^C] \right), \quad (2.1)$$

where $\lambda = \kappa g_{\text{YM}}^2$ and $A, B, \dots = 1, 2, 3$. The equations of motion are read as

$$P_A = \frac{i}{2g_{\text{YM}}^2} \varepsilon_{ABC} [X^B, X^C], \quad (2.2)$$

$$P^A = \kappa X^A. \quad (2.3)$$

Substituting Eq. (2.2) into the master action (2.1) leads to the matrix version of Maxwell-Chern-Simons action [14]

$$S_{\text{MCS}} = -\frac{1}{g_{\text{YM}}^2} \text{Tr} \left(\frac{1}{4} [X^A, X^B]^2 + \frac{i\lambda}{3} \varepsilon_{ABC} X^A [X^B, X^C] \right), \quad (2.4)$$

while Eq. (2.3) leads to the matrix version of massive Chern-Simons theory

$$S_{\text{mCS}} = \kappa \text{Tr} \left(\frac{i}{3!} \varepsilon_{ABC} X^A [X^B, X^C] - \frac{\lambda}{2} X_A X^A \right). \quad (2.5)$$

Thus we establish the matrix version of the duality between topologically massive electrodynamics and a self-dual massive model [23]. Therefore, it is enough to solve either Eq. (2.4) or Eq. (2.5) to get physical spectra.

From the action (2.5), one can see that the equations of motion are given by the $SO(3 - p, p)$ Lie algebra with $p = 0, 1, 2$

$$[X^A, X^B] = -i\lambda \varepsilon^{AB}{}_C X^C. \quad (2.6)$$

We are interested in deriving a two-dimensional manifold from the Lie algebra (2.6) where the Casimir invariant is given by¹

$$g_{AB} X^A X^B \equiv (-)^{\#} R^2. \quad (2.7)$$

We will consider three cases depending on the choice of metric g_{AB} : (I) $g_{AB} = \text{diag}(1, 1, 1)$ with $\# = 0$, (II) $g_{AB} = \text{diag}(-1, 1, 1)$ with $\# = 0$, and (III) $g_{AB} = \text{diag}(-1, 1, -1)$ with $\# = 1$. They describe a two-dimensional manifold M of radius R given by Eq. (2.7) in the classical limit: (I) sphere S^2 , (II) de Sitter space dS_2 , and (III) anti-de Sitter space AdS_2 , which may be represented by the cosets $SO(3)/SO(2)$, $SO(2, 1)/SO(1, 1)$, and $SO(1, 2)/SO(1, 1)$, respectively. See Sec. IV for the coset space realization of two-dimensional hypersurface M .

We will first clarify how the Lie algebra (2.6) arises from the quantization of two-dimensional (orientable) manifolds [25, 29]. Let M be an orientable two-manifold and $\omega \in \Omega^2(M)$ a volume form. Then ω is nondegenerate (since $\omega \neq 0$ everywhere) and obviously closed, i.e., $d\omega = 0$. Therefore, any orientable two-manifold M is a symplectic manifold. A unique feature in two dimensions is that a symplectic two-form is just a volume form. Hence any two volume forms ω and ω' on a two-dimensional manifold M , defining the same orientation and having the same total volume, will be related by an exact two-form; $\omega' = \omega + dA$. This is a well-known result on volume forms due to Moser [18]. (For a noncompact manifold, we would need to introduce a compact support of symplectic form.) In particular, every closed symplectic two-manifold is determined up to local isotopic deformations by its genus and total volume. This implies that a nontrivial deformation of two-dimensional manifolds will be encoded only in volume and topology changes up to volume-preserving metric

¹It is well known that the Lie algebra (2.6) can be represented by differential operators as tangent vectors on some manifold, which is actually the result we want to realize using the map (1.8). Without imposing the Casimir invariant (2.7), one gets a three-dimensional manifold, e.g., S^3 from $SU(2)$ algebra. In our case, imposing Eq. (2.7), we will get a two-dimensional manifold instead. As will be discussed in the Appendix, the $SO(3 - p, p)$ Lie algebra in Eq. (2.6) will then be interpreted as the Lorentz algebra of an ambient three-dimensional space, which is precisely the three-dimensional version of Eq. (3.21).

(shape) deformations. We will see that this feature still persists in a two-dimensional NC manifold.

To begin with, let us introduce a local Darboux chart $(U; y^1, y^2)$ centered at $p \in M$ and valid on a neighborhood U such that $\omega|_U = \frac{1}{2} B_{ab} dy^a \wedge dy^b = -dy^1 \wedge dy^2$. The Poisson bracket for $f, g \in C^\infty(M)$ is then defined in terms of local coordinates y^a ($a = 1, 2$)

$$\{f, g\}_\theta = \theta^{ab} \frac{\partial f}{\partial y^a} \frac{\partial g}{\partial y^b}, \quad (2.8)$$

where $\theta^{12} = 1$. We will consider the two-dimensional manifold M as a hypersurface embedded in $\mathbb{R}^{3-p,p}$ and described by $L^A = L^A(y)$, $A = 1, 2, 3$, satisfying the relation (2.7). For example, one can choose $y^a = (\cos\theta, \varphi)$ for S^2 , $y^a = (\sinh t, \varphi)$ for dS_2 , and $y^a = (t, \sinh x)$ for AdS_2 as follows.

(I) S^2 of unit radius:

$$\begin{aligned} L^1 &= \sqrt{1 - y^2} \cos\varphi, & L^2 &= \sqrt{1 - y^2} \sin\varphi, \\ L^3 &= y, \end{aligned} \quad (2.9)$$

where $y = \cos\theta$.

(II) dS_2 of unit radius:

$$\begin{aligned} L^1 &= -y, & L^2 &= \sqrt{1 + y^2} \sin\varphi, \\ L^3 &= \sqrt{1 + y^2} \cos\varphi, \end{aligned} \quad (2.10)$$

where $y = \sinh t$.

(III) AdS_2 of unit radius:

$$\begin{aligned} L^1 &= \sqrt{1 + y^2} \cosh t, & L^2 &= -y, \\ L^3 &= \sqrt{1 + y^2} \sinh t, \end{aligned} \quad (2.11)$$

where $y = \sinh x$.

It is easy to see that the above coordinate system $L^A(y) \in C^\infty(M)$ satisfies a linear Poisson structure under the Poisson bracket (2.8)

$$\{L^A, L^B\}_\theta = -\varepsilon^{AB}{}_C L^C. \quad (2.12)$$

The coordinate system $L_A(y) = g_{AB} L^B(y) \in C^\infty(M)$ satisfying the constraint (2.7) can be mapped to vector fields $V_A^{(0)}(y) = V_A^{(0)a}(y) \frac{\partial}{\partial y^a} \in \Gamma(TM)$ according to Eq. (1.6) as

$$V_A^{(0)} = \theta^{ab} \frac{\partial L_A}{\partial y^b} \frac{\partial}{\partial y^a}. \quad (2.13)$$

The two-dimensional metric on M is then determined by the vector fields (2.13) where the inverse metric is given by

$$\mathbb{G}_{(0)}^{ab} = (\det \mathbb{G}_{(0)}^{(0)})^{-1} g^{AB} V_A^{(0)a} V_B^{(0)b} \quad (2.14)$$

and so the two-dimensional (emergent) metric reads as

$$\mathbb{G}_{ab}^{(0)} = g_{AB} \frac{\partial L^A}{\partial y^a} \frac{\partial L^B}{\partial y^b}. \quad (2.15)$$

One can easily check that the resulting metric $ds^2 = \mathbb{G}_{ab}^{(0)} dy^a dy^b$ is equivalent to the induced metric from the standard flat metric $ds^2 = g_{AB} dL^A dL^B$ on $\mathbb{R}^{3-p,p}$:

$$(I): ds^2 = \frac{dy^2}{1 - y^2} + (1 - y^2) d\varphi^2 = d\theta^2 + \sin^2\theta d\varphi^2, \quad (2.16)$$

$$\begin{aligned} (II): ds^2 &= -\frac{dy^2}{1 + y^2} + (1 + y^2) d\varphi^2 \\ &= -dt^2 + \cosh^2 t d\varphi^2, \end{aligned} \quad (2.17)$$

$$\begin{aligned} (III): ds^2 &= -(1 + y^2) dt^2 + \frac{dy^2}{1 + y^2} \\ &= -\cosh^2 x dt^2 + dx^2. \end{aligned} \quad (2.18)$$

As it should be, we see here that the metric (2.15) determined by the vector fields in Eq. (2.13) is just the induced metric on a two-dimensional surface M embedded in $\mathbb{R}^{3-p,p}$. Let us now consider a generic fluctuation of the surface M around the vacuum geometry (I)–(III) described by

$$X^A(y) = L^A(y) + A^A(y). \quad (2.19)$$

The fluctuating coordinate system (2.19) satisfies the following Poisson bracket relation:

$$\{X^A, X^B\}_\theta = -\varepsilon^{AB}{}_C X^C + F^{AB}, \quad (2.20)$$

where

$$F^{AB} = \{L^A, A^B\}_\theta - \{L^B, A^A\}_\theta + \{A^A, A^B\}_\theta + \varepsilon^{AB}{}_C A^C. \quad (2.21)$$

Note that the field strength F^{AB} in Eq. (2.20) cannot be arbitrary since the Poisson algebra (2.20) should satisfy the Jacobi identity, $\varepsilon_{ABC} \{X^A, \{X^B, X^C\}_\theta\}_\theta = 0$. This constraint can be solved by taking the field strength F^{AB} in Eq. (2.20) as the form

$$F^{AB}(X) = \varepsilon^{ABC} \frac{\partial F(X)}{\partial X^C} \quad (2.22)$$

with an arbitrary smooth function $F(X)$ defined in $\mathcal{M} = \mathbb{R}^{3-p,p}$ because we have

$$\begin{aligned} \frac{1}{2} \varepsilon_{ABC} \{X^A, \{X^B, X^C\}_\theta\}_\theta &= \left\{ X^A, \frac{\partial F(X)}{\partial X^A} \right\}_\theta \\ &= \frac{\partial^2 F(X)}{\partial X^A \partial X^B} \{X^A, X^B\}_\theta = 0. \end{aligned}$$

Then the Poisson bracket relation (2.20) can be written as follows:

$$\{X^A, X^B\}_\theta = \varepsilon^{ABC} \frac{\partial G(X)}{\partial X^C}, \quad (2.23)$$

where the polynomial $G(X)$ is defined in $\mathcal{M} = \mathbb{R}^{3-p,p}$ and

given by

$$G(X) = F(X) - \frac{1}{2}g_{AB}X^AX^B + \rho. \quad (2.24)$$

It is interesting to notice that, for $f, g \in C^\infty(M)$,

$$\begin{aligned} \{f(X), g(X)\}_\theta &= \frac{\partial f(X)}{\partial X^A} \frac{\partial g(X)}{\partial X^B} \{X^A, X^B\}_\theta \\ &= \varepsilon^{ABC} \frac{\partial G(X)}{\partial X^A} \frac{\partial f(X)}{\partial X^B} \frac{\partial g(X)}{\partial X^C} \\ &\equiv \{G(X), f(X), g(X)\}_{\text{NP}}, \end{aligned} \quad (2.25)$$

where $\{f(X), g(X), h(X)\}_{\text{NP}}$ is the Nambu-Poisson bracket for arbitrary functions $f, g, h \in C^\infty(\mathcal{M})$. The Nambu-Poisson bracket satisfies some fundamental identity (see Eq. (3.2) in [30])

$$\begin{aligned} \{f_1, f_2, \{f_3, f_4, f_5\}_{\text{NP}}\}_{\text{NP}} &= \{\{f_1, f_2, f_3\}_{\text{NP}}, f_4, f_5\}_{\text{NP}} \\ &\quad + \{f_3, \{f_1, f_2, f_4\}_{\text{NP}}, f_5\}_{\text{NP}} \\ &\quad + \{f_3, f_4, \{f_1, f_2, f_5\}_{\text{NP}}\}_{\text{NP}}. \end{aligned} \quad (2.26)$$

Then one can easily see that the Jacobi identity for the Poisson bracket (2.23) is actually the statement of the fundamental identity (2.26) since

$$\begin{aligned} \{f, \{g, h\}_\theta\}_\theta + \{g, \{h, f\}_\theta\}_\theta + \{h, \{f, g\}_\theta\}_\theta \\ = \{G, f, \{G, g, h\}_{\text{NP}}\}_{\text{NP}} + \{G, g, \{G, h, f\}_{\text{NP}}\}_{\text{NP}} \\ + \{G, h, \{G, f, g\}_{\text{NP}}\}_{\text{NP}} \\ = -\{\{f, G, G\}_{\text{NP}}, g, h\}_{\text{NP}} = 0. \end{aligned} \quad (2.27)$$

In order to allow a general fluctuation including topology and volume changes of the two-dimensional surface M , suppose that the function $F(X)$ in Eq. (2.22) is an arbitrary polynomial in three variables in $\mathcal{M} = \mathbb{R}^{3-p,p}$. The two-dimensional surface M embedded in \mathcal{M} will be defined by zeros of the polynomial (2.24), i.e., $M = G^{-1}(\{0\})$ and $X^A(y)$ in Eq. (2.19) will be a local parameterization of M in terms of Darboux coordinates y^a . For example, a Riemann surface Σ_g of genus g is described by

$$G(\vec{x}) = (P(x) + y^2)^2 + z^2 - \mu^2, \quad \vec{x} = (x, y, z) \in \mathbb{R}^3, \quad (2.28)$$

with the polynomial $P(x) = x^{2k} + a_{2k-1}x^{2k-1} + \dots + a_1x + a_0$ where the polynomial $P - \mu$ has two simple roots and the polynomial $P + \mu$ has $2g$ simple roots ($\mu > 0$) [25]. The unperturbed surfaces in (I)–(III) correspond to the polynomial (2.24) with $F(X) = 0$, i.e., $M = G_{F=0}^{-1}(\{0\})$ where $\rho = (-1)^\# R^2$. After determining the embedding coordinate (2.19) by solving the polynomial equation $G(X) = 0$ as illustrated in the simple cases (I)–(III), the metric of the two-dimensional surface $M = G^{-1}(\{0\})$, according to the map (1.8), will be given by the vector fields

$$V_A = \theta^{ab} \frac{\partial X_A(y)}{\partial y^b} \frac{\partial}{\partial y^a}. \quad (2.29)$$

The resulting metric $ds^2 = \mathfrak{G}_{ab}(y)dy^a dy^b$ where

$$\mathfrak{G}_{ab} = g_{AB} \frac{\partial X^A}{\partial y^a} \frac{\partial X^B}{\partial y^b} \quad (2.30)$$

will again be equivalent to the induced metric on M embedded in the three-dimensional spacetime $ds^2 = g_{AB}dX^A dX^B$ whose embedding is defined by the polynomial (2.24).

If we consider a generic fluctuation described by an arbitrary polynomial (2.24), we expect that the perturbation (2.19) falls into one of the three classes; (A) metric preserving coordinate transformations generated by flat connections, (B) volume-preserving metric deformations; and (C) volume-changing deformations. From the analysis in Eq. (1.10) we well understand for the case (A) what is going on there. The gauge field fluctuation in Eq. (2.19) should belong to a pure gauge, i.e., $F^{AB} = 0$. To check this result, consider a pure gauge ansatz $A^A(y) = g^{-1}(y) \times \{L^A, g(y)\}_\theta$. One can calculate the corresponding field strength (2.21)

$$F^{AB} = \{A^A, A^B\}_\theta \quad (2.31)$$

and the Casimir invariant (2.7)

$$g_{AB}(X^A X^B - L^A L^B) = g_{AB} A^A A^B, \quad (2.32)$$

where $g_{AB} L^A L^B = 0$ was used. The nonvanishing terms, $\mathcal{O}(\theta^3)$ and $\mathcal{O}(\theta^2)$, in Eq. (2.31) and Eq. (2.32), respectively, can be neglected in the commutative limit and eventually will disappear in the NC space (2.37) as will be shown later. The case (B) corresponds to the metric change generated by a general vector field X satisfying $\mathcal{L}_X B + dA = 0$. In this case the vector field X is not a Hamiltonian vector field and it in general contains a harmonic part in $H^1(M)$. Therefore, it could be possible that the metric deformation generated by the nontrivial vector field X will in general accompany a topology change of the two-dimensional surface M . The topology change will be triggered by a higher order, e.g. quartic, polynomial $F(X)$ in Eq. (2.24) [25]. Finally, as a simple example of the case (C), a volume change of the two-dimensional surface M is described by the gauge field $A^A(y) = \alpha L^A(y)$ and $F^{AB}(y) = -\alpha(1 + \alpha)\varepsilon^{AB}{}_C L^C = -\alpha\varepsilon^{AB}{}_C X^C$. In this case the Poisson bracket relation (2.20) is given by

$$\{X^A, X^B\}_\theta = -(1 + \alpha)\varepsilon^{AB}{}_C X^C. \quad (2.33)$$

That is, the volume change can be done by turning on the polynomial $F(X) = -\frac{\alpha}{2}g_{AB}X^A X^B$ in Eq. (2.24). Therefore, the volume change in Eq. (2.7), $R \rightarrow (1 + \alpha)R$, can also be interpreted as the change of coupling constant in Eq. (2.6), $\lambda \rightarrow (1 + \alpha)\lambda$, or the change of noncommutativity in Eq. (2.8), $\theta^{ab} \rightarrow (1 + \alpha)\theta^{ab}$.

Because the Lie algebra (2.6) arises as the equations of motion of the action (2.1), it is necessary to generalize the action (2.1) in order to describe a general two-dimensional surface defined by the polynomial (2.24). The generalized action will be defined by

$$S_G = \text{Tr} \left(\frac{g_{\text{YM}}^2}{2} P_A P^A + \lambda P^A \frac{\partial G(X)}{\partial X^A} + \frac{i\kappa}{3!} \varepsilon_{ABC} X^A [X^B, X^C] \right). \quad (2.34)$$

The equations of motion are now given by

$$P_A = -\kappa \frac{\partial G(X)}{\partial X^A}, \quad (2.35)$$

$$-\left[P^B \frac{\partial^2 G(X)}{\partial X^A \partial X^B} \right] = \frac{i}{2g_{\text{YM}}^2} \varepsilon_{ABC} [X^B, X^C],$$

where $[P^B \frac{\partial^2 G(X)}{\partial X^A \partial X^B}]$ is a formal expression of the matrix ordering under the trace for the variation $P^B \frac{\delta}{\delta X^A} (\frac{\partial G}{\partial X^B})$. The previous equations of motion, (2.2) and (2.3), are given by the polynomial (2.24) with $F(X) = 0$. Of course a vacuum manifold defined by the new action (2.34) should be newly determined by solving the equations of motion (2.35).

A two-dimensional NC space can be obtained by quantizing the symplectic manifold $(M, \omega = -dy^1 \wedge dy^2)$, i.e., by replacing the Poisson bracket (2.8) by a star commutator

$$\{f, g\}_\theta \rightarrow -i[\hat{f}, \hat{g}]_\star \quad (2.36)$$

and the ordinary product in $C^\infty(M)$ by the star product in NC \star -algebra \mathcal{A}_θ . Then the local Darboux coordinates y^a ($a = 1, 2$) satisfy the commutation relation

$$[y^a, y^b]_\star = i\theta^{ab}. \quad (2.37)$$

The fluctuation in Eq. (2.19) now becomes an element in \mathcal{A}_θ given by

$$\hat{X}^A(y) = \hat{L}^A(y) + \hat{A}^A(y), \quad (2.38)$$

where $\hat{L}^A(y)$ is a background solution satisfying the constraint $(-1)^\# R^2 = g_{AB} \hat{L}^A \star \hat{L}^B$ and $[\hat{L}^A, \hat{L}^B]_\star = -i\varepsilon^{AB} \hat{L}^C$ obtained from Eq. (2.12) by the quantization (2.36). (See [29] for the deformation quantization of hyperbolic planes.) Then one can calculate the star commutator

$$\begin{aligned} [\hat{X}^A, \hat{X}^B]_\star &= [\hat{L}^A(y) + \hat{A}^A(y), \hat{L}^B(y) + \hat{A}^B(y)]_\star \\ &= -i\varepsilon^{AB} \hat{L}^C + [\hat{L}^A, \hat{A}^B]_\star - [\hat{L}^B, \hat{A}^A]_\star \\ &\quad + [\hat{A}^A, \hat{A}^B]_\star + i\varepsilon^{AB} \hat{A}^C \\ &= -i\varepsilon^{AB} \hat{X}^C(y) + i\hat{F}^{AB}(y). \end{aligned} \quad (2.39)$$

Substituting the above expression into the action (2.5) leads to the action for the fluctuations

$$\hat{S}_{\text{mCS}} = -\frac{\kappa}{12\pi|\theta|} \int d^2y (\varepsilon_{ABC} \hat{X}^A \star \hat{F}^{BC} + \lambda \hat{X}_A \star \hat{X}^A). \quad (2.40)$$

The equations of motion derived from the variation with respect to \hat{A}^A say that the fluctuations should be a flat connection, i.e., $\hat{F}^{AB} = 0$, already inferred from Eq. (2.39).

In order to treat the generalized action (2.34), the Jacobi identity, $\varepsilon_{ABC} [\hat{X}^A, [\hat{X}^B, \hat{X}^C]_\star]_\star = 0$, can be solved in a similar way as the commutative case by the form

$$\hat{F}^{AB}(\hat{X}) = \varepsilon^{ABC} \frac{\partial \hat{F}(\hat{X})}{\partial \hat{X}^C}. \quad (2.41)$$

The derivative $\frac{\partial \hat{F}(\hat{X})}{\partial \hat{X}^C}$ will be defined with the symmetric Weyl ordering [25]. Then one can evaluate the commutator $[\hat{X}^A, \frac{\partial \hat{F}(\hat{X})}{\partial \hat{X}^A}]_\star$ by a successive application of the Leibniz rule $[\hat{X}^A, \hat{f} \star \hat{g}]_\star = \hat{f} \star [\hat{X}^A, \hat{g}]_\star + [\hat{X}^A, \hat{f}]_\star \star \hat{g}$ such that each term finally has a form $\hat{F}_1(\hat{X}) \star [\hat{X}^A, \hat{X}^B]_\star \star \hat{F}_2(\hat{X})$. If we formally denote the resulting expression as the form

$$\begin{aligned} \frac{1}{2} \varepsilon_{ABC} [\hat{X}^A, [\hat{X}^B, \hat{X}^C]_\star]_\star &= i \left[\hat{X}^A, \frac{\partial \hat{F}(\hat{X})}{\partial \hat{X}^A} \right]_\star \\ &= i \left\{ \frac{\partial^2 \hat{F}(\hat{X})}{\partial \hat{X}^A \partial \hat{X}^B} \star [\hat{X}^A, \hat{X}^B]_\star \right\}, \end{aligned} \quad (2.42)$$

it turns out that the polynomial $\frac{\partial^2 \hat{F}(\hat{X})}{\partial \hat{X}^A \partial \hat{X}^B}$ is symmetric with respect to $(A \leftrightarrow B)$ and so Eq. (2.42) identically vanishes. Therefore, the star commutator (2.39) takes the form [25]

$$[\hat{X}^A, \hat{X}^B]_\star = -i\varepsilon^{ABC} \frac{\partial \hat{G}(\hat{X})}{\partial \hat{X}^C}, \quad (2.43)$$

where the polynomial $\hat{G}(\hat{X})$ is the star product version of Eq. (2.24) given by

$$\hat{G}(\hat{X}) = \hat{F}(\hat{X}) - \frac{1}{2} g_{AB} \hat{X}^A \star \hat{X}^B + \rho. \quad (2.44)$$

Suppose that we have solved the polynomial equation $\hat{G}(\hat{X}) = 0$ whose solution is given by $\hat{X}_A = g_{AB} \hat{X}^B = \hat{X}_A(y)$. (See [25] for explicit solutions for tori and deformed spheres.) Now one can define an inner derivation of the NC \star algebra $(\mathcal{A}_\theta, [\cdot, \cdot]_\star)$ as in Eq. (1.8) by considering an adjoint action of $\hat{X}_A(y) = g_{AB} \hat{X}^B(y)$ as follows:

$$\begin{aligned} \hat{V}_A[\hat{f}](y) &\equiv \text{ad}_{\hat{X}_A}[\hat{f}](y) = -i[\hat{X}_A(y), \hat{f}(y)]_\star \\ &= V_A^a(y) \frac{\partial f(y)}{\partial y^a} + \mathcal{O}(\theta^3). \end{aligned} \quad (2.45)$$

The leading term in Eq. (2.45) is exactly equal to the vector fields $V_A(y) = V_A^a(y) \frac{\partial}{\partial y^a}$ in Eq. (2.29). We may identify \hat{V}_A with generalized tangent vectors defined on a two-dimensional fuzzy manifold described by the polynomial (2.44).

As was shown in Eq. (1.3), the symplectomorphism can be identified with NC $U(1)$ gauge transformations. Flat connections, i.e., $\hat{F}^{AB}(y) = 0$ in which case $\hat{F}(\hat{X}) = 0$, are given by $\hat{A}^A(y) = \hat{g}^{-1}(y) \star [\hat{L}^A(y), \hat{g}(y)]_\star$ or $\hat{X}^A(y) = \hat{g}^{-1}(y) \star \hat{L}^A(y) \star \hat{g}(y)$ with any invertible $\hat{g}(y) \in \mathcal{A}_\theta$. So the equations of motion (2.39) are the same as before and the solution (2.38) of flat connections preserves the area (2.7), say, $g_{AB} \hat{X}^A \star \hat{X}^B = (-)^\# R^2$. Also note that the remaining terms in Eqs. (2.31) and (2.32) are completely cured in the NC space (2.37) as we remarked before. Because the embedding (2.44) has not been changed, it is a natural consequence that a pure gauge fluctuation does not change a two-dimensional metric of fuzzy manifold \hat{M} as we already noted in Eq. (1.10).

Now we want to discuss some interesting aspects of our construction. As we observed above, a pure gauge fluctuation does not change the two-dimensional metric $ds^2 = \mathfrak{G}_{ab} dy^a dy^b$ and belongs to the same representation, i.e., $g_{AB} L^A L^B = g_{AB} X^A X^B = (-)^\# R^2$ for $L^A(y)$ and $X^A(y) = L^A(y) + A^A(y)$ in $C^\infty(M)$. This means that there exists a global Lorentz transformation in three dimensions such that $X^A = \Lambda^A_B L^B$ where $\Lambda^A_B \in \text{SO}(3-p, p)$. In other words the metric \mathfrak{G}_{ab} is invariant under the Lorentz transformation in ambient spaces as expected. It is interesting to notice that a local gauge transformation in two dimensions can be interpreted as a global Lorentz transformation in three-dimensional target spacetime. More generally, one may represent a generic fluctuation of gauge fields in X^A as a general coordinate transformation, that is, $L^A(y) \mapsto X^A(y) = X^A(L(y))$. Then the vector fields V_A^L and V_A^X in TM for the smooth functions $L^A(y)$ and $X^A(y)$ are defined by Eq. (2.29) and they are related by $V_A^X = \frac{\partial X_A}{\partial L_B}(y) V_B^L$ thanks to the chain rule $\{X_A(L), f\}_\theta = \frac{\partial X_A}{\partial L_B} \{L_B, f\}_\theta$. According to Eq. (2.30), the two-dimensional metric can then be written as

$$\mathfrak{G}_{ab} = G_{AB} \frac{\partial L^A}{\partial y^a} \frac{\partial L^B}{\partial y^b}, \quad (2.46)$$

where $G_{AB}(y) = \frac{\partial X^C}{\partial L^A}(y) \frac{\partial X^D}{\partial L^B}(y) g_{CD}$. Thus a generic fluctuation possibly changing the volume as well as topology [turning on a nontrivial $F(X) \neq 0$] can be interpreted as a general coordinate transformation supported on the two-dimensional surface M . Of course this is consistent with the fact that the metric $ds^2 = \mathfrak{G}_{ab} dy^a dy^b$ is the induced metric on a submanifold M embedded in $\mathbb{R}^{3-p,p}$.

It is well known [23] that the massive Chern-Simons gauge theory in three dimensions has a physical degree of freedom. One may wonder which mode in the action (2.5) corresponds to the physical one. Note that the gauge field dynamics in three dimensions need not be subject to the constraint (2.7). Because gauge field fluctuations preserving a two-dimensional area and satisfying the equations of motion (2.6) are flat connections and also pure gauges, the only remaining physical mode satisfying the same Lie

algebra (2.6) is an area changing fluctuation as we observed in Eq. (2.33). Because the area change can also be interpreted as the change of coupling constant or noncommutativity, it would be intriguing to recall that a similar feature also arises in the AdS/CFT correspondence [12] where the size of bulk spacetime is related to the coupling constant of gauge theory.

III. EMERGENT GEOMETRY FOR SNYDER SPACETIME

Now we want to generalize the analysis for the two-dimensional cases to higher dimensions, in particular, to four-dimensional manifolds with constant curvature as an emergent geometry from some matrix model. Let us start with the following IKKT matrix model with a mass deformation [26]:

$$S_{\text{mIKKT}} = \text{Tr} \left(-\frac{1}{4} [X^a, X^b]^2 + \frac{(d-1)\kappa}{2} X_a X^a \right), \quad (3.1)$$

where X^a are $N \times N$ Hermitian matrices and $a, b = 1, \dots, d \geq 2$. One can rewrite the action (3.1) as the form

$$S_\kappa = \text{Tr} \left(\frac{1}{4} M_{ab} M^{ab} - \frac{1}{2} M_{ab} [X^a, X^b] + \frac{(d-1)\kappa}{2} X_a X^a \right) \quad (3.2)$$

by introducing Lagrange multipliers M_{ab} which are $N \times N$ anti-Hermitian matrices. In spite of the mass deformation with $\kappa \neq 0$, the matrix action (3.2) respects the $U(N)$ gauge symmetry given by

$$(X^a, M_{ab}) \rightarrow U(X^a, M_{ab})U^\dagger \quad (3.3)$$

with $U \in U(N)$. The equations of motion are given by

$$[X^a, X^b] = M^{ab}, \quad (3.4)$$

$$[M^{ab}, X_b] + (d-1)\kappa X^a = 0, \quad (3.5)$$

where Eq. (3.5) becomes the equations of motion derived from the action (3.1) when substituting $[X^a, X^b]$ for M^{ab} . One can easily check that the above equations of motion can be obtained from the Snyder algebra [27]:

$$\begin{aligned} [X^a, X^b] &= M^{ab}, \\ [X^a, M^{bc}] &= \kappa(g^{ac} X^b - g^{ab} X^c), \\ [M^{ab}, M^{cd}] &= \kappa(g^{ac} M^{bd} - g^{ad} M^{bc} - g^{bc} M^{ad} + g^{bd} M^{ac}), \end{aligned} \quad (3.6)$$

where the last equation can be derived from the other two applying the Jacobi identity. Therefore, if the matrices (X^a, M^{bc}) satisfy the Snyder algebra (3.6), they automatically satisfy the equations of motion, (3.4) and (3.5). Here the deformation parameter κ carries the physical dimension of $(\text{length})^2$ since we will consider X^a as “matrix coordinates.”

Because we consider the action (3.2) as a massive deformation of the IKKT matrix model (1.11), we regard the matrices M_{ab} in the action (3.2) as Lagrange multipliers and so these can be integrated out. The resulting action of course recovers the original action (3.1). Thus the number of dynamical coordinates remains the same as the undeformed case. Actually it will be shown later that the matrix X^a as a dynamical coordinate is mapped to a NC gauge field and M^{ab} to its field strength. Therefore, the emergent geometry for the mass-deformed case can be derived by essentially the same way as the undeformed case, except that the deformed case in general admits a Poisson structure only instead of a symplectic structure. But this is not a difficulty since a Poisson structure is enough to formulate emergent geometry from large N matrices or NC gauge fields, as will be shown below. Note that Poisson manifolds are a more general class of manifolds which contains symplectic manifolds as a special class.

Now the problem is how to generalize the emergent geometry picture for the undeformed case (1.11) to the mass-deformed case (3.2) where the vacuum geometry will be nontrivial, i.e., curved, since $M^{ab} = \text{constants}$ cannot be a vacuum solution unlike the $\kappa = 0$ case. We showed that the generators X^a in the Snyder algebra (3.6) satisfy the equations of motion (3.4) and (3.5). In order to map the matrix algebra $(M_N, [\cdot, \cdot])$ defining the action (3.2) to a NC \star algebra $(\mathcal{A}_\theta, [\cdot, \cdot]_\star)$, we will show that the Snyder algebra (3.6) can be obtained by the deformation quantization of a Poisson manifold [31] whose Poisson tensor is given by $\Pi = \frac{1}{2} L^{ab}(x) \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b}$. In other words, we want to show that the Schouten bracket [32] for the Poisson tensor Π vanishes, i.e.,

$$[\Pi, \Pi]_S \equiv \left(L^{da} \frac{\partial L^{bc}}{\partial x^d} + L^{db} \frac{\partial L^{ca}}{\partial x^d} + L^{dc} \frac{\partial L^{ab}}{\partial x^d} \right) \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b} \wedge \frac{\partial}{\partial x^c} = 0 \quad (3.7)$$

if the Poisson bracket $\{x^a, x^b\}_\Pi = L^{ab}(x) = \langle \Pi, dx^a \wedge dx^b \rangle$ satisfies the Snyder algebra (3.6). It is easy to see that the Jacobi identity $\{\{x^a, x^b\}_\Pi, x^c\}_\Pi + \{\{x^b, x^c\}_\Pi, x^a\}_\Pi + \{\{x^c, x^a\}_\Pi, x^b\}_\Pi = 0$ is satisfied due to the second algebra in Eq. (3.6). From the Jacobi identity, we immediately get the result (3.7) and so the two-vector field Π is a Poisson tensor.

The Poisson tensor Π of a Poisson manifold M induces a bundle map $\Pi^\#: T^*M \rightarrow TM$ by

$$A \mapsto \Pi^\#(A) = L^{ab}(x) A_a(x) \frac{\partial}{\partial x^b} \quad (3.8)$$

for $A = A_a(x) dx^a \in T_x^*M$, which is called the anchor map of Π [32]. The rank of the Poisson structure at a point $x \in M$ is defined as the rank of the anchor map at this point. If the rank equals the dimension of the manifold at each point, the Poisson structure reduces to a symplectic structure which is also called nondegenerate. The nondegener-

ate Poisson structure uniquely determines the symplectic structure defined by the two-form $\omega = \frac{1}{2} \omega_{ab}(x) dx^a \wedge dx^b = \Pi^{-1}$ and the condition (3.7) is equivalent to the statement that the two-form ω is closed, $d\omega = 0$. In this case the anchor map $\Pi^\#: T^*M \rightarrow TM$ is a bundle isomorphism as we discussed in Sec. I. To define a Hamiltonian vector field $\Pi^\#(df)$ of a smooth function $f \in C^\infty(M)$, what one really needs is a Poisson structure which reduces to a symplectic structure for the nondegenerate case. Given a smooth Poisson manifold (M, Π) , the map $f \mapsto X_f = \Pi^\#(df)$ is a homomorphism [32] from the Lie algebra $C^\infty(M)$ of smooth functions under the Poisson bracket to the Lie algebra of smooth vector fields under the Lie bracket. In other words, the Lie algebra homomorphism (1.7) is still true even for any Poisson manifold.

Like the Darboux theorem in symplectic manifolds, the Poisson geometry also enjoys a similar property known as the splitting theorem proved by Weinstein [33]. The splitting theorem states that a d -dimensional Poisson manifold is locally equivalent to the product of \mathbb{R}^{2n} equipped with the canonical symplectic structure with \mathbb{R}^{d-2n} equipped with a Poisson structure of rank zero at the origin. That is, the Poisson manifold (M, Π) is locally isomorphic (in a neighborhood of x) to the direct product $S \times N$ of a symplectic manifold $(S, \sum_{i=1}^n dq^i \wedge dp_i)$ with a Poisson manifold $(N_x, \{\cdot, \cdot\}_N)$ whose Poisson tensor vanishes at x .

Note that not every Snyder space can be obtained by the quantization of a symplectic manifold (M, ω) in contrast to the two-dimensional orientable hyperspaces in Sec. II. If M is a compact symplectic manifold, the second de Rham cohomology group $H^2(M)$ is nontrivial and so the only n -sphere that admits a symplectic form is the two-sphere. For example, let $S^4 = \{(u, v, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R} : |u|^2 + |v|^2 = t(2-t)\}$. Then the bivector field $\Pi = uv \partial_u \wedge \partial_v - uv^* \partial_u \wedge \partial_{v^*} - u^* v \partial_{u^*} \wedge \partial_v + u^* v^* \partial_{u^*} \wedge \partial_{v^*}$ is a Poisson tensor, that is, $[\Pi, \Pi]_S = 0$, and $\Pi \wedge \Pi = 4|u|^2 |v|^2 \partial_u \wedge \partial_v \wedge \partial_{u^*} \wedge \partial_{v^*}$. Therefore, the Poisson tensor Π vanishes on a subspace of either $u = 0$ or $v = 0$ and the Poisson structure becomes degenerate there. This is the reason why we have to rely on a Poisson structure rather than a symplectic structure to formulate emergent geometry from the Snyder algebra (3.6).

Because any Poisson manifold can be quantized via deformation quantization [31], the anchor map (3.8) can be lifted to a NC manifold as in Eq. (1.8). As we noticed before, it is enough to have a Poisson structure to achieve the map $C^\infty(M) \rightarrow \Gamma(TM): f \mapsto X_f = \Pi^\#(df)$ such as Eq. (1.6). So let us take the limit $N \rightarrow \infty$ of the Snyder algebra (3.6) and suppose that the Poisson manifold (M, Π) is quantized via deformation quantization, i.e.,

$$\{x^a, x^b\}_\Pi = L^{ab}(x) \rightarrow [\hat{x}^a, \hat{x}^b]_\star = i \hat{L}^{ab}(\hat{x}), \quad (3.9)$$

where $\hat{L}^{ab}(\hat{x}) \in \mathcal{A}_\theta$ are assumed to be dimensionful operators of $(\text{length})^2$ satisfying the Snyder algebra (3.6).

Let us consider a vacuum solution of the mass-deformed matrix model (3.2) as the Snyder space defined by Eq. (3.9). Now we will regard the background solution $\hat{x}^a \in \mathcal{A}_\theta$ in (3.9) as NC fields but from now on we will omit the hat for notational simplicity. Consider fluctuations of the large N matrices $X^a \equiv \kappa g^{ab} \hat{D}_b(x)$ around the vacuum solution (3.9) as follows:

$$\hat{D}_a(x) = \hat{D}_a^{(0)}(x) + \hat{A}_a(x), \quad (3.10)$$

where $\hat{D}_a^{(0)}(x) = \frac{1}{\kappa} g_{ab} x^b$. The background solution $(\kappa \hat{D}_a^{(0)}(x), \hat{L}_{ab}(x) \equiv \kappa^2 \hat{Q}_{ab})$ satisfies the Snyder algebra (3.6). Using the above variables, one can calculate the star commutator

$$\begin{aligned} [\hat{D}_a, \hat{D}_b]_\star &= [\hat{D}_a^{(0)}(x) + \hat{A}_a(x), \hat{D}_b^{(0)}(x) + \hat{A}_b(x)]_\star \\ &= i\hat{Q}_{ab} + [\hat{D}_a^{(0)}, \hat{A}_b]_\star - [\hat{D}_b^{(0)}, \hat{A}_a]_\star + [\hat{A}_a, \hat{A}_b]_\star \\ &\equiv i\hat{F}_{ab}. \end{aligned} \quad (3.11)$$

One can check that the field strength defined in Eq. (3.11) covariantly transforms under the gauge transformation $\delta \hat{A}_a = -i([\hat{D}_a^{(0)}, \hat{\lambda}]_\star + [\hat{A}_a, \hat{\lambda}]_\star)$, viz.,

$$\delta \hat{F}_{ab} = -i[\hat{F}_{ab}, \hat{\lambda}]_\star. \quad (3.12)$$

Note that we need the background part \hat{Q}_{ab} in \hat{F}_{ab} to maintain the gauge covariance (3.12). Using the result (3.11), we get the action for the fluctuations after integrating out the M -fields in Eq. (3.2)

$$\begin{aligned} \hat{S}_\kappa &= \frac{\kappa^4}{4} \text{Tr}_{\mathcal{H}} g^{ac} g^{bd} \hat{F}_{ab} \star \hat{F}_{cd} \\ &\quad + \frac{(d-1)\kappa^3}{2} \text{Tr}_{\mathcal{H}} g^{ab} \hat{D}_a \star \hat{D}_b, \end{aligned} \quad (3.13)$$

where the trace $\text{Tr}_{\mathcal{H}}$ is defined over the Hilbert space \mathcal{H} associated with a representation space of the NC \star algebra (3.9). It might be remarked that, in spite of the mass term, the action (3.13) respects the NC $U(1)$ gauge symmetry acting on $(\hat{D}_a, \hat{F}_{ab}) \rightarrow \hat{U} \star (\hat{D}_a, \hat{F}_{ab}) \star \hat{U}^\dagger$ where $\hat{U} \in \mathcal{A}_\theta$.

Because $\hat{D}_a = \frac{1}{\kappa} g_{ab} X^b$, one can rewrite the Snyder algebra (3.6) in terms of gauge theory variables:

$$\begin{aligned} [\hat{D}_a, \hat{D}_b]_\star &= i\hat{F}_{ab} = \kappa^{-2} M_{ab}, \\ [\hat{D}_a, \hat{F}_{bc}]_\star &= -i\kappa^{-1} (g_{ac} \hat{D}_b - g_{ab} \hat{D}_c), \\ [\hat{F}_{ab}, \hat{F}_{cd}]_\star &= -i\kappa^{-1} (g_{ac} \hat{F}_{bd} - g_{ad} \hat{F}_{bc} - g_{bc} \hat{F}_{ad} \\ &\quad + g_{bd} \hat{F}_{ac}). \end{aligned} \quad (3.14)$$

Because $-i[\hat{D}_a, \hat{F}_{bc}]_\star = -i[x_a/\kappa, \hat{F}_{bc}]_\star - i[\hat{A}_a, \hat{F}_{bc}]_\star \equiv \hat{D}_a \hat{F}_{bc}$, one can easily check that the Bianchi identity,

$\hat{D}_{[a} \hat{F}_{bc]} = 0$, and the equations of motion, $\hat{D}_a \hat{F}^{ab} = (d-1)\kappa^{-1} \hat{D}^b$, are directly derived from the second algebra in Eq. (3.14). Note that the last equation in Eq. (3.14) can be obtained from the other two applying the Jacobi identity. From a gauge theory point of view, it is a bizarre relation since the field strength \hat{F}_{ab} of an arbitrary gauge field \hat{A}_a behaves like an angular momentum operator in d dimensions. This kind of behavior is absent in an undeformed case, $\kappa = 0$. The theory will strongly constrain the behavior of gauge fields and so there might be some hidden integrability.

A Hamiltonian vector field $X_f = \Pi^\#(df)$ for a smooth function $f \in C^\infty(M)$ is defined by the anchor map (3.8) as follows [32]:

$$X_f(g) = -\langle \Pi, df \wedge dg \rangle = -L^{ab}(x) \frac{\partial f}{\partial x^a} \frac{\partial g}{\partial x^b} = \{g, f\}_\Pi. \quad (3.15)$$

Because the Poisson manifold (M, Π) has been quantized in Eq. (3.9), the correspondence between the Lie algebras $(C^\infty(M), \{\cdot, \cdot\}_\Pi)$ and $(\Gamma(TM), [\cdot, \cdot])$ can be lifted to the NC \star algebra $(\mathcal{A}_\theta, [\cdot, \cdot]_\star)$ as in Eq. (1.8). That is, we can map NC fields in \mathcal{A}_θ to vector fields in $\Gamma_\theta(\widehat{TM})$, \mathcal{A}_θ -valued sections of a generalized tangent bundle \widehat{TM} . For example, $\hat{D}_a(x)$ in Eq. (3.10) are mapped to the following vector fields in \widehat{TM} :

$$\begin{aligned} ad_{\hat{D}_a}[\hat{f}](x) &= -i[\hat{D}_a(x), \hat{f}(x)]_\star \\ &= -L^{\mu\nu}(x) \frac{\partial D_a(x)}{\partial x^\nu} \frac{\partial f(x)}{\partial x^\mu} + \dots \\ &\equiv V_a^\mu(x) \frac{\partial f(x)}{\partial x^\mu} + \dots = V_a[f](x) + \mathcal{O}(L^3), \end{aligned} \quad (3.16)$$

where the leading order leads to the usual vector fields $V_a \in TM$ in Eq. (3.15).

We might express from the outset the star product using different NC coordinates \hat{y}^a defined by $[\hat{y}^a, \hat{y}^b]_\star = i\tilde{L}^{ab}(\hat{y})$. In terms of the new $\tilde{\star}$ product, the adjoint action defining an inner derivation in \mathcal{A}_θ is then given by

$$\begin{aligned} ad_{\hat{D}_a}[\hat{f}](\hat{y}) &= -i[\hat{D}_a(\hat{y}), \hat{f}(\hat{y})]_{\tilde{\star}} \\ &= -\tilde{L}^{\mu\nu}(\hat{y}) \frac{\partial D_a(\hat{y})}{\partial \hat{y}^\nu} \frac{\partial f(\hat{y})}{\partial \hat{y}^\mu} + \dots \\ &\equiv \tilde{V}_a^\mu(\hat{y}) \frac{\partial f(\hat{y})}{\partial \hat{y}^\mu} + \dots = \tilde{V}_a[f](\hat{y}) + \mathcal{O}(\tilde{L}^3). \end{aligned} \quad (3.17)$$

Noting that the star products, \star and $\tilde{\star}$, are related by a coordinate transformation $x^a \mapsto y^a = y^a(x)$ [31], in other words,

$$\tilde{L}^{\mu\nu}(y) = L^{ab}(x) \frac{\partial y^\mu(x)}{\partial x^a} \frac{\partial y^\nu(x)}{\partial x^b},$$

one can easily check [2] using the chain rule that the vector fields defined by Eq. (3.17) are diffeomorphic to those in Eq. (3.16) as expected, i.e.,

$$\tilde{V}_a^\mu(y) = V_a^\nu(x) \frac{\partial y^\mu(x)}{\partial x^\nu}. \quad (3.18)$$

We are particularly interested in the background geometry defined by Eq. (3.9). In this case, the vector fields for the background gauge fields $\hat{D}_a^{(0)}(x)$ are given by

$$\begin{aligned} ad_{\hat{D}_a^{(0)}}[\hat{f}](x) &= -i[\hat{D}_a^{(0)}(x), \hat{f}(x)]_\star = V_a^{(0)\mu}(x) \frac{\partial f(x)}{\partial x^\mu} + \cdots \\ &\equiv V_a^{(0)}[f](x) + \mathcal{O}(L^3). \end{aligned} \quad (3.19)$$

Using the Snyder algebra for $(\hat{D}_a^{(0)}, \hat{Q}_{ab})$ and the relation

$$\begin{aligned} ad_{[\hat{D}_a^{(0)}, \hat{D}_b^{(0)}]_\star} &= i[V_a^{(0)}, V_b^{(0)}] + \mathcal{O}(L^3) = iad_{\hat{Q}_{ab}} \\ &\equiv iS_{ab}^{(0)} + \mathcal{O}(L^3), \end{aligned} \quad (3.20)$$

one can see that $(V_a^{(0)}, S_{ab}^{(0)}) \in \Gamma(TM_{\text{back}})$ also satisfy the Snyder algebra (3.6) where the Lie algebra in $\Gamma(TM_{\text{back}})$ is defined by the Lie bracket, e.g., $[V_a^{(0)}, V_b^{(0)}] = S_{ab}^{(0)}$.

We want to find the representation of the Snyder algebra (3.6) in terms of differential operators [27], i.e., vector fields in $\Gamma(TM_{\text{back}})$. In order to find an explicit expression of vector fields $V_a^{(0)} \in \Gamma(TM_{\text{back}})$, first notice that the Snyder algebra (3.6) can be understood as the Lorentz algebra in $(d+1)$ dimensions with the identification $M^{d+1,a} = \sqrt{\kappa}X^a$

$$\begin{aligned} [M^{AB}, M^{CD}] &= \kappa(g^{AC}M^{BD} - g^{AD}M^{BC} - g^{BC}M^{AD} \\ &\quad + g^{BD}M^{AC}), \end{aligned} \quad (3.21)$$

where $A, B, \dots = 1, \dots, d+1$. Therefore, the equivalence between the Snyder algebra (3.6) in d dimensions and the Lorentz algebra $SO(d+1-p, p)$ in $(d+1)$ dimensions implies that the Snyder space as an emergent geometry defined by the action (3.2) can be obtained as a d -dimensional hypersurface M embedded in $\mathbb{R}^{d+1-p, p}$. For example, in the $d=2$ case, the Lorentz algebra (3.21) is equivalent to the Lie algebra (2.6) with the identification $M^{AB} = -i\lambda \varepsilon_C^{AB} X^C$, ($A, B, C = 1, 2, 3$) where $\kappa = -\lambda^2 \det g_{AB}$ and so $[X^A, X^B] = M^{AB}$. And the Lie algebra (2.6) describes a two-dimensional hypersurface foliated by the quadratic form (2.7) in $\mathbb{R}^{3-p, p}$. See the Appendix for the details.

Similarly we will consider, in particular, four-dimensional hypersurfaces M for three cases with $p = 0, 1, 2$. Let us consider a homogeneous quadratic form as an invariant of the Lorentz algebra (3.21)

$$g_{AB}x^Ax^B = (-1)^\# R^2 \quad (3.22)$$

and the ambient space metric g_{AB} will be taken as a five-dimensional flat Euclidean or Lorentzian metric given by (I) $g_{AB} = \text{diag}(1, 1, 1, 1, 1)$ with $\# = 0$, (II) $g_{AB} = \text{diag}(-1, 1, 1, 1, 1)$ with $\# = 0$, and (III) $g_{AB} = \text{diag}(-1, 1, 1, 1, -1)$ with $\# = 1$. Then they describe (I) S^4 , (II) dS_4 , and (III) AdS_4 of radius R given by Eq. (3.22) in a continuum limit. It should be remarked that the case (I), a four-sphere S^4 , admits only a Poisson structure instead of a nondegenerate symplectic structure² and so its quantization has to be described in terms of deformation quantization of Poisson manifold as we explained before. Although we do not know whether the other cases, (II) and (III), admit a nondegenerate Poisson, i.e., symplectic, structure, the arguments followed by Eq. (3.21) will be completely sensible even with the Poisson structure only.

Suppose that $x^A = x^A(x)$, $A = 1, 2, \dots, 5$ satisfy Eq. (3.22) and are local parameterizations of M in terms of local coordinates x^a . We will identify $x^A = x^a$, $A = 1, \dots, 4$, with the Poisson coordinates in Eq. (3.9), which is the background solution $X_{\text{back}}^a = x^a = \kappa g^{ab} D_b^{(0)}(x)$ in Eq. (3.10). The vector fields $(V_a^{(0)}, S_{ab}^{(0)}) \in \Gamma(TM_{\text{back}})$ in Eqs. (3.19) and (3.20) satisfying the Snyder algebra can then be understood as differential Lorentz generators of $SO(5-p, p)$,

$$S_{AB}^{(0)} = \kappa \left(x_B \frac{\partial}{\partial x^A} - x_A \frac{\partial}{\partial x^B} \right), \quad (3.23)$$

where $V_a^{(0)} \equiv S_{5,a}^{(0)}/\sqrt{\kappa}$ and the five-dimensional metric g_{AB} (to define $x_A = g_{AB}x^B$) is a standard flat metric. According to the identification, the vector fields $V_a^{(0)}$ in Eq. (3.19) can be represented by the coordinates x^A as follows [27]:

$$V_a^{(0)} = V_a^{(0)A}(x) \frac{\partial}{\partial x^A} = S_{5,a}^{(0)}/\sqrt{\kappa} = \sqrt{\kappa} \left(x_a \frac{\partial}{\partial x^5} - x_5 \frac{\partial}{\partial x^a} \right) \quad (3.24)$$

and so we get the result $V_a^{(0)b} = -\sqrt{\kappa} \delta_a^b x_5$ and $V_a^{(0)5} = \sqrt{\kappa} x_a$. Then it is obvious that $S_{ab}^{(0)} = [V_a^{(0)}, V_b^{(0)}] = \kappa(x_b \frac{\partial}{\partial x^a} - x_a \frac{\partial}{\partial x^b})$ are the generators of the four-dimensional Lorentz group, i.e., $S_{ab}^{(0)} \in SO(4)$ or $SO(3,1)$ and $(V_a^{(0)}, S_{ab}^{(0)})$ satisfy the Snyder algebra (3.6).

As will be shown in the Appendix, the Lie algebra (2.6) is the Snyder algebra in two dimensions whose generators are given by $(X^1, X^2, M^{12} = \pm \frac{i}{\lambda} X^3)$. In this case the Lie algebra (2.6) describes a two-dimensional hypersurface

²Instead one can consider a bundle over S^4 with fibre S^2 , which is the Kähler coset space $SO(5)/U(2) \simeq S^4 \times S^2$ [34]. Then S^4 may be described by the complex coordinate system of $SO(5)/U(2)$, where a symplectic structure manifests and the second de Rham cohomology group $H^2(S^4 \times S^2)$ is definitely nontrivial.

embedded in three-dimensional space whose metric is given by Eq. (2.15). Therefore, in order to define a four-dimensional metric determined by the Snyder algebra (3.6), we will consistently extend the two-dimensional case and so the metric is defined by the vector fields (3.24) as follows:

$$\begin{aligned} ds^2 &= \mathbb{G}_{ab}^{(0)} dx^a \otimes dx^b = (\det \mathbb{G}_{ab}^{(0)}) g_{AB} V_a^{(0)A} V_b^{(0)B} dx^a \otimes dx^b \\ &= (\det \mathbb{G}_{ab}^{(0)}) (g_{ab} x_5^2 + g_{55} x_a x_b) dx^a \otimes dx^b, \end{aligned} \quad (3.25)$$

where $\det \mathbb{G}_{ab}^{(0)} = \frac{1}{x_5^2}$ and we put $R = \kappa = 1$ for simplicity. Of course Eq. (3.25) describes a four-dimensional maximally symmetric space with a constant curvature, e.g., \mathbf{S}^4 , $d\mathbf{S}_4$, or AdS_4 depending on the signature of the five-dimensional metric g_{AB} . Because $x^5 = \pm \sqrt{1 - g_{55} g_{ab} x^a x^b}$, the metric (3.25) can be rewritten as the following form:

$$\begin{aligned} ds^2 &= \left(g_{ab} + g_{55} \frac{x_a x_b}{x_5^2} \right) dx^a \otimes dx^b \\ &= g_{AB} \frac{\partial x^A}{\partial x^a} \frac{\partial x^B}{\partial x^b} dx^a \otimes dx^b = g_{AB} dx^A \otimes dx^B. \end{aligned} \quad (3.26)$$

Note that the final result (3.26) is completely parallel to the two-dimensional one, e.g., (2.15).

Therefore we get an interesting result. The mass-deformed IKKT matrix model (3.2) in d dimensions is completely described by the Snyder algebra (3.6) which is equivalent to the Lorentz algebra $\text{SO}(d+1-p, p)$, Eq. (3.21), in $(d+1)$ dimensions. We found that a vacuum geometry of the Snyder algebra is a constant curvature space. For example, the metric (3.25) in four dimensions describes \mathbf{S}^4 , $d\mathbf{S}_4$, and AdS_4 depending on the choice of the five-dimensional metric g_{AB} . Thus the equivalence between the Snyder algebra (3.6) in d dimensions and the Lorentz algebra (3.21) in $(d+1)$ dimensions is beautifully realized as a well-known geometrical result that a constant curvature space in d dimensions such as \mathbf{S}^d , $d\mathbf{S}_d$, and AdS_d can be embedded in a flat Euclidean or Lorentzian spacetime in $(d+1)$ dimensions. In particular, this result clearly illustrates how a nontrivial curved spacetime emerges from the zero-dimensional (i.e., background-independent) matrix model (3.2) through the correspondence (1.8) between NC \star algebra $(\mathcal{A}_\theta, [\cdot, \cdot]_\star)$ and $\Gamma_\theta(\widehat{TM})$, generalized vector fields. We will discuss in Sec. IV how the constant curvature spacetimes in Eq. (3.26) can be described by the coset space realization of the Snyder algebra (3.6).

We can further deduce consistent pictures about emergent geometry by closely following the two-dimensional case we observed in the previous section. Consider a generic fluctuation in Eq. (3.10). If the fluctuation is a flat connection, i.e., $\hat{A}_a(x) = \hat{g}^{-1}(x) \star [\hat{D}_a^{(0)}, \hat{g}(x)]_\star$, then $\hat{D}_a(x) = \hat{g}^{-1}(x) \star \hat{D}_a^{(0)} \star \hat{g}(x)$ and $\hat{F}_{ab}(x) = \hat{g}^{-1}(x) \star \hat{Q}_{ab} \star \hat{g}(x)$. One can immediately see from Eq. (3.14)

that the Snyder algebra for the operators $(\hat{D}_a, \hat{F}_{ab})$ is simply a gauge transformation of the Snyder algebra for the operators $(\hat{D}_a^{(0)}, \hat{Q}_{ab})$. Therefore, the resulting geometry determined by the vector fields (3.16) will not be changed and the constraint (3.22) will be preserved. So the coordinate change in terms of flat connections should be a Killing symmetry of the background geometry (3.25) as was explained in Eq. (1.10) and correspond to a global Lorentz transformation in higher dimensions, which was precisely the case for two-dimensional geometries. For example, from Eq. (3.22) or Eq. (3.26), one can deduce that $x^A \rightarrow x'^A = \Lambda^A_B x^B$ where $\Lambda^A_B \in \text{SO}(5-p, p)$.

We observed that a higher dimensional manifold in general emerges from a NC \star -algebra \mathcal{A}_θ defined by a Poisson structure rather than a symplectic structure. Another notable difference from the two-dimensional case is that the underlying action (3.2) contains fluctuations by nonflat connections and so nontrivial metric deformations. This means that the action (3.2) describes a fluctuating geometry, not a rigid geometry. The Snyder algebra (3.14) clearly shows that the action (3.13) allows such fluctuations by nonflat connections as an on-shell solution. Indeed the algebra (3.14) can be understood as the Lorentz algebra (3.21) after the identification $M_{AB} = i\kappa^2 \hat{F}_{AB} = i\kappa^2 (\hat{F}_{ab}, \hat{F}_{d+1,a}) \equiv -\frac{i}{\sqrt{\kappa}} \hat{D}_a$.

Suppose that the fluctuations (3.10) in commutative limit are described by smooth functions $z^a(x) = x^a + \kappa A^a(x)$ where x^a describe the vacuum geometry in Eq. (3.25). Then one can map the solution $D_a = g_{ab} z^b / \kappa \in C^\infty(M)$ to vector fields in $\Gamma(TM)$ according to Eq. (3.16). Let us denote the resulting vector fields as $(V_a, S_{ab} = [V_a, V_b])$ which satisfy the Snyder algebra as easily inferred from Eq. (3.14). The resulting Snyder algebra can be lifted to the Lorentz algebra in five dimensions given by

$$S_{AB} = \kappa \left(z_B \frac{\partial}{\partial z^A} - z_A \frac{\partial}{\partial z^B} \right), \quad (3.27)$$

where $V_a \equiv S_{5,a} / \sqrt{\kappa}$ and $z^A = z^A(x)$ are five-dimensional coordinates satisfying $g_{AB} z^A z^B = (-1)^\# R^2$. Following the same procedure as Eqs. (3.25) and (3.26), the metric of fluctuating surface M can be derived as³

³We feel some remarks are necessary to correctly understand Eq. (3.28) and to avoid any confusion. The equivalence principle in general relativity guarantees that there always exists a locally inertial frame at an arbitrary point P in spacetime where the metric becomes locally flat, i.e., $ds^2|_P = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$. But the local inertial frame $\xi^\alpha = \xi^\alpha(x)$ is valid only on a local coordinate patch and cannot be globally extended over all spacetime unless the spacetime is flat. Similarly, Eq. (3.14) implies that it is always possible to choose a local coordinate z^a such that the metric at P locally looks like the background geometry (3.25). But we have to notice that the local coordinates $z^a(x) = x^a + \kappa A^a(x)$ depend on dynamical gauge fields satisfying the equations of motion or the Snyder algebra (3.14) and so should not be regarded as a globally constant curvature spacetime as if the local inertial frame does not mean a flat spacetime.

$$\begin{aligned}
ds^2 &= \mathbb{G}_{ab} dz^a \otimes dz^b = \left(g_{ab} + g_{55} \frac{z_a z_b}{z_5^2} \right) dz^a \otimes dz^b \\
&= \left(g_{ab} + g_{55} \frac{z_a z_b}{z_5^2} \right) \frac{\partial z^a}{\partial x^c} \frac{\partial z^b}{\partial x^d} dx^c \otimes dx^d \\
&= \left(g_{ab} + g_{55} \frac{x_a x_b}{x_5^2} \right) dx^a \otimes dx^b \\
&\quad + (\text{deformations of } \mathcal{O}(A)), \tag{3.28}
\end{aligned}$$

and

$$\begin{aligned}
ds^2 &= \left(g_{ab} + g_{55} \frac{z_a z_b}{z_5^2} \right) dz^a \otimes dz^b \\
&= g_{AB} \frac{\partial z^A}{\partial x^a} \frac{\partial z^B}{\partial x^b} dx^a \otimes dx^b = g_{AB} dz^A \otimes dz^B. \tag{3.29}
\end{aligned}$$

If the solution (3.10) is understood as a general coordinate transformation $x^A \mapsto z^A = z^A(x)$ in $(d+1)$ dimensions, one may notice that Eq. (3.29) is certainly a higher dimensional analogue of the two-dimensional result (2.46).

Now let us recapitulate why the emergent geometry we have examined so far is completely consistent with all the rationale inferred from the algebraic point of view. We are interested in the emergent geometry derived from the mass-deformed IKKT matrix model (3.2). We observed that the equations of motion can be derived from the Snyder algebra (3.6). An essential point is that the Snyder algebra (3.6) in d dimensions can be lifted to the $(d+1)$ -dimensional Lorentz algebra (3.21). So the d -dimensional Snyder algebra can be represented by the $(d+1)$ -dimensional Lorentz generators with the constraint $g_{AB} z^A z^B = (-1)^{\#} R^2$. As we know, the Lorentz algebra (3.21) represents a global symmetry of $(d+1)$ -dimensional flat spacetime. Therefore, the emergent gravity determined by the Snyder algebra (3.6) can always be embedded into $(d+1)$ -dimensional flat spacetime although the d -dimensional geometry is highly nontrivial. From the d -dimensional point of view, the geometry of hypersurface M is emergent from dynamical gauge fields as the map (3.16) definitely implies. One may clearly see this picture from Eq. (3.29). First recall that $z^a(x) = x^a + \kappa A^a(x)$ where $A^a(x)$ describe fluctuations around the background spacetime whose metric is given by Eq. (3.25). But the last result of Eq. (3.29) shows that the dynamical fluctuations of the manifold M can again be embedded into the $(d+1)$ -dimensional flat spacetime, but its embedding is now described by the “dynamical” coordinates $z^A(x) = x^A + \kappa A^A(x)$.

Like the two-dimensional case, one may consider a nonlinear deformation of the Snyder algebra by replacing the mass term in the action (3.2) by a general polynomial as follows:

$$S_G = \text{Tr} \left(\frac{1}{4} M_{ab} M^{ab} - \frac{1}{2} M_{ab} [X^a, X^b] + \frac{\kappa}{2} G(X) \right). \tag{3.30}$$

Then the equations of motion (3.5) are replaced by

$$[M^{ab}, X_b] + \kappa \left[\frac{\partial G(X)}{\partial X_a} \right] = 0, \tag{3.31}$$

where the second term is a formal expression of the matrix ordering under the trace Tr as Eq. (2.35). Equation (3.31) could be derived by considering the nonlinear version of the Snyder algebra (3.6)

$$[X^a, M^{bc}] = \kappa f^{abcd} \left[\frac{\partial G(X)}{\partial X_d} \right], \tag{3.32}$$

where $f^{abcd} = g^{ac} g^{bd} - g^{ab} g^{cd}$ has been chosen to recover the linear Snyder algebra with $G(X) = (d-1)\kappa X_a X^a$. As long as the polynomial $G(X)$ is explicitly given, the commutator $[M^{ab}, M^{cd}]$ can be calculated by applying the Jacobi identity

$$\begin{aligned}
[M^{ab}, M^{cd}] &= [M^{ab}, [X^c, X^d]] \\
&= [[M^{ab}, X^c], X^d] - [[M^{ab}, X^d], X^c] \tag{3.33}
\end{aligned}$$

and using the algebra (3.32). The right-hand side of Eq. (3.33) can eventually be arranged into the form $\kappa G^{ac}(X) M^{bd} + \dots$ using the commutation relation (3.32). Therefore, the nonlinear deformation of the Snyder algebra described by the action (3.30) seems to work. So it will be interesting to investigate whether the nonlinear Snyder algebra can still have a higher dimensional interpretation like the linear case and what kind of vacuum geometry arises from a given polynomial $G(X)$.

IV. DISCUSSION AND CONCLUSION

Here we discuss the fact that the constant curvature space described by the Snyder algebra (3.6) can be represented as a coset space G/H . In other words, the d -dimensional hypersurface M is a homogeneous space. To be specific, we have the following coset realization of M :

$$\begin{aligned}
S^d &= \text{SO}(d+1)/\text{SO}(d), \\
\text{Ad}S_d &= \text{SO}(d, 1)/\text{SO}(d-1, 1), \\
\text{Ad}S_d &= \text{SO}(d-1, 2)/\text{SO}(d-1, 1). \tag{4.1}
\end{aligned}$$

Taking G to be a Lie group as in Eq. (4.1), the coset manifold endows a Riemannian structure as we already know. Split the Lie algebra of G as $\mathbb{G} = \mathbb{H} \oplus \mathbb{K}$ where \mathbb{H} is the Lie algebra of H and \mathbb{K} contains the coset generators. The structure constants of G are defined by [28]

$$\begin{aligned}
[H_i, H_j] &= f_{ij}^k H_k, \quad H_i \in \mathbb{H}, \\
[H_i, K_a] &= f_{ia}^j H_j + f_{ia}^b K_b, \quad K_a \in \mathbb{K}, \\
[K_a, K_b] &= f_{ab}^i H_i + f_{ab}^c K_c. \tag{4.2}
\end{aligned}$$

If $f_{ia}^j = 0$, the coset space G/H is said to be reductive and, if $f_{ab}^c = 0$, it is called symmetric.

In order to realize the coset space (4.1) from the Snyder algebra (3.6), it is obvious how to identify the generators in \mathbb{H} and \mathbb{K} : $K_a = iX_a \in \mathbb{K}$ and $H_i = M_{ab} \in \mathbb{H}$. From this identification, we see that the coset space (4.1) is symmetric as well as reductive, which is a well-known fact. Therefore, it will be interesting to see how the emergent geometry from the Snyder algebra (3.6) can be constructed from the Riemannian geometry of the coset space G/H . The whole geometry of G/H can be constructed in terms of coset representatives

$$L(y) = e^{y^a K_a} \quad (a = 1, \dots, \dim G - \dim H), \quad (4.3)$$

where the local coordinates y^a parameterize the coset gH for any $g \in G$. Under left multiplication by a generic element g of G , the coset representative (4.3) will be transformed to another representative $L(y')$ of the form

$$gL(y) = L(y')h, \quad h \in H, \quad (4.4)$$

where y' and h depend on y and g and on the way of choosing representatives.

Consider the Lie algebra valued one-form

$$V(y) = L^{-1}(y)dL(y) = V^a(y)K_a + \Omega^i(y)H_i. \quad (4.5)$$

The one-form $V^a(y) = V^a_\mu(y)dy^\mu$ is a covariant frame (vielbein) on G/H and $\Omega^i_\mu(y)dy^\mu$ is called the H connection. Under left multiplication by a constant $g \in G$, the one-form (4.5) transforms according to Eq. (4.4) as

$$V(y') = hL^{-1}(y)g^{-1}d(gL(y)h^{-1}) = hV(y)h^{-1} + hdh^{-1}. \quad (4.6)$$

One can check using Eq. (4.6) that the left action of G on $V^a(y)$ is equivalent to a $SO(d)$ or $SO(d-1, 1)$ rotation on $V^a(y)$ ($d = \dim G/H$) [28]. The metric of the coset space G/H can be written in terms of the vielbeins in Eq. (4.5) as

$$\mathcal{G}_{\mu\nu}^{(0)}(y) = g_{ab}V^a_\mu(y)V^b_\nu(y), \quad (4.7)$$

where g_{ab} is the flat coset metric and the metric (4.7) is invariant under the left action of G due to the property (4.6).

Because the metric (3.25) describes the coset manifolds (4.1), it will be equivalent to the G -invariant metric (4.7). Note that the metric (3.25) is also G -invariant as Eq. (3.26) definitely shows. So let us check the Riemannian structure of the coset spaces (4.1). The differential properties of the one-form (4.5) are expressed by the Maurer-Cartan equation

$$dV + V \wedge V = 0. \quad (4.8)$$

Using Eq. (4.2), one can decompose the Maurer-Cartan Eq. (4.8) as

$$dV^a + \frac{1}{2}f^a_{bc}V^b \wedge V^c + f^a_{ib}\Omega^i \wedge V^b = 0, \quad (4.9)$$

$$d\Omega^i + \frac{1}{2}f^i_{ab}V^a \wedge V^b + f^i_{ja}\Omega^j \wedge V^a + \frac{1}{2}f^i_{jk}\Omega^j \wedge \Omega^k = 0. \quad (4.10)$$

In our case the above equations are much simpler because $f^a_{bc} = f^i_{ja} = 0$. Combining Eq. (4.9) together with the torsion free condition $DV^a = dV^a + \omega^a_b \wedge V^b = 0$ yields the spin connection on G/H

$$\omega^a_b = f^a_{ib}\Omega^i. \quad (4.11)$$

The Riemann curvature tensor is defined in terms of ω^a_b by

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b. \quad (4.12)$$

Substituting (4.11) into (4.12) and using Eq. (4.10) leads to the curvature tensors

$$\begin{aligned} R^a_b &= -\frac{1}{2}f^a_{ib}f^i_{cd}V^c \wedge V^d + (f^a_{ic}f^c_{jb} - \frac{1}{2}f^k_{ij}f^a_{kb})\Omega^i \wedge \Omega^j \\ &= \frac{1}{2}R^a_{bcd}V^c \wedge V^d, \end{aligned} \quad (4.13)$$

where the second term in Eq. (4.13) vanishes because of the Jacobi identity $[[K_a, H_i], H_j] + [[H_i, H_j], K_a] + [[H_j, K_a], H_i] = 0$.

Comparing the coset algebra (4.2) with the Snyder algebra (3.6) leads to the identification of the structure constant $f^a_{id} = g^{ae}f_{eid}$ for $i = [bc]$

$$f_{a[bc]d} = f_{abcd} = g_{ac}g_{bd} - g_{ab}g_{cd}. \quad (4.14)$$

Then the Riemann curvature tensor (4.13) of coset manifold G/H is given by⁴

$$\begin{aligned} R_{abcd} &= -f_{aib}f^i_{cd} = -f_{aefb}f^{ef}_{cd} \\ &= g_{55}(g_{ac}g_{bd} - g_{ad}g_{bc}). \end{aligned} \quad (4.15)$$

As was shown in Eq. (3.25), a vacuum geometry of the Snyder algebra (3.6) is also given by an Einstein manifold of constant curvature and is precisely the same as Eq. (4.15). Therefore, we confirm that the vacuum geometry of the Snyder algebra (3.6) is described by the G -invariant metric (4.7) of the coset space G/H . But we have to notice that the Snyder algebra (3.14) is in general defined by dynamical gauge fields fluctuating around the vacuum manifold G/H as Eq. (3.28) clearly shows. One might already notice that the generators in Eq. (4.2) are constant matrices while those in Eq. (3.6) are in general

⁴According to the identification (3.21), the first Snyder algebra for four-dimensional anti-de Sitter space is given by $[X^a, X^b] = \kappa[M^{5,a}, M^{5,b}] = \kappa g^{55}M^{ab}$. Thus the anti-de Sitter space will be equally cared by the replacement $f^i_{ab} \rightarrow g_{55}f^i_{ab}$ in the algebra (4.2). That is the reason why the g_{55} factor comes in Eq. (4.15).

mapped to NC fields in \mathcal{A}_θ as in Eq. (3.14). Therefore, it should be interesting to directly derive Einstein's equations [4] to incorporate all possible deformations induced by gauge fields from the Snyder algebra, whose metric may be G -invariant as always, as we checked in Eq. (3.29). We hope to address this issue in the near future.

Let us conclude with some remarks about the significance of emergent geometry based on the results we have obtained. The emergence usually means the arising of novel and coherent structures, patterns, and properties through the collective interactions of more fundamental entities, for example, the superconductivity in condensed matter system or the organization of life in biology. In our case, we are talking about the emergence of a much more bizarre object: gravity. A stringent point of emergent gravity is to require that spacetime should also be emergent simultaneously according to the picture of general relativity.

What does the emergence of spacetime mean? It means that the emergent gravity should necessarily be background independent where the prior existence of any spacetime structure is not *a priori* assumed but should be defined by fundamental ingredients in quantum gravity theory. We have already exhibited such examples with the matrix actions (1.11), (2.1), and (3.2).

Let us pick up the simplest example (2.5) to illuminate how some geometry emerges from a background-independent theory. Note that the action (2.5) is a “zero-dimensional” matrix model. In order to define the action (2.5), we did not introduce any kind of spacetime structure. We only have three Hermitian matrices (as objects) which are subject to the algebraic relations (2.6) and (2.7) (as morphisms).⁵ From these algebraic relations between objects, we can derive a geometry by mapping the matrix algebra to a Poisson algebra or a NC \star algebra, as was shown in Sec. II. Depending on the choice of an algebraic relation characterized by the signature of g_{AB} , we get a different geometry. The underlying argument should be familiar, in particular, with the representation theory of Lie groups and Lie algebras.

A profound aspect of emergent geometry is that a background-independent formulation can be realized with matrix models, as we illustrated with the actions (1.11), (2.1), and (3.2). In this approach, an operator algebra, e.g., \star algebra defined by NC gauge fields, defines a relational fabric between NC gauge fields, whose prototype at a macroscopic world emerges as a smooth spacetime geometry. In this scheme, the geometry is a derived concept defined by the algebra. One has to specify an underlying algebra to talk about a corresponding geometry. Furthermore, a smooth geometry is doomed in a deep

NC space, whereas an algebra between objects plays a more fundamental role. Therefore, the motto of emergent gravity is that an algebra defines a geometry.

As we observed in Eq. (1.7), the map between a Poisson algebra $(C^\infty(M), \{-, -\}_\theta)$ and the Lie algebra $(\Gamma(TM), [-, -])$ of vector fields is a Lie algebra homomorphism. This means that a geometric structure determined by the Lie algebra $(\Gamma(TM), [-, -])$ is faithfully inherited from the Poisson algebra $(C^\infty(M), \{-, -\}_\theta)$. Thus the map between an underlying algebra and its emergent geometry should be structure-preserving, i.e., a homomorphism. This homomorphism is also true even for a general Poisson structure. Actually it should be required for consistency of emergent gravity. If not, one could not say that a geometry can be derived from an algebra.

In our case, this implies that an algebraic structure in a matrix theory will be encoded in a geometric structure of emergent gravity. Note, as we showed in Sec. III, the maximally symmetric spaces in Eq. (4.1) can be derived from the Snyder algebra (3.6) by applying the map (3.16). And recall that those d -dimensional symmetric spaces can always be embedded in a $(d + 1)$ -dimensional flat spacetime. If so, a natural question is how this geometric property is encoded in the Snyder algebra (3.6). As Eq. (3.21) shows, the geometric property is precisely realized as the fact that the d -dimensional Snyder algebra can be arranged into the Lorentz algebra in $(d + 1)$ -dimensional flat spacetime. Although the equivalence between the d -dimensional Snyder algebra and the $(d + 1)$ -dimensional Lorentz algebra is a well-known fact, it is a nice nontrivial check that the algebraic structure of the Snyder algebra has been consistently encoded in the geometric property of emergent spacetime since the emergent gravity has to respect the homomorphism from an algebra to a geometry for consistency.

As a completely different direction, we may consider the matrices (X^a, M^{ab}) as independent dynamical coordinates, which satisfy the $(d + 1)$ -dimensional Lorentz algebra (3.21). As an example, a three-dimensional sphere S^3 appears in this way from the SU(2) algebra (2.6) as we discussed in the footnote 2. In this case there are $d(d + 1)/2$ coordinates in total and so we will get some $d(d + 1)/2$ -dimensional manifold from the algebra (3.6) or (3.21). Although we do not know what the underlying Poisson structure is in this case, we guess that the resulting emergent geometry derived from the Lorentz algebra (3.21) would be a group manifold of $SO(d + 1 - p, p)$ as can be inferred from the three-dimensional case. To clarify this issue will be an interesting future work.

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⁵Indeed g_{AB} is nothing more than a symbol for the algebraic characterization of “zero-dimensional” matrices although it will be realized as a three-dimensional metric in the end.

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APPENDIX: TWO-DIMENSIONAL SNYDER ALGEBRA

Here we will show that the two-dimensional version of the Snyder algebra (3.6) is precisely equal to the three-dimensional $SO(3 - p, p)$ Lie algebra (2.6).

In two dimensions, the Snyder algebra (3.6) reads as

$$\begin{aligned} [X^1, X^2] &= M^{12}, & [X^1, M^{12}] &= -\kappa g^{11} X^2, \\ [X^2, M^{12}] &= \kappa g^{22} X^1. \end{aligned} \quad (\text{A1})$$

If one defines $M^{12} \equiv \pm i\lambda X^3$ ($= -i\lambda \varepsilon_3^{12} X^3$), one can immediately see that the Snyder algebra (A1) can be written as the form of the Lie algebra (2.6) with $\kappa = -\lambda^2 \det g_{AB}$. Conversely, if one defines $X^A \equiv \frac{i}{2\lambda} \varepsilon^A_{BC} M^{BC}$ ($A, B, C = 1, 2, 3$), the Snyder algebra (A1) takes the form of the three-dimensional Lorentz algebra (3.21). Note that the two-dimensional Snyder algebra (A1) is the equation of motion derived from the action (3.2), which can be rewritten as the action (2.5) for the three-dimensional Lie algebra with the above identification. It might be remarked that the three dimensions is special in the sense that an antisymmetric rank-2 tensor is dual to a vector, i.e., $M^{AB} = -i\lambda \varepsilon^{AB}_C X^C$ and so the Lorentz alge-

bra (3.21) can be expressed as the form (2.6) only in three dimensions.

As we discussed in Sec. II, the quadratic form $C_2 \equiv \sum_{A,B=1}^3 g_{AB} X^A X^B$ is a Casimir invariant of $SO(3 - p, p)$ Lie algebra, i.e.,

$$[X^A, C_2] = 0, \quad \forall A = 1, 2, 3. \quad (\text{A2})$$

Because $X^A = \frac{i}{2\lambda} \varepsilon^A_{BC} M^{BC}$, Eq. (A2) can be rewritten as

$$[M^{AB}, C_2] = 0, \quad \forall M^{AB} \in SO(3 - p, p). \quad (\text{A3})$$

This means that C_2 is a Lorentz invariant, which can also be derived using the commutation relation

$$[X^A, M^{BC}] = \kappa(g^{AC} X^B - g^{AB} X^C). \quad (\text{A4})$$

The invariance (A2) implies that C_2 is a multiple of the identity element of the algebra such as Eq. (2.7). From the viewpoint (A3), C_2 is an invariant under $SO(3 - p, p)$ Lorentz transformations. Therefore, the Casimir invariant (2.7) can simultaneously be interpreted as a Lorentz invariant which reduces to the three-dimensional version of Eq. (3.22), i.e. $\sum_{A,B=1}^3 g_{AB} x^A x^B = (-1)^{\#} R^2$, in a classical limit.

In summary, it was shown that the three-dimensional $SO(3 - p, p)$ Lie algebra (2.6) is isomorphic to the two-dimensional version of the Snyder algebra (3.6) where the embedding condition (3.22) can be identified with the Casimir invariant (2.7).

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