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IDENTIFICATION OF TWO-PHASE FREE BOUNDARY ARISING IN PLASMA PHYSICS*

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Abstract. We try to estimate the shape and the location of two-phase free boundary which has been studied in [A. Friedman and Y. Liu, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4), 22 (1995), pp. 375–448] to model a stationary magnetohydrodynamics system. A sufficient condition is obtained to check whether a test disk is included in the plasma region D surrounded by a two-phase free boundary. In the test disk technique, only two simply verifiable conditions are used and indispensableness of the conditions is demonstrated using an example. The technique is applicable to select some of test disks placed in the domain Ω, which gives a rough guess on the shape of plasma region. Next we draw some geometrical properties of plasma region D when the domain Ω possesses a kind of convexity. It is proved that if Ω itself contains the mirror image of the right portion $\{x \in \Omega : x \cdot \xi > t\}$ of the domain with respect to a line $\{x : x \cdot \xi = t\}$ for all $t > t_0$, then so does the plasma region.

Key words. two-phase free boundary, test disk technique, dumbbell shaped domain, moving plane method, symmetric convex domain

AMS subject classifications. Primary, 35R35; Secondary, 31A25, 31B20, 76W05

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1. Introduction.

A two-phase free boundary problem is a mathematical model to find an interface between two disjoint domains on which solutions satisfy different types of governing equations. Such problems arise in various physical and engineering systems and an example we consider originates from a magnetohydrodynamics system which consists of vacuum and plasma region [6, 8]. During the last 20 years, significant progress in the study of free boundary problems has been made and many results regarding existence and regularity of the solutions have been obtained [2, 3, 4, 5]. However, information about global shape of the interfaces is more desired in many practical situations than regularity results.

Our main interest is to develop new techniques to estimate the size, the location, and some geometric properties of the region D surrounded by a two-phase free boundary. In particular, we consider a free boundary problem in a toroidally symmetric tokamak machine with two-dimensional cross section Ω. Not much has been known in this direction and many fundamental questions are still yet to be answered. For example, we still don't know whether D is convex provided Ω is convex. Though this particular question in the case of one-phase free boundary has been studied in several papers [1, 7, 10], the arguments cannot be directly applicable to our two-phase problem. Identifying the exact shape of the free boundary is not an easy task and it is partially due to the global dependency on the geometry of Ω and counterexamples of uniqueness of the plasma region.

We start with mathematical description of our free boundary problem and readers interested in the derivation of this problem and its physical meaning may consult our

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previous paper [8], the paper by Friedman and Liu [6], and its references. Suppose Ω is a bounded domain in $\mathbf{R}^n$ with $\mathcal{C}^2$ boundary $\partial\Omega$ and c, μ are given positive and nonnegative constants, respectively. In [6], Friedman and Liu considered a free boundary problem to find a solution u, a positive constant λ, and an interface between the plasma region D and the vacuum region satisfying the following equations:

$$\Delta u = 0 \text{ in } \{x : u(x) > 0\} \text{ with } u|_{\partial\Omega} = c, \tag{1.1}$$

$$\Delta u + \lambda u = 0 \text{ in } \{x : u(x) \le 0\} \text{ with } \int_{\{u\le 0\}} u^2 = 1, \tag{1.2}$$

$$|\nabla u^+|^2 - |\nabla u^-|^2 = \mu^2 \text{ along the interface } \partial D, \tag{1.3}$$

where the plasma region D is the interior of the set $\{x \in \Omega : u(x) \le 0\}$ and where ∇u^- and ∇u^+ denote the nontangential limits of ∇u from D and from the vacuum region $\Omega\backslash D$, respectively. It is proved that there exists a minimizer u_μ of the following minimizing problem:

$$\mathcal{M}_\mu^\Omega \left| \begin{array}{ll} \text{Minimize} & J_\mu(u) := \int_\Omega |\nabla u|^2 dx - \mu^2\, |\{x \in \Omega : u(x) \le 0\}| \\ \text{within the class} & \mathcal{K} = \{u \in H^1(\Omega) \ : \ u|_{\partial\Omega} = c,\ \int_{\{u\le 0\}} u^2 = 1\}, \end{array}\right.$$

