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A SUBGROUP OF THE FULL POSET-ISOMETRY GROUP*

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Abstract. Let P be a poset on $[n]$. We construct a subgroup $\mathcal{G}_P$ of the full poset-isometry group $\mathrm{Iso}_P(F_q^n)$ and present the structure of $\mathcal{G}_P$ as well as its size. We also find poset-metric spaces satisfying $\mathcal{G}_P = \mathrm{Iso}_P(F_q^n)$ by using a characterization of $\mathcal{G}_P$.

Key words. isometry group, poset-isometry, poset-metric, $\mathcal{NRT}$-metric

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1. Introduction. Let F_q be a finite field with q elements, and let F_q^n be the vector space of n-tuples over F_q. Brualdi, Graves, and Lawrence [2] introduced a poset-metric on F_q^n associated with a partially ordered set, and since then it has been extensively studied in [1, 3, 4, 7, 6]. We briefly introduce basic notions of a poset-metric. Let P be a partially ordered set (simply, poset) on the set $[n] = \{1, 2, \ldots, n\}$ with the order relation $\preceq$. A subset I of P is called an order ideal if $j \in I$ and $i \preceq j$ imply $i \in I$. For an arbitrary subset A of P, we denote by $\langle A\rangle$ the smallest order ideal of P containing A. We say that $i \in A$ is maximal if $i \preceq j$ and $j \in A$ imply $i = j$. A bijection $\sigma : P \to P$ is called an automorphism if σ and σ^{-1} preserve the order relation of P. The group of automorphisms of P is denoted by $\mathrm{Aut}(P)$.

Let $P = ([n], \preceq)$ be a poset on the set $[n]$ of coordinate positions of vectors in F_q^n. The P-weight of a vector u in F_q^n is defined by $w_P(u) = |\langle \mathrm{supp}(u)\rangle|$, where $\mathrm{supp}(u)$ is the set of nonzero coordinate positions of a vector u and $|X|$ denotes the size of a finite set X. For u and v in F_q^n, the P-distance $d_P(u, v)$ between u and v is defined by $d_P(u, v) = w_P(u - v)$. It is well known [2] that $d_P(\cdot, \cdot)$ determines a metric on F_q^n, called the P-metric or the poset-metric. We denote such a P-metric space by (F_q^n, d_P). If P is an antichain, then d_P coincides with the Hamming metric. More generally, an $\mathcal{NRT}$-metric space associated with the poset of a disjoint union of chains of equal length is an important subject of studying uniform distributions of points in the unit cube [1, 5, 8, 9].

A map of F_q^n into F_q^n is called a P-isometry if it preserves the P-distance. It is easy to see that every P-isometry is a bijection. We denote by $\mathrm{LIso}_P(F_q^n)$ the group of linear P-isometries of F_q^n and call it the linear isometry group of (F_q^n, d_P). The group of linear P-isometries of F_q^n was completely established in [6]. It is now of interest to determine the isometry group (not a linear one) of a poset-metric space. We denote by $\mathrm{Iso}_P(F_q^n)$ the group of P-isometries of F_q^n and call it the isometry group of (F_q^n, d_P). In [7], as the first attempt of the study, the authors described the isometry group of a product of $\mathcal{NRT}$-metric spaces. In general, the problem of determining the poset-isometry group is not easy even for the case of the full **N**-metric space. In this paper,

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we show that nonlinear poset-isometries are rather rich by constructing a subgroup $\mathcal{G}_P$ (see section 3) such that $\mathrm{LIso}_P(F_q^n) \leq \mathcal{G}_P \leq \mathrm{Iso}_P(F_q^n)$.

In section 2, we develop the properties of the poset-isometries on any poset-metric space. In section 3, we construct a group $\mathcal{G}_P$ and present the structure of $\mathcal{G}_P$ as well as its size. By using a characterization of $\mathcal{G}_P$, we find poset-metric spaces for which $\mathcal{G}_P = \mathrm{Iso}_P(F_q^n)$.

2. Properties of the poset-isometries. In this section, we will develop auxiliary results which will be useful in the next section. Let u be a vector in F_q^n. In what follows, we denote by $\{e_1, e_2, \ldots, e_n\}$ the canonical basis of F_q^n, by M_u the set of maximal elements of $\mathrm{supp}(u)$, and by u_l the lth coordinate component of u. We remark that if T is a P-isometry of F_q^n which fixes the origin, then it preserves the P-weight. Obviously, $\mathrm{supp}(u+v) \subseteq \mathrm{supp}(u) \cup \mathrm{supp}(v)$ for all $u, v \in F_q^n$. We will use those facts without further mention.

LEMMA 2.1. *Let T be a P-isometry of F_q^n which fixes the origin. Let α be a nonzero element in F_q and j in $[n]$. Then we have $T(u) = T(u+\alpha e_j) + T(v)$ for some v in F_q^n. Moreover, we have $w_P(v) = |\langle j \rangle|$.*

Proof. Let T be a P-isometry of F_q^n which fixes the origin, and let α be a nonzero element in F_q. Since T is a bijection, we can write $T^{-1}(T(u) - T(u+\alpha e_j)) = v$ for some $v \in F_q^n$, or, equivalently, $T(u) = T(u+\alpha e_j) + T(v)$. It follows that $w_P(v) = w_P(T(v)) = w_P(T(u) - T(u+\alpha e_j)) = d_P(u, u+\alpha e_j) = |\langle j \rangle|$. This proves the lemma. □

LEMMA 2.2. *Let T be a P-isometry of F_q^n which fixes the origin, and let u be a nonzero vector in F_q^n. If $j \in \mathrm{supp}(T(u))$, then we have $|\langle j \rangle| \leq |\langle i \rangle|$ for some $i \in \mathrm{supp}(u)$.*

Proof. Let T be a P-isometry of F_q^n which fixes the origin and $\mathrm{supp}(u) = \{i_1, i_2, \ldots, i_s\}$. It follows from Lemma 2.1 that

$$T(u) = T(u - e_{i_1}) + T(v^{(1)}) \tag{1}$$

for some nonzero vector $v^{(1)}$ and $w_P(v^{(1)}) = |\langle i_1 \rangle|$. In the same way, we have

$$T(u - e_{i_1}) = T(u - e_{i_1} - e_{i_2}) + T(v^{(2)}) \tag{2}$$

for some nonzero vector $v^{(2)}$ and $w_P(v^{(2)}) = |\langle i_2 \rangle|$. By substituting (2) into (1), we have $T(u) = T(u - e_{i_1} - e_{i_2}) + T(v^{(1)}) + T(v^{(2)})$ and $w_P(v^{(r)}) = |\langle i_r \rangle|, r = 1, 2$. Continuing these processes, we see that

$$T(u) = T(e_{i_s}) + T(v^{(s-1)}) + T(v^{(s-2)}) + \cdots + T(v^{(1)}) \tag{3}$$

and $w_P(v^{(r)}) = |\langle i_r \rangle|, r = 1, 2, \ldots, s-1$. Since $j \in \mathrm{supp}(T(u))$, it follows from (3) that

$$\begin{aligned} j \in \mathrm{supp}(T(u)) &\subseteq \mathrm{supp}(T(e_{i_s})) \cup \mathrm{supp}(T(v^{(s-1)})) \\ &\cup \mathrm{supp}(T(v^{(s-2)})) \cup \cdots \cup \mathrm{supp}(T(v^{(1)})). \end{aligned}$$

If $j \in \mathrm{supp}(T(e_{i_s}))$, then $|\langle j \rangle| \leq |\langle \mathrm{supp}(T(e_{i_s})) \rangle| = w_P(T(e_{i_s})) = w_P(e_{i_s}) = |\langle i_s \rangle|$. If $j \in \mathrm{supp}(T(v^{(r)}))$ for some $r = 1, 2, \ldots, s-1$, then we have $|\langle j \rangle| \leq |\langle \mathrm{supp}(T(v^{(r)})) \rangle| = w_P(T(v^{(r)})) = w_P(v^{(r)}) = |\langle i_r \rangle|$. In either case, we have the desired result. □

An order ideal of P is principal if it contains a unique maximal element.

LEMMA 2.3. *Let T be a P-isometry of F_q^n which fixes the origin and $i \in [n]$. Then for $\alpha \in F_q$, the order ideal generated by $\mathrm{supp}(T(\alpha e_i))$ is principal. Moreover, for nonzero $\alpha \in F_q$, the size of $\langle \mathrm{supp}(T(\alpha e_i)) \rangle$ is the same as that of $\langle i \rangle$.*

Proof. Let T be a P-isometry of F_q^n which fixes the origin. We will show that the order ideal generated by $\mathrm{supp}(T(\alpha e_i))$ is principal. If α is the zero element in F_q, then there is nothing to prove. Assume that α is a nonzero element in F_q^n. Let $T(\alpha e_i) = v$ for some $v \in F_q^n$. We first claim that there is an element $j \in \mathrm{supp}(T(\alpha e_i))$ such that $|\langle j\rangle| = |\langle i\rangle|$. Assume to the contrary. Then by applying Lemma 2.2, we have $|\langle j\rangle| < |\langle i\rangle|$ for every j in $\mathrm{supp}(T(\alpha e_i)) = \mathrm{supp}(v)$. Since $i \in \mathrm{supp}(\alpha e_i) = \mathrm{supp}(T^{-1}(v))$ and T^{-1} is also a P-isometry of F_q^n which fixes the origin, we apply Lemma 2.2, and so we have $|\langle i\rangle| \le |\langle l\rangle|$ for some l in $\mathrm{supp}(v)$. It follows from those observations that $|\langle l\rangle| < |\langle i\rangle| \le |\langle l\rangle|$, a contradiction. This proves the claim. Next, we prove that the element j is unique. We assume that there are two distinct elements j_1, j_2 in $\mathrm{supp}(v)$ satisfying $|\langle j_1\rangle| = |\langle j_2\rangle| = |\langle i\rangle|$. Then we have $|\langle i\rangle| = w_P(T(\alpha e_i)) = |\langle \mathrm{supp}(v)\rangle| \ge |\langle j_1, j_2\rangle| = |\langle j_1\rangle \cup \langle j_2\rangle| = 2|\langle i\rangle| - |\langle j_1\rangle \cap \langle j_2\rangle|$. This yields that $|\langle j_1\rangle \cap \langle j_2\rangle| \ge |\langle i\rangle| = |\langle j_1\rangle|$ and so we have $j_1 = j_2$, a contradiction. This proves the lemma. ☐

Let T be a P-isometry of F_q^n which fixes the origin. It follows from Lemma 2.3 that T induces a map $\phi_T : P \to P$, which is defined by

$$\phi_T(i) = M_{T(e_i)}. \tag{4}$$

Then ϕ_T is a well-defined map by Lemma 2.3. We call ϕ_T the map induced by T. It follows from Lemma 2.3 that for a nonzero element α in F_q and $i \in [n]$,

$$\langle \mathrm{supp}(T(\alpha e_i))\rangle = \langle \phi_T(i)\rangle \quad \text{and} \quad |\langle \phi_T(i)\rangle| = |\langle i\rangle|, \tag{5}$$

where ϕ_T is the map induced by T. This means that $T(e_j)$ can be written in the form

$$T(e_j) = \sum_{i \in \langle \phi_T(j)\rangle} a_{ij} e_i, \tag{6}$$

where $a_{ij} \in F_q$ and $a_{lj} \neq 0$ for $l = \phi_T(j)$.

PROPOSITION 2.4. *Let $P = ([n], \preceq)$ be a poset, and let T be a P-isometry of F_q^n, which fixes the origin. Then the map $\phi_T : P \to P$ defined by $\phi_T(i) = M_{T(e_i)}$ is an automorphism of P.*

Proof. We need to prove that ϕ_T is a bijective map of P onto P, and that ϕ_T and ϕ_T^{-1} preserve the order relation of P.

Bijectivity: Assume that $\phi_T(i) = \phi_T(j)$ but $i \neq j$. It follows from Lemma 2.3 that

$$\langle \mathrm{supp}(T(e_i))\rangle = \langle \phi_T(i)\rangle = \langle \phi_T(j)\rangle = \langle \mathrm{supp}(T(e_j))\rangle. \tag{7}$$

By using (6) and (7), we have

$$d_P(T(e_i), T(e_j)) \le w_P(T(e_i)) = w_P(e_i) = |\langle i\rangle|. \tag{8}$$

It follows from $i \neq j$ and (8) that $|\langle i, j\rangle| = d_P(e_i, e_j) = d_P(T(e_i), T(e_j)) \le |\langle i\rangle|$, which yields that $i = j$, a contradiction. This proves that ϕ_T is an injective map of a finite set P into P and hence it is bijective.

Order preservation: Notice that if $i \neq j$, then by using (6) and the bijectivity of ϕ_T, we have

$$d_P(T(e_i), T(e_j)) = w_P(T(e_i) - T(e_j)) = |\langle \phi_T(i), \phi_T(j)\rangle|. \tag{9}$$

Assume that $i \prec j$. Then we have

$$|\langle \phi_T(j)\rangle| \overset{(5)}{=} |\langle j\rangle| = |\langle i,j\rangle| = d_P(e_i,e_j) = d_P(T(e_i),T(e_j)) \overset{(9)}{=} |\langle \phi_T(i),\phi_T(j)\rangle|.$$

This proves that $\langle \phi_T(j)\rangle = \langle \phi_T(i),\phi_T(j)\rangle$ and so we have $\phi_T(i) \prec \phi_T(j)$. Conversely, assume that $\phi_T(i) \prec \phi_T(j)$. Then $|\langle i,j\rangle| = |\langle \phi_T(i),\phi_T(j)\rangle| = |\langle \phi_T(j)\rangle| = |\langle j\rangle|$. This proves that $\langle i,j\rangle = \langle j\rangle$, and we conclude that $i \prec j$. Hence the result follows. ∎

Let $u = (u_1,u_2,\ldots,u_n) \in F_q^n$ and $\sigma \in \mathrm{Aut}(P)$. We define $u_\sigma = \sum_{i=1}^n u_i e_{\sigma(i)}$.

LEMMA 2.5. *Let T be a P-isometry of F_q^n which fixes the origin, and let u be a vector in F_q^n. If $\langle \mathrm{supp}(u)\rangle = \langle i\rangle$, then $\langle \mathrm{supp}(T(u))\rangle = \langle \phi_T(i)\rangle$, where ϕ_T is the automorphism of P induced by T.*

Proof. Let T be a P-isometry of F_q^n which fixes the origin and $\langle \mathrm{supp}(u)\rangle = \langle i\rangle$. As in the proof of Lemma 2.1, we have

$$T(u) = T(u_i e_i) + T(v_{\phi_T}) \tag{10}$$

for some $v_{\phi_T} \in F_q^n$ and ϕ_T is the automorphism of P induced by T. Moreover, by using (10), we have

$$w_P(v_{\phi_T}) = w_P(u - u_i e_i) < w_P(u). \tag{11}$$

The last inequality in (11) follows from the fact that i is a maximal element of $\mathrm{supp}(u)$. It follows from (5) and (10) that $\phi_T(i)$ is in $\mathrm{supp}(T(u_i e_i)) = \mathrm{supp}(T(u) - T(e_{v_{\phi_T}}))$ $\subseteq \mathrm{supp}(T(u)) \cup \mathrm{supp}(T(v_{\phi_T}))$. We claim that $\phi_T(i) \in \mathrm{supp}(T(u))$. Otherwise, we should have $\phi_T(i) \in \mathrm{supp}(T(v_{\phi_T}))$. By assumption and Lemma 2.2, we have

$$w_P(u) = |\langle i\rangle| = |\langle \phi_T(i)\rangle| \le |\langle l\rangle| \tag{12}$$

for some $l \in \mathrm{supp}(v_{\phi_T})$. By combining (11) and (12), we have that $w_P(u) \le |\langle l\rangle|$ $\le w_P(v_{\phi_T}) = w_P(u - u_i e_i) < w_P(u)$, a contradiction. This shows that $\langle \phi_T(i)\rangle \subseteq \langle \mathrm{supp}(T(u))\rangle$. But $|\langle \mathrm{supp}(T(u))\rangle| = w_P(T(u)) = w_P(u) = |\langle \mathrm{supp}(u)\rangle| = |\langle i\rangle| = |\langle \phi_T(i)\rangle|$. This proves the lemma. ∎

By $\Gamma_u, u \in F_q^n$ we denote the set $\{i \in M_u : |\langle i\rangle| \le |\langle j\rangle| \text{ for all } j \in M_u\}$.

LEMMA 2.6. *Let T be a P-isometry of F_q^n which fixes the origin, and let u be a nonzero vector in F_q^n. If $i \in \Gamma_u$, then we have $T(u) = T(u - u_i e_i) + T(v_{\phi_T})$ for some $v_{\phi_T} \in F_q^n$, where ϕ_T is the automorphism of P induced by T. Moreover, we have $\langle \mathrm{supp}(v_{\phi_T})\rangle = \langle j\rangle$ and $\langle \mathrm{supp}(T(v_{\phi_T}))\rangle = \langle \phi_T(j)\rangle$ for some $j \in \Gamma_u$.*

Before proving the lemma, it should be mentioned that j indeed equals i. See the proof of Theorem 2.7.

Proof. Let T be a P-isometry of F_q^n which fixes the origin and $M_u = \{j_1,j_2,\ldots,j_t\}$. It follows from Lemma 2.1 that for $i \in \Gamma_u$,

$$T(u) = T(u - u_i e_i) + T(v_{\phi_T}), \tag{13}$$

$w_P(v_{\phi_T}) = |\langle i\rangle|$, and ϕ_T is the automorphism of P induced by T. By Lemma 2.5, it is sufficient to show that $\langle \mathrm{supp}(v_{\phi_T})\rangle = \langle j\rangle$ for some $j \in \Gamma_u$. We have two cases to consider. The first case is when Γ_u is a proper subset of M_u, say, $|\langle j_1\rangle| = |\langle j_2\rangle| = \cdots = |\langle j_s\rangle| < |\langle j_{s+1}\rangle| \le \cdots \le |\langle j_t\rangle|$. We claim that (i) $j_r \notin \mathrm{supp}(v_{\phi_T})$ for all r, $r = s+1, s+2, \ldots, t$, and (ii) $j_r \in \mathrm{supp}(v_{\phi_T})$ for some r, $r = 1,2,\ldots,s$. Proof of claim (i): Assume to the contrary that $j_r \in \mathrm{supp}(v_{\phi_T})$ for some r, $r = s+1, s+2, \ldots, t$. Then $|\langle j_r\rangle| \le |\langle \mathrm{supp}(v_{\phi_T})\rangle| = w_P(v_{\phi_T}) = |\langle i\rangle|$, a contradiction to the fact that $i \in \Gamma_u$.

Proof of claim (ii): Assume to the contrary that $j_r \notin \mathrm{supp}(v_{\phi_T})$ for all $r = 1, 2, \ldots, s$. It follows from (13) and the P-distance preservation of T that

$$w_P(u - u_{j_1}e_{j_1}) = w_P(u - v_{\phi_T}). \tag{14}$$

By using assumption and claim (i), we have $w_P(u - v_{\phi_T}) \geq w_P(u)$. By combining this and (14), we have $w_P(u - u_{j_1}e_{j_1}) \geq w_P(u)$. This is a contradiction to the fact that $j_1 \in M_u$. Hence claim (ii) is proved. Now, by using claim (ii), $w_P(v_{\phi_T}) = |\langle i\rangle|$, and $i \in \Gamma_u$, we see that $\langle \mathrm{supp}(v_{\phi_T})\rangle = \langle j_r\rangle$ for some r, $r = 1, 2, \ldots, s$. Hence the first case is proved. The second case is when $\Gamma_u = M_u$, that is, $|\langle j_1\rangle| = |\langle j_2\rangle| = \cdots = |\langle j_t\rangle|$. The proof is similar to the first case. This proves the lemma. ∎

We are ready to state the main result of this section.

THEOREM 2.7. *Let $P = ([n], \preceq)$ be a poset, and let T be a P-isometry of F_q^n which fixes the origin. Then we have $\langle \mathrm{supp}(T(u))\rangle = \langle \phi_T(\mathrm{supp}(u))\rangle$ for any vector u in F_q^n, where ϕ_T is the automorphism of P induced by T.*

Proof. Let T be a P-isometry of F_q^n which fixes the origin, and let u be a vector in F_q^n. If u is the zero vector, then we are done. We proceed by induction on the size of $\mathrm{supp}(u)$, which we have already shown in Lemma 2.3 in the case when the size of $\mathrm{supp}(u)$ equals one. Assume that $\langle \mathrm{supp}(T(u))\rangle = \langle \phi_T(\mathrm{supp}(u))\rangle$ for any u of support size s $(< n)$. Let $v \in F_q^n$ with $|\mathrm{supp}(v)| = s + 1$. Choose $l \in \Gamma_v$. It follows from Lemma 2.6 that

$$T(v) = T(v - v_l e_l) + T(v_{\phi_T}), \tag{15}$$

$\langle \mathrm{supp}(T(v_{\phi_T}))\rangle = \langle \phi_T(j)\rangle$ for some $j \in \Gamma_v$, and ϕ_T is the automorphism of P induced by T. Let us write $M_v = \{k_1(= l), k_2, \ldots, k_t\}$. By using (15) and the induction hypothesis, we see that

$$\begin{aligned}\langle \mathrm{supp}(T(v))\rangle &\subseteq \langle \mathrm{supp}(T(v - v_l e_l))\rangle \cup \langle \mathrm{supp}(T(v_{\phi_T}))\rangle\\ &= \langle \phi_T(\mathrm{supp}(v - v_l e_l))\rangle \cup \langle \phi_T(j)\rangle\\ &= \langle \phi_T(k_2), \phi_T(k_3), \ldots, \phi_T(k_t), \phi_T(j)\rangle. \end{aligned}\tag{16}$$

So we have

$$\begin{aligned}|\langle k_1, k_2, \ldots, k_t\rangle| = w_P(v) = w_P(T(v)) &\leq |\langle \phi_T(k_2), \ldots, \phi_T(k_t), \phi_T(j)\rangle|\\ &= |\langle k_2, \ldots, k_t, j\rangle|.\end{aligned}\tag{17}$$

Since $j \in \Gamma_v$, it follows from (17) that $j = k_1(= l)$. It then follows from (16) and (17) that $\langle \mathrm{supp}(T(v))\rangle = \langle \phi_T(k_1), \phi_T(k_2), \ldots, \phi_T(k_t)\rangle = \langle \phi_T(\mathrm{supp}(v))\rangle$. This proves the theorem. Moreover, the remark in Lemma 2.6 is automatically proved. ∎

3. Construction of $\mathcal{G}_P$ and its application. Let $P = ([n], \preceq)$ be a poset. In this section, we will construct a subgroup $\mathcal{G}_P$ such that $\mathrm{LIso}_P(F_q^n) \leq \mathcal{G}_P \leq \mathrm{Iso}_P(F_q^n)$. We start with definitions and notations. For a subset A of P, we define the upset $\uparrow A$ (resp., $\upharpoonleft A$) to be the set $\uparrow A = \bigcup_{i\in A}\{j \in P : j \succeq i\}$ (resp., $\upharpoonleft A = \bigcup_{i\in A}\{j \in P : j \succ i\}$). If $A = \{l\}$, then we denote $\upharpoonleft \{l\}$ (resp., $\upharpoonleft \{l\}$) by $\uparrow l$ (resp., $\upharpoonleft l$) simply. We write $l \parallel A$ if l is incomparable to every element of A.

Let $u = (u_1, u_2, \ldots, u_n)$ be a vector in F_q^n, and let A be a subset of P. We define u^A as $u^A = \sum_{i\in A} u_i e_i$. By the definition of u^A, we have $\mathrm{supp}(u^{\uparrow l}) = \{j \in\uparrow l : u_j \neq 0\}$. For $l \in [n]$, let $G_l : F_q^n \to F_q$ be a map. Define a map $T_{(G_1,G_2,\ldots,G_n;\phi)} : F_q^n \to F_q^n$ by

$$T_{(G_1,G_2,\ldots,G_n;\phi)}(u) = \sum_{l=1}^{n} G_l(u)e_{\phi(l)}, \tag{18}$$

where $\phi \in \mathrm{Aut}(P)$. Assume that G_l for $l \in [n]$ satisfy the following two conditions:

(C_1) $G_l's$ depend only on $\uparrow l$, that is, $G_l(u) = G_l(v)$ if $u^{\uparrow l} = v^{\uparrow l}$. Thus we can write G_l as $G_l(u) = G_l(u^{\uparrow l})$ for every $u \in F_q^n$.

(C_2) Each value $G_l(u^{\uparrow l})$ is taken to be $G_{l,u^{\uparrow l}}(u_l)$, where $G_{l,u^{\uparrow l}}$, depending on $u^{\uparrow l}$, is a bijection of F_q.

Therefore, our map $T_{(G_1,G_2,\ldots,G_n;\phi)}$ becomes

$$T_{(G_1,G_2,\ldots,G_n;\phi)}(u) = \sum_{l=1}^{n} G_l(u^{\uparrow l})e_{\phi(l)} = \sum_{l=1}^{n} G_{l,u^{\uparrow l}}(u_l)e_{\phi(l)}. \tag{19}$$

LEMMA 3.1. *Let $P = ([n], \preceq)$ be a poset. Then we have that $u^{\uparrow l} = u_l e_l$ if $l \in M_u$, and $u^{\uparrow l} = \mathbf{0}$ if $l \notin \langle M_u \rangle$.*

Proof. (i) Let $l \in M_u$. By the definition of $u^{\uparrow l}$, the first part is true. (ii) Let $l \notin \langle M_u \rangle$. Assume that $u^{\uparrow l} \neq \mathbf{0}$. Then there is an element $j \in \uparrow l$ such that $u_j \neq 0$. So we have $l \preceq j \in \langle M_u \rangle$, a contradiction to $l \notin \langle M_u \rangle$. This proves the lemma. ∎

For a given G_l with $l \in [n]$, let $T_{(G_1,G_2,\ldots,G_n;\phi)}$ be a map which satisfies the conditions C_1 and C_2. Let $\phi \in \mathrm{Aut}(P)$. Recall that $u_\phi = \sum_{i=1}^{n} u_i e_{\phi(i)}$. We define $\widetilde{\phi}(u) = u_\phi$. Obviously, $\widetilde{\phi}$ is a linear P-isometry of F_q^n. It follows from (19) that

$$\widetilde{\phi} \circ T_{(G_1,G_2,\ldots,G_n;id)} = T_{(G_1,G_2,\ldots,G_n;\phi)}. \tag{20}$$

We will prove below that $T_{(G_1,G_2,\ldots,G_n;\phi)}$ is a P-isometry of F_q^n. From (20), it is sufficient to show that $T_{(G_1,G_2,\ldots,G_n;id)}$ is a P-isometry of F_q^n.

LEMMA 3.2. *Let $P = ([n], \preceq)$ be a poset. Then $T_{(G_1,G_2,\ldots,G_n;\phi)}$, which satisfy that the conditions C_1 and C_2 are P-isometries of F_q^n.*

Proof. As mentioned previously, for a given G_l with $l \in [n]$, we will prove that $T = T_{(G_1,G_2,\ldots,G_n;id)}$ is a P-isometry of F_q^n. We have $T(u) - T(v) = \sum_{l=1}^{n}(G_l(u^{\uparrow l}) - G_l(v^{\uparrow l}))e_l$. We will compute the coefficient $G_l(u^{\uparrow l}) - G_l(v^{\uparrow l})$ of e_l in $T(u) - T(v)$. We may choose u and v in F_q^n such that $M_{(u-v)}$ is not empty because if $M_{(u-v)}$ is empty, then $u = v$, and so the claim is true. We need to prove that the coefficient of e_l is a nonzero element in F_q if $l \in M_{(u-v)}$, and is the zero element in F_q if $l \notin \langle M_{(u-v)} \rangle$. If we proved these statements, then we would have that $T(u) - T(v) = \sum_{l \in M_{(u-v)}}(G_l(u^{\uparrow l}) - G_l(v^{\uparrow l}))e_l + \sum_{l \in \langle M_{(u-v)} \rangle^*}(G_l(u^{\uparrow l}) - G_l(v^{\uparrow l}))e_l$ and so we have $\langle \mathrm{supp}(T(u) - T(v)) \rangle = \langle M_{(u-v)} \rangle$. We can easily deduce from this that T is in $\mathrm{Iso}_P(F_q^n)$. Note that $M_{(u-v)}$ coincides with the set of maximal elements of the disjoint union of the following three sets;

$$\{i \in M_u \setminus M_v : u_i \neq v_i\},\ \{i \in M_v \setminus M_u : u_i \neq v_i\},\ \text{ and } \{i \in M_u \cap M_v : u_i \neq v_i\}. \tag{21}$$

Notice also that $G_l(u_l e_l) = G_{l,\mathbf{0}}(u_l)$ and $G_l(\mathbf{0}) = G_{l,\mathbf{0}}(0)$. It follows from the condition C_2, Lemma 3.2, and (21) that the coefficient $G_l(u^{\uparrow l}) - G_l(v^{\uparrow l})$ of e_l in $T(u) - T(v)$ becomes

$$G_l(u^{\uparrow l}) - G_l(v^{\uparrow l}) = \begin{cases} G_l(u_l e_l) - G_l(\mathbf{0}) \neq 0 & \text{if } l \in M_u \setminus \langle M_v \rangle \text{ and } u_l \neq v_l, \\ G_l(\mathbf{0}) - G_l(v_l e_l) \neq 0 & \text{if } l \in M_v \setminus \langle M_u \rangle \text{ and } u_l \neq v_l, \\ G_l(u_l e_l) - G_l(v_l e_l) \neq 0 & \text{if } l \in M_u \cap M_v \text{ and } u_l \neq v_l. \end{cases}$$

Thus if $l \in M_{(u-v)}$, then we have the desired result. Next, in the case of $l \notin \langle M_{(u-v)} \rangle$, we have $u^{\uparrow l} = v^{\uparrow l}$ by Lemma 3.1, and so $G_l(u^{\uparrow l}) = G_l(v^{\uparrow l})$. This proves the lemma. ∎

We denote by $\mathcal{G}_P(F_q^n)$ (or simply $\mathcal{G}_P$) the set of P-isometries $T_{(G_1,G_2,\ldots,G_n;\phi)}$ which satisfy the conditions C_1 and C_2. Our construction shows that $\mathcal{G}_P$ is in fact a subgroup of $\mathrm{Iso}_P(F_q^n)$. Indeed, let $T_{(G_1,G_2,\ldots,G_n;\phi)}$ and $T_{(H_1,H_2,\ldots,H_n;\varphi)} \in \mathcal{G}_P$. Then we can write $(T_{(G_1,G_2,\ldots,G_n;\phi)} \circ T_{(H_1,H_2,\ldots,H_n;\varphi)})(u) = T_{(K_1,K_2,\ldots,K_n;\psi)}(x(u))$ for some $T_{(K_1,K_2,\ldots,K_n;\psi)} \in \mathcal{G}_P$ and some $x(u) \in F_q^n$ which depend on u. It is now sufficient to show that if $u \neq v$, then $x(u) \neq x(v)$. This claim is easily proved by using the facts that the composition of two P-isometries of F_q^n is also a P-isometry of F_q^n and every P-isometry of F_q^n is a bijection. Similarly, we can prove that the inverse of $T_{(G_1,G_2,\ldots,G_n;\phi)}$ is also in $\mathcal{G}_P$. If $T = T_{(G_1,G_2,\ldots,G_n;\phi)} \in \mathcal{G}_P$ is linear, then T is expressed by $T(u) = \sum_{l=1}^n \sum_{k\in\uparrow l} u_k G_l(e_k) e_{\phi(l)}$ because each G_l is linear. We denote by δ_{ij} Kronecker's delta. In particular, for each $j \in [n]$, we have

$$T(e_j) = \sum_{l=1}^{n} \sum_{k\in\uparrow l} \delta_{kj} G_l(e_k) e_{\phi(l)} = \sum_{l\in[n], j\in\uparrow l} G_l(e_j) e_{\phi(l)} = \sum_{l\in\langle j\rangle} G_l(e_j) e_{\phi(l)}, \tag{22}$$

where $G_l(e_j) \in F_q$, and $G_j(e_j) = G_{j,\mathbf{0}}(1) \neq 0$ by condition C_2. The relation (22) coincides with Theorem 1.2 in [6], and we have $\mathrm{LIso}_P(F_q^n) \leq \mathcal{G}_P \leq \mathrm{Iso}_P(F_q^n)$.

Let us put $\mathcal{T}_P = \{T_{(G_1,G_2,\ldots,G_n;id)} : G_l'\text{s satisfy the conditions } C_1 \text{ and } C_2\}$ and $\mathcal{A}_P = \{\widetilde{\phi} : \widetilde{\phi}(u) = u_\phi \text{ for } \phi \in \mathrm{Aut}(P)\}$. By (20), we see that $\widetilde{\phi}$ is in $\mathcal{G}_P$. Thus $\mathcal{A}_P$, which is identified with $\mathrm{Aut}(P)$, is a subgroup of $\mathcal{G}_P$. We will show that the intersection of $\mathcal{T}_P$ and $\mathcal{A}_P$ is a trivial one. Let $T_{(G_1,G_2,\ldots,G_n;\phi)} \in \mathcal{G}_P \cap \mathcal{A}_P$. Then $\widetilde{\phi} \circ T_{(G_1,G_2,\ldots,G_n;id)} = \widetilde{\varphi}$ for some $\varphi \in \mathrm{Aut}(P)$. Since $T_{(G_1,G_2,\ldots,G_n;id)} = \widetilde{\phi}^{-1} \circ \widetilde{\varphi}$ is linear, we see that $T_{(G_1,G_2,\ldots,G_n;id)} = \mathrm{id}$ by using (22) and Corollary 1.3 in [6]. This proves the claim. Moreover, from this we can easily verify that every element in $\mathcal{G}_P$ is different and that $\mathcal{T}_P$ is a subgroup of $\mathcal{G}_P$. Thus $\mathcal{G}_P$ is the direct product of $\mathcal{A}_P$ and $\mathcal{T}_P$. Since for each $l \in [n]$ and any $u^{\uparrow l} \in F_q^n$ the map $G_{l,u^{\uparrow l}}$ is a permutation on F_q, the size of $\mathcal{T}_P$ is equal to $\prod_{l=1}^n (q!)^{q^{|\uparrow l|}}$. Let us write $\downarrow l = \langle l\rangle \setminus \{l\}$. We remark that the size of $\mathrm{LIso}_P(F_q^n)$ is equal to $|\mathrm{Aut}(P)|((q-1)!)^n \prod_{l=1}^n q^{|\downarrow l|}$. See [6] for details. We denote by $\mathcal{G}_P^{\mathbf{0}}$ the group of P-isometries of F_q^n in $\mathcal{G}_P$ which fix the origin. From the above observations and Lemma 3.2, we lead to the following theorem.

THEOREM 3.3. *Let $P = ([n], \preceq)$ be a poset. Then $\mathcal{G}_P$ is the direct product of $\mathcal{A}_P$ and $\mathcal{T}_P$ such that $\mathrm{LIso}_P(F_q^n) \leq \mathcal{G}_P \leq \mathrm{Iso}_P(F_q^n)$. Moreover, we have*

$$|\mathcal{G}_P| = |\mathrm{Aut}(P)| \cdot \prod_{l=1}^{n} (q!)^{q^{|\uparrow l|}}, \quad |\mathcal{G}_P^{\mathbf{0}}| = |\mathrm{Aut}(P)| \cdot \prod_{l=1}^{n} (q-1)!(q!)^{q^{|\uparrow l|}-1},$$

$$|\mathcal{G}_P^{\mathbf{0}}|/|\mathrm{LIso}_P(F_q^n)| = q^{\sum_{l=1}^n (q^{|\uparrow l|}-1-|\downarrow l|)} \prod_{l=1}^{n} ((q-1)!)^{q^{|\uparrow l|}-1}.$$

PROPOSITION 3.4. *Let $P = ([n], \preceq)$ be a poset for which $\mathcal{G}_P = \mathrm{Iso}_P(F_q^n)$. Then the group of P-isometries of F_q^n is the semidirect product $\mathcal{A}_P \ltimes \mathcal{T}_P$. In particular, we have that $|\mathrm{Iso}_P(F_q^n)| = |\mathrm{Aut}(P)| \cdot \prod_{l=1}^n (q!)^{q^{|\uparrow l|}}$.*

Proof. By Theorem 3.3, it is sufficient to show that $\mathcal{T}_P$ is a normal subgroup of $\mathrm{Iso}_P(F_q^n)$. Let $T_{(H_1,H_2,\ldots,H_n;\phi)} \in \mathrm{Iso}_P(F_q^n)$ and $\mathrm{id} \neq T_{(G_1,G_2,\ldots,G_n;id)} \in \mathcal{T}_P$. By noting that ${T_{(H_1,H_2,\ldots,H_n;\phi)}}^{-1} \circ T_{(G_1,G_2,\ldots,G_n;id)} \circ T_{(H_1,H_2,\ldots,H_n;\phi)} = {T_{(H_1,H_2,\ldots,H_n;id)}}^{-1} \circ \widetilde{\phi}^{-1} \circ T_{(G_1,G_2,\ldots,G_n;id)} \circ \widetilde{\phi} \circ T_{(H_1,H_2,\ldots,H_n;id)}$, we need to prove that $\widetilde{\phi}^{-1} \circ T_{(G_1,G_2,\ldots,G_n;id)} \circ \widetilde{\phi}$ is in $\mathcal{T}_P$. Assume that it is not true. Since the intersection of $\mathcal{T}_P$ and $\mathcal{A}_P$ is a trivial one, we have $\mathcal{T}_P \ni T_{(G_1,G_2,\ldots,G_n;id)} \in \widetilde{\phi}\mathcal{A}_P\widetilde{\phi}^{-1} = \mathcal{A}_P$, which implies that $T_{(G_1,G_2,\ldots,G_n;id)} = \mathrm{id}$, a contradiction. This proves the proposition. ∎

Let T be a P-isometry of F_q^n. We define $T_{\mathbf{0}}(u) = T(u) - T(\mathbf{0})$. Then $T_{\mathbf{0}}$ is a P-isometry of F_q^n which fixes the origin.

LEMMA 3.5. *Let $P = ([n], \preceq)$ be a poset and $T = T_{(G_1,G_2,\ldots,G_n;\varphi)} \in \mathcal{G}_P$. Then $T_{\mathbf{0}}$ is expressed by $T_{\mathbf{0}}(u) = \sum_{l=1}^n (G_{\varphi^{-1}\circ\phi_{T_{\mathbf{0}}}(l)}(u^{\uparrow\varphi^{-1}\circ\phi_{T_{\mathbf{0}}}(l)}) - G_{\varphi^{-1}\circ\phi_{T_{\mathbf{0}}}(l)}(\mathbf{0}))e_{\phi_{T_{\mathbf{0}}}(l)}$, where $\phi_{T_{\mathbf{0}}}$ is the automorphism of P induced by $T_{\mathbf{0}}$. Hence we have $T_{\mathbf{0}} \in \mathcal{G}_P$.*

Proof. Let $T(u) = \sum_{l=1}^n G_l(u^{\uparrow l})e_{\varphi(l)} \in \mathcal{G}_P$. By Theorem 2.7, we can write $T_{\mathbf{0}}$ as $T_{\mathbf{0}}(u) = \sum_{l=1}^n T_{\mathbf{0},l}(u)e_{\phi_{T_{\mathbf{0}}}(l)}$, where $\phi_{T_{\mathbf{0}}}$ is the automorphism of P induced by $T_{\mathbf{0}}$. From these two expressions, we have $T_{\mathbf{0},\phi_{T_{\mathbf{0}}}{}^{-1}(l)}(u) = G_{\varphi^{-1}(l)}(u^{\uparrow\varphi^{-1}(l)}) - G_{\varphi^{-1}(l)}(\mathbf{0})$ for all $l \in [n]$. Hence the result follows from replacing l by $T_{\mathbf{0}}(l)$ in the above equation. ∎

Let us put $\mathcal{C}_{1,P}$ as

$$\bigcap_{l\in[n]} \{T \in \mathrm{Iso}_P(F_q^n) : \langle \mathrm{supp}(T_{\mathbf{0}}(x(l)))\rangle \subseteq \langle \mathrm{supp}(T_{\mathbf{0}}(u^{\uparrow l} + x(l)) - T_{\mathbf{0}}(u^{\uparrow l}))\rangle \text{ for all } u \in F_q^n\}$$

whenever $\mathrm{supp}(x(l))$ is a subset of the complement of $\uparrow l$. Note that the inclusion in the set is in fact an equality since the size of their two order ideals is the same. And put $\mathcal{C}_{2,P}$ as

$$\bigcap_{l\in[n]} \{T \in \mathrm{Iso}_P(F_q^n) : \phi_{T_{\mathbf{0}}}(l) \in \langle \mathrm{supp}(T_{\mathbf{0}}(u_l e_l + u^{\uparrow l}) - T_{\mathbf{0}}(v_l e_l + u^{\uparrow l}))\rangle \text{ for all } u \in F_q^n \text{ with } u_l \neq v_l\}.$$

THEOREM 3.6. *Let $P = ([n], \preceq)$ be a poset. Then we have that $\mathcal{G}_P = \mathcal{C}_{1,P} \cap \mathcal{C}_{2,P}$.*

Proof. ($\subseteq$): Let $T \in \mathcal{G}_P$ and $L = T_{\mathbf{0}}$. By Lemma 3.5, we can write L as $L(u) = \sum_{l=1}^n L_l(u^{\uparrow l})e_{\phi_L(l)}$, where L_l is a bijection on the lth coordinate position and ϕ_L is the automorphism of P induced by L. In the proof of Lemma 3.2, we showed that if $T' = T'_{(G_1,G_2,\ldots,G_n;id)} \in \mathcal{G}_P$, then $\langle \mathrm{supp}(T'(u) - T'(v))\rangle = \langle \mathrm{supp}(u - v)\rangle$ for all $u, v \in F_q^n$. Recall that $u_\phi = \sum_{i=1}^n u_i e_{\phi(i)}$ and $\widetilde{\phi}(u) = u_\phi$ for $\phi \in \mathrm{Aut}(P)$. By replacing T' by $\widetilde{\phi_L}^{-1} \circ L$, we have $\langle \mathrm{supp}(\widetilde{\phi_L}^{-1} \circ L(u) - \widetilde{\phi_L}^{-1} \circ L(v))\rangle = \langle \mathrm{supp}(u - v)\rangle$. Since $\widetilde{\phi_L}$ is linear and $\mathrm{supp}(\widetilde{\phi_L}(u)) = \phi_L(\mathrm{supp}(u))$, we have $\langle \mathrm{supp}(L(u) - L(v))\rangle = \langle \phi_L(\mathrm{supp}(u - v))\rangle$. From the above equality, we easily see that $T \in \mathcal{C}_{1,P} \cap \mathcal{C}_{2,P}$. ($\supseteq$) : Let $T \in \mathcal{C}_{1,P} \cap \mathcal{C}_{2,P}$. We will prove that there are maps $G_l : F_q^n \to F_q$ and $l \in [n]$ such that

(I) $T(u) = \sum_{l=1}^n G_l(u^{\uparrow l})e_{\phi_{T_{\mathbf{0}}}(l)}$, where $\phi_{T_{\mathbf{0}}}$ is the automorphism of P induced by $T_{\mathbf{0}}$;

(II) for each $l \in [n]$ and any $u^{\uparrow l} \in F_q^n$, the map $G_{l,u^{\uparrow l}} : F_q \to F_q$ defined by $G_{l,u^{\uparrow l}}(u_l) = G_l(u^{\uparrow l})$ is a bijection. Proof of part (I): Let us write $T(u) = \sum_{l=1}^n T_l(u)e_l$ and $L = T_{\mathbf{0}}$. According to Theorem 2.7, we can write $L(u) = \sum_{j=1}^n L_l(u)e_{\phi_L(l)}$, where ϕ_L equals the automorphism of P induced by L. Then $T(u) = \sum_{l=1}^n (L_l(u) + T_{\phi_L(l)}(\mathbf{0}))e_{\phi_L(l)}$. Since $T \in \mathcal{G}_{1,P}$, we have

$$\langle \phi_L(\mathrm{supp}(x(l)))\rangle = \langle \mathrm{supp}(L(x(l)))\rangle \subseteq \langle \mathrm{supp}(L(u^{\uparrow l} + x(l)) - L(u^{\uparrow l}))\rangle, \tag{23}$$

where $\mathrm{supp}(x(l)) \subseteq (\uparrow l)^c$ for all $l \in [n]$. Now, we need to prove that there are maps T_l such that for each $l \in [n]$ the T_l does not depend upon the coordinate positions in the complement of $\uparrow l$, that is, $L_l(u^{\uparrow l}) = L_l(u^{\uparrow l} + x(l))$ whenever $\mathrm{supp}(x(l))$ is a subset of the complement of $\uparrow l$. We may assume that $x(l) \neq \mathbf{0}$ for all $l \in [n]$. Assume to the contrary that $L_l(u^{\uparrow l} + x(l)) \neq L_l(u^{\uparrow l})$ for some $l \in [n]$. Then

$$\phi_L(l) \in \langle \mathrm{supp}(L(u^{\uparrow l} + x(l)) - L(u^{\uparrow l})) \rangle. \tag{24}$$

It follows from (23) and (24) that $\langle \phi_L(l), \phi_L(\mathrm{supp}(x(l))) \rangle \subseteq \langle \mathrm{supp}(L(u^{\uparrow l} + x(l)) - L(u^{\uparrow l})) \rangle$. By using this and the P-distance preservation of T, we see that

$$\begin{aligned} |\langle \mathrm{supp}(x(l)) \rangle| &= |\langle \mathrm{supp}(L(u^{\uparrow l} + x(l)) - L(u^{\uparrow l})) \rangle| \\ &\geq |\langle \phi_L(l), \phi_L(\mathrm{supp}(x(l))) \rangle| \\ &= |\langle l, \mathrm{supp}(x(l)) \rangle|, \end{aligned}$$

which yields that $l \in \mathrm{supp}(x(l))$, a contradiction to the choice of $x(l)$. This proves part (I). Proof of part (II): By part (I), there are maps $G_l : F_q^n \to F_q$ such that $T(u) = \sum_{l=1}^{n} G_l(u^{\uparrow l}) e_{\phi_{T_\mathbf{0}}(l)}$, where $\phi_{T_\mathbf{0}}$ is the automorphism of P induced by $T_\mathbf{0}$. Let $L = T_\mathbf{0}$. Assume that $u_l \neq v_l$. Put $a = u_l e_l + u^{\uparrow l}$ and $b = v_l e_l + u^{\uparrow l}$. We will prove that $G_{l,u^{\uparrow l}}(u_l) = G_l(a) \neq G_l(b) = G_{l,u^{\uparrow l}}(v_l)$, or, equivalently, the $\phi_T(l)$th component of $T(a)$ is different from that of $T(b)$. Since $T \in \mathcal{C}_{2,P}$, we have $\phi_L(l) \in \langle \mathrm{supp}(L(a) - L(b)) \rangle$, which yields that the $\phi_T(l)$th component of $T(a) - T(b)$ is not vanishing. This proves the second part. ∎

LEMMA 3.7. *Let $P = ([n], \preceq)$ be a poset. Let $l \in [n]$, let u be a vector in F_q^n, and let T be a P-isometry of F_q^n. Then, for any vector x with* $\mathrm{supp}(x) \subseteq (\uparrow l)^c$*, we have*

$$\{\phi_{T_\mathbf{0}}(k) : k \in M_x \setminus \langle M_{u^{\uparrow l}} \rangle^*\} \subseteq \mathrm{supp}(T_\mathbf{0}(u^{\uparrow l} + x) - T_\mathbf{0}(u^{\uparrow l})),$$

where $\phi_{T_\mathbf{0}}$ is the automorphism of P induced by $T_\mathbf{0}$.

Proof. Let $L = T_\mathbf{0}$. If either $u^{\uparrow l} = \mathbf{0}$ or $x = \mathbf{0}$, then it is obvious. Assume that $u^{\uparrow l} \neq \mathbf{0}$ and that $x \neq \mathbf{0}$. It follows from the location of elements in $M_{(u^{\uparrow l} + x)}$ and Theorem 2.7 that

$$L(u^{\uparrow l} + x) = \sum_{k \in M_{u^{\uparrow l}}} u_{k,l} e_{\phi_L(k)} + \sum_{k \in M_x \setminus \langle M_{u^{\uparrow l}} \rangle^*} x_{k,l} e_{\phi_L(k)} + \sum_{k \in \langle M_{u^{\uparrow l}} \rangle^* \cup \langle M_x \rangle^*} v_{k,l} e_{\phi_L(k)}, \tag{25}$$

where $u_{k,l}, x_{k,l} \neq 0$, $v_{k,l} \in F_q$, and ϕ_L is the automorphism of P induced by L. Similarly, we have

$$L(u^{\uparrow l}) = \sum_{k \in M_{u^{\uparrow l}}} u'_{k,l} e_{\phi_L(k)} + \sum_{k \in \langle M_{u^{\uparrow l}} \rangle^*} v'_{k,l} e_{\phi_L(k)}, \tag{26}$$

where $u'_{k,l} \neq 0$ and $v'_{k.l} \in F_q$. By subtracting (26) from (25), we have

$$\begin{aligned} L(u^{\uparrow l} + x) - L(u^{\uparrow l}) = &\sum_{k \in M_{u^{\uparrow l}}} (u_{k,l} - u'_{k,l}) e_{\phi_L(k)} + \sum_{k \in M_x \setminus \langle M_{u^{\uparrow l}} \rangle^*} x_{k,l} e_{\phi_L(k)} \\ &+ \sum_{k \in \langle M_{u^{\uparrow l}} \rangle^* \cup \langle M_x \rangle^*} v_{k,l} e_{\phi_L(k)} - \sum_{k \in \langle M_{u^{\uparrow l}} \rangle^*} v'_{k,l} e_{\phi_L(k)}. \end{aligned} \tag{27}$$

Hence the result follows from $x_{k,l} \neq 0$. ∎

LEMMA 3.8. *Let $P = ([n], \preceq)$ be a poset, let u be a vector in F_q^n, and let T be a P-isometry of F_q^n. Then L_u defined by $L_u(x) = T(u + x) - T(u)$ is a P-isometry of F_q^n which fixes the origin.*

Proof. The proof is straightforward. ☐

We say that P is a tree poset if it is a tree with its root as the maximum element.

THEOREM 3.9. *Let $P = ([n], \preceq)$ be a poset. Then we have that $\mathcal{G}_P = \mathrm{Iso}_P(F_q^n)$ if one of the following conditions holds:*

(i) *The automorphism group of P is a trivial one.*

(ii) *P is the disjoint union of dual tree posets.*

Proof. Let $T \in \mathrm{Iso}_P(F_q^n)$ and $L = T_{\mathbf{0}}$. We will use Theorem 3.6, i.e., $\mathcal{G}_P = \mathcal{C}_{1,P} \cap \mathcal{C}_{2,P}$. (i) It follows from Lemma 3.8 that $L_{u^{\uparrow l}}$ defined by $L_{u^{\uparrow l}}(x) = L(u^{\uparrow l}+x) - L(u^{\uparrow l})$ is a P-isometry of F_q^n which fixes the origin. By Theorem 2.7 and assumption for P, we have $\langle \mathrm{supp}(L(x))\rangle = \langle \mathrm{supp}(x)\rangle = \langle \mathrm{supp}(L_{u^{\uparrow l}}(x))\rangle = \langle \mathrm{supp}(L(u^{\uparrow l}+x) - L(u^{\uparrow l}))\rangle$. For $u_l \neq v_l$, we also have $\langle l\rangle = \langle \mathrm{supp}(L_{u^{\uparrow l}}((v_l - u_l)e_l))\rangle = \langle \mathrm{supp}(L(u^{\uparrow l}) - L((v_l - u_l)e_l + u^{\uparrow l}))\rangle$. Thus T is in $\mathcal{C}_{1,P} \cap \mathcal{C}_{2,P} = \mathcal{G}_P$.

(ii) Assume that $\mathrm{supp}(x) \subseteq (\uparrow l)^c$. To show $T \in \mathcal{C}_{1,P}$, we divide the problem into two cases.

Case 1. $l \parallel M_x$. By using the structure for P, we have $M_x \cap \langle M_{u^{\uparrow l}}\rangle^* = \varnothing$. It then follows from Theorem 2.7 and Lemma 3.7 that $\langle \mathrm{supp}(L(x))\rangle = \langle \{\phi_L(k) : k \in M_x\}\rangle \subseteq \langle \mathrm{supp}(L(u^{\uparrow l}+x) - L(u^{\uparrow l}))\rangle$. Hence the first case is proved.

Case 2. l is comparable to j for some $j \in M_x$. We may assume that $u^{\uparrow l} \neq \mathbf{0}$. By assumption for P, the size of $M_x \cap \langle M_{u^{\uparrow l}}\rangle^*$ is only one, say,

$$M_x \cap \langle M_{u^{\uparrow l}}\rangle^* = \{j\}. \tag{28}$$

We claim that for k with $j \prec k$, the $\phi_L(k)$th component of $L(u^{\uparrow l}+x) - L(u^{\uparrow l})$ is vanishing. Assume that it is not true, that is, $\phi_L(k) \in \langle \mathrm{supp}(L(u^{\uparrow l}+x) - L(u^{\uparrow l}))\rangle$. We have that

$$\begin{aligned}\langle \mathrm{supp}(L(x))\rangle &= \langle \phi_L(\mathrm{supp}(x))\rangle \text{ (by Theorem 2.7)}\\ &= \langle \phi_L(j), \phi_L(M_x \setminus \langle M_{u^{\uparrow l}}\rangle^*)\rangle \text{ (by (28))}\\ &\subseteq \langle \phi_L(k), \phi_L(M_x \setminus \langle M_{u^{\uparrow l}}\rangle^*)\rangle \text{ (by } j \prec k)\\ &\subseteq \langle \mathrm{supp}(L(u^{\uparrow l}+x) - L(u^{\uparrow l}))\rangle \text{ (by assumption and Lemma 3.7).}\end{aligned}$$

By using Lemma 3.8, we see that all order ideals in the above are the same. From this and $j \prec k$, we deduce that $k \in M_x$, a contradiction. This proves the claim. It follows from the claim and (27) that

$$L(u^{\uparrow l}+x) - L(u^{\uparrow l}) = \sum_{k \in M_x \setminus \langle M_{u^{\uparrow l}}\rangle^*} x_{k,l} e_{\phi_L(k)} + \sum_{k \in \langle j\rangle \cup \langle M_x\rangle^*} v_{k,l} e_{\phi_L(k)} - \sum_{k \in \langle j\rangle} v'_{k,l} e_{\phi_L(k)}.$$

In the above equation, if $v_{j,l} = v'_{j,l}$ ($j \in M_x$), then $|\langle \mathrm{supp}(x)\rangle| = |\langle \mathrm{supp}(L(u^{\uparrow l}+x) - L(u^{\uparrow l}))\rangle| > |\phi_L(\langle \mathrm{supp}(x)\rangle)| = |\langle \mathrm{supp}(x)\rangle|$, a contradiction. This shows that $T \in \mathcal{C}_{1,P}$. It remains to prove that $T \in \mathcal{C}_{2,P}$. Put $a = u_l e_l + u^{\uparrow l}$ and $b = v_l e_l + u^{\uparrow l}$ and $u_l \neq v_l$. It follows from the P-distance preservation of L and $u_l \neq v_l$ that

$$|\langle \mathrm{supp}(L(a) - L(b))\rangle| = d_P(L(a), L(b)) = d_P(a,b) = |\langle l\rangle| = |\langle \phi_L(l)\rangle|. \tag{29}$$

There are two cases to consider.

Case 1. $u^{\uparrow l} = \mathbf{0}$. In this case, $a = u_l e_l$ and $b = v_l e_l$. By Theorem 2.7, we have

$$\langle \mathrm{supp}(L(a))\rangle = \langle \phi_L(l)\rangle = \langle \mathrm{supp}(L(b))\rangle. \tag{30}$$

Hence the result follows from (29) and (30).

Case 2. $u^{\uparrow l} \neq \mathbf{0}$. In this case, $M_a = M_b$. By using Theorem 2.7 and $M_a = M_b$, we can write

$$L(a) - L(b) = \sum_{i \in M_a} c_i e_{\phi_L(i)} + \sum_{i \in \langle M_a \rangle^*} c_i e_{\phi_L(i)}, \tag{31}$$

where $c_i \in F_q$. Notice that l is comparable to every element of $\langle M_a \rangle$ by the structure for P. Assume that $c_j \neq 0$ for $j \in \langle M_a \rangle \setminus \langle l \rangle$. Then $l \prec j$ and $\phi_L(j) \in \langle \mathrm{supp}(L(a) - L(b)) \rangle$. From $\phi_L(j) \in \langle \mathrm{supp}(L(a) - L(b)) \rangle$ and (29), we have $|\langle j \rangle| = |\langle \phi_L(j) \rangle| \leq |\langle \mathrm{supp}(L(a) - L(b)) \rangle| = |\langle l \rangle|$, a contradiction to $l \prec j$. This shows that $L(a) - L(b)$ in (31) is rewritten as $L(a) - L(b) = \sum_{i \in \langle l \rangle} c_i e_{\phi_L(i)}$. By using this and (29), we have $c_l \neq 0$. Thus $T \in \mathcal{C}_{2,P}$. This proves (ii). ∎

We remark that Theorem 3.9(ii) extends the result obtained in [7].

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