where $|D|$ denotes Lebesgue measure of set D. It is also proved that the minimizer u_μ^Ω satisfies (1.1)–(1.3) and for $n = 2$ the boundary of the plasma region ∂D_μ^Ω is smooth. Throughout this paper, u_μ^Ω denotes a minimizer of $\mathcal{M}_\mu^\Omega$ and $D_\mu^\Omega = interior\{u_\mu^\Omega \le 0\}$ denotes the corresponding plasma region. The subscript$_\mu$ or the superscript$^\Omega$ will be omitted when there is not confusion. Note that the plasma region D_μ^Ω is not a single valued function with respect to μ and Ω but depends on the choice of u_μ^Ω since there might exist many minimizers u_μ for given μ and Ω.

We present in Lemma 2.1 that $|D_\mu|$ increases as μ increases, regardless of the choices of u_μ, and the energy functional J_μ is differentiable almost everywhere as shown below,

$$J_\mu(u_\mu) := J_0(u_0) - \int_0^\mu 2\lambda |D_\lambda| d\lambda.$$

In order to investigate the location and the size of D_μ, it is natural to check if a test ball in the domain is included in D_μ. Our main result in Theorem 2.2 is about a sufficient condition for a ball B to be included in the plasma region D_μ. We state the theorem in two-dimensional case although the arguments could be easily extended on higher dimension provided that the free boundary is sufficiently smooth. Let B be an open disk contained in Ω of size large enough to satisfy $\frac{|\partial B|}{|B|} \le \mu$. Suppose the solution h_B of the Dirichlet–Laplace problem, $\Delta h_B = 0$ in $\Omega \setminus B$ with the boundary data $h_B = 1$ on $\partial\Omega$ and $h_B = 0$ on ∂B, satisfies a testing condition

$$|\nabla h_B| \le \mu \text{ on } \partial B;$$

then the test disk B is included in the plasma region D_μ

$$B \subset D_\mu.$$

Two brief comments can be made regarding the theorem. First, the theorem contains useful tools to guess the plasma region by placing many test disks on the domain and selecting some of them. Second, the size limit condition $\frac{|\partial B|}{|B|} \le \mu$ in the theorem is indispensable and a theorem conjecture without this condition has a counterexample which is shown in Example 2.3.

In section 3, we investigate some geometrical properties of plasma region D^{Ω}. This kind of work is possible when there is some limitation on the domain Ω and two natural domain properties are maybe symmetry and convexity. Our previous paper [8] gives some results on symmetric convex domains. In order to improve such results, we introduce a new concept which is named mirror covering domain. A domain Ω is called a mirror covering domain with respect to a line $T_\xi(t_0)$ if Ω itself contains the mirror image of the right portion $\{x \in \Omega : x \cdot \xi > t\}$ of the domain with respect to a line $\{x : x \cdot \xi = t\}$ for all $t > t_0$. In Theorem 3.1, we obtain mirror image covering properties which says that if the domain Ω is a mirror covering domain with respect to $T_\xi(t_0)$, then so does the plasma region. Our previous result in [8] proves a similar theorem, that is, if Ω is symmetric and convex with respect to x_2-axis, then so is D_μ. Our new theorem significantly improves our old result and is also applicable to more general domains which need not to be symmetric. The proof of the theorem is based on the moving plane method which was used in the paper by J. Serrin [11] who deals with one-phase free boundary.

2. The size and the location of the plasma region. The problem $\mathcal{M}_\mu^\Omega$ might have more than one solution (see [8]) and, in such a case, u_μ denotes any possible minimizers and D_μ denotes corresponding plasma region.

LEMMA 2.1. *For $\mu_1 < \mu_2$, $J(\mu) := J_\mu(u_\mu)$ and the corresponding energy $\int_\Omega |\nabla u_\mu|^2$ satisfies the following inequalities:*

$$(\mu_2^2 - \mu_1^2)|D_{\mu_1}| < J(\mu_1) - J(\mu_2) < (\mu_2^2 - \mu_1^2)|D_{\mu_2}|, \tag{2.1}$$

$$\int_\Omega |\nabla u_{\mu_1}|^2 < \int_\Omega |\nabla u_{\mu_2}|^2, \tag{2.2}$$

and the energy functional J is uniquely characterized by

$$J(\mu) = J(0) - \int_0^\mu 2\lambda |D_\lambda| d\lambda. \tag{2.3}$$

Proof. It is obvious that $J_{\mu_2}(u_{\mu_1}) \ne J_{\mu_2}(u_{\mu_2})$. Suppose not; u_{μ_1} is also a minimizer of the problem $\mathcal{M}_{\mu_2}^\Omega$ and therefore from (1.3), u_{μ_1} has to satisfy

$$|\nabla u_{\mu_1}^+|^2 - |\nabla u_{\mu_1}^-|^2 = \mu_2^2 \quad \text{on } \partial D_{\mu_1},$$

which is a contradiction since a minimizer u_{μ_1} for $\mathcal{M}_{\mu_1}^\Omega$ has μ_1^2 gradient square jump. Similarly, $J_{\mu_1}(u_{\mu_1}) \ne J_{\mu_1}(u_{\mu_2})$. Since u_{μ_1} and u_{μ_2} are minimizers of $\mathcal{M}_{\mu_1}^\Omega$ and $\mathcal{M}_{\mu_2}^\Omega$, respectively,

$$J_{\mu_1}(u_{\mu_1}) < J_{\mu_1}(u_{\mu_2}) = J_{\mu_2}(u_{\mu_2}) + (\mu_2^2 - \mu_1^2)|D_{\mu_2}|, \tag{2.4}$$

$$J_{\mu_2}(u_{\mu_2}) < J_{\mu_2}(u_{\mu_1}) = J_{\mu_1}(u_{\mu_1}) - (\mu_2^2 - \mu_1^2)|D_{\mu_1}|. \tag{2.5}$$

These inequalities give the lower and the upper bounds of $J(\mu_1) - J(\mu_2)$ in (2.1) which states that $J(\mu)$ is monotone decreasing and Lipschitz continuous and D_μ is increasing with respect to μ. (Note: $|D_\mu|$ could depend on the choice of a minimizer.)

To prove (2.2), we rewrite (2.4) in terms of $\int_\Omega |\nabla u_{\mu_1}|^2$ and $\int_\Omega |\nabla u_{\mu_2}|^2$.

$$\int_\Omega |\nabla u_{\mu_1}|^2 - \mu_1^2 |D_{\mu_1}| < \int_\Omega |\nabla u_{\mu_2}|^2 - \mu_1^2 |D_{\mu_2}| \tag{2.6}$$

and $|D_{\mu_1}| < |D_{\mu_2}|$ proves (2.2).

Since $|D_\mu|$ is monotone increasing and $\lim_{\mu\to\infty} |D_\mu| = \Omega$, $|D_\mu|$ as a function of μ is continuous except on countably many points. Therefore, (2.1) proves that $J(\mu)$ is differentiable almost everywhere and $J(\mu) = J(0) - \int_0^\mu 2\lambda |D_\lambda| d\lambda$. ∎

We assume $c = 1$ and $n = 2$ for simplicity in the later part of the section. For a given open subset F of Ω with smooth boundary, let h_F be the solution of the Dirichlet problem

$$\Delta h_F = 0 \quad \text{in} \quad \Omega \setminus \bar{F},$$
$$h_F|_F \equiv 0\,, \quad h_F|_{\partial\Omega} = 1,$$

and $\lambda(F)$ be the smallest eigenvalue of Δ for the Dirichlet problem in the domain F.

THEOREM 2.2. *Let B be an open disk contained in Ω with $\frac{|\partial B|}{|B|} \le \mu$. Suppose that the harmonic function h_B satisfies the estimate*

$$|\nabla h_B^+| \le \mu \text{ on } \partial B,$$

where ∇h_B^+ denotes the gradient of h_B from the outside of B. Then

$$B \subset D_\mu.$$

Proof. For simplicity of the notation, let $u := u_\mu$ and $D := D_\mu$. To derive a contradiction, assume $D^* := D \cup B \ne D$. Let $\tilde{u}$ be a function defined as the normalized first eigenfunction of Δ with Dirichlet boundary condition in D^* and the harmonic function h_{D^*} in $\Omega \setminus \overline{D}^*$; then it satisfies

$$J_\mu(\tilde{u}) = J_\mu(h_{D^*}) + \lambda(D^*). \tag{2.7}$$

Therefore, the fact that u is a minimizer of J_μ leads

$$J_\mu(\tilde{u}) - J_\mu(u) = J_\mu(h_{D^*}) + \lambda(D^*) - J_\mu(h_D) - \lambda(D) \ge 0.$$

Since $\lambda(D^*) < \lambda(D)$,

$$J_\mu(h_{D^*}) - J_\mu(h_D) > 0. \tag{2.8}$$

On the other hand, it is easy to derive the following inequalities from the assumption $|\nabla h_B^+| \le \mu$ on ∂B using the maximum principle and the Hopf lemma,

$$|\nabla h_{D^*}^+| < \mu \quad \text{on } \partial B \cap \partial D^* \quad \text{and} \quad |\nabla h_{D^*}^+| < |\nabla u^+| \quad \text{on } \partial D \cap \partial D^*. \tag{2.9}$$

Using integration by parts over the region where u is harmonic,

$$\int_{\Omega\setminus\bar{D}} |\nabla u|^2 = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\partial D} |\nabla u^+|, \tag{2.10}$$

where ν denotes the unit out normal vector to the boundary. Similarly, we obtain

$$\int_{\Omega\setminus\bar{D}^*} |\nabla h_{D^*}|^2 = \int_{\partial D^*} |\nabla h_{D^*}^+|. \tag{2.11}$$

Using (2.9), (2.10), and (2.11),

$$
\begin{aligned}
(2.12)\quad & J_\mu(h_{D^*}) - J_\mu(h_D) \\
&= \int_{\partial D^*} |\nabla h_{D^*}^+| - \int_{\partial D} |\nabla u^+| + \mu^2|D| - \mu^2|D^*| \\
&= \int_{\partial D\cap\partial D^*} \left(|\nabla h_{D^*}^+| - |\nabla u^+|\right) + \int_{\partial B\setminus\bar{D}} |\nabla h_{D^*}^+| - \int_{\partial D\cap B} |\nabla u^+| - \mu^2|E| \\
&< \int_{\partial D\cap\partial D^*} \left(|\nabla h_{D^*}^+| - |\nabla u^+|\right) + \mu\left(|\partial B\setminus\bar{D}| - |\partial D\cap B| - \mu|E|\right) \\
&\le \mu\left(|\partial B\setminus\bar{D}| - |\partial D\cap B| - \mu|E|\right),
\end{aligned}
$$

where $E = B\setminus D \neq \emptyset$.

In order to derive a contradiction using (2.8) and (2.12), it suffices to prove the following inequality:

$$I := |\partial B\setminus\bar{D}| - |\partial D\cap B| - \mu|E| \le 0.$$

This quantity is purely geometric and we can, without loss of generality, assume that the arc $\Gamma = \partial B\setminus\bar{D}$ has only one connected component since we can estimate total I by adding the values of I for each components in the case of multicomponent Γ. For simplicity, we assume that B is centered at the origin with radius ρ and the arc Γ is in the range $\theta = 0$ and $\theta = \alpha$. Let $L(t,\theta)$ be the ray joining $(t\cos\theta, t\sin\theta)$ to $(\rho\cos\theta, \rho\sin\theta)$ and $r(\theta)$ be the smallest nonnegative number such that $L(t,\theta)$ does not intersect D for all $r(\theta) < t < \rho$. It is easy to see that the $(r(\theta)\cos\theta, r(\theta)\sin\theta)$ lies on the set $\partial D\cap B$ and

$$
\begin{aligned}
|\partial D\cap B| &\ge \int_0^\alpha r(\theta)d\theta, \\
|E| &\ge \int_0^\alpha \frac{1}{2}\left[\rho^2 - r^2(\theta)\right] d\theta.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I &\le \alpha\rho - \int_0^\alpha r(\theta)d\theta - \mu\int_0^\alpha \frac{1}{2}\left[\rho^2 - r^2(\theta)\right] d\theta \\
&= \int_0^\alpha \left[\rho - r(\theta)\right]\left[1 - \mu\frac{\rho + r(\theta)}{2}\right] d\theta \\
&\le 0.
\end{aligned}
$$

The last inequality is true since the assumption $\frac{|\partial B|}{|B|} = \frac{2}{\rho} \le \mu$ implies $\frac{1}{2}\mu\rho \ge 1$. This completes the proof. ∎

In Theorem 2.2, the condition $\frac{|\partial B|}{|B|} \le \mu$ is quite unusual and one might think this condition should be removed. We will, however, show that the condition is indispensable in the theorem by constructing a disk B such that $B\cap D_\mu = \emptyset$ even though $|\nabla h_B| \le \mu$ on ∂B for given $\Omega, \mu = 1$. (μ is set to be 1 for simplicity of the description.)

Let u^Ω denote a minimizer of the problem $\mathcal{M}_\mu^\Omega$ as usual and let $\tilde{u}^\Omega$ denote a minimizer of the functional

$$(2.13)\qquad \tilde{J}^\Omega(\phi) = \int_\Omega |\nabla\phi|^2 - \mu^2|\{\phi = 0\}|$$

within the class $\tilde{\mathcal{K}} = \{\phi \in H^1(\Omega) : \phi = 1 \text{ on } \partial\Omega\}$. This new problem is identical to $\mathcal{M}_\mu^\Omega$ except that $\int_{\{u\leq 0\}} u^2 = 1$ condition is missing. Note that the solutions of these problems are known when Ω is a disk B_a of radius a. (Detailed computation can be found in [8].) In summary, the minimizers u^{B_a} and $\tilde{u}^{B_a}$ are positive outside of a disk of radius r_c and the corresponding energy functional values can be explicitly computed as follows:

$$J_{\mu=1}^{B_a}(u^{B_a}) = \inf_{0<r<a} \left(\frac{2\pi}{\log a/r} + \frac{\lambda(B_1)}{r^2} - \pi r^2 \right)$$

and

$$\tilde{J}_{\mu=1}^{B_a}(\tilde{u}^{B_a}) = \inf_{0<r<a} \left(\frac{2\pi}{\log a/r} - \pi r^2 \right).$$

For example, $J_1^{B_e} \approx 6.045$ with $r_c \approx 1.550$, $\tilde{J}_1^{B_e} = 0$ with $\tilde{u}^{B_e} \equiv 1$ for $a = e \approx 2.718$ and $J_1^{B_{2e}} \approx -30.989$ with $r_c \approx 4.315$, $\tilde{J}_1^{B_{2e}} \approx -31.300$ with $r_c \approx 4.311$ for $a = 2e \approx 5.437$.

EXAMPLE 2.3. *Let Ω be a dumbbell shaped domain consists of two disks and a narrow connecting bridge:*

$$\Omega = B_L \cup B_R \cup T_\epsilon, \tag{2.14}$$

where $B_L = B_e(0,0)$ is a disk centered at the origin and of radius e, $B_R = B_{2e}(4e,0)$ of radius $2e$, and T_ϵ is a narrow bridge $\{(x_1,x_2) : e-\epsilon < x_1 < 2e+\epsilon, |x_2| < \epsilon\}$. If ϵ is sufficiently small, a test disk $B = B_1(0,0)$ satisfies

$$|\nabla h_B| \leq \mu = 1. \tag{2.15}$$

However, the intersection of the plasma region D_μ and the test disk B is empty

$$D_\mu \cap B = \emptyset. \tag{2.16}$$

Proof. It is easy to show that the gradient of h_B on ∂B is bounded by $\mu = 1$. Define $w(x) = \log|x|$ in $1 \leq |x| \leq e$. Then $\Delta w = 0$ in $B_L \backslash B$ and $w|_{\partial B_L} = 1, w|_{\partial B} = 0$. From the maximum principle, $h_B \leq w$ in $B_L \setminus \bar{B}$, therefore, $|\nabla h_B(x)| \leq |\nabla w(x)| = 1$ on $|x| = 1$ by the Hopf lemma. Hence the disk B satisfies the condition $|\nabla h_B| \leq \mu = 1$.

Next we want to show that B does not intersect with D_μ. Note that this may happen since one of two conditions in Theorem 2.2 is missing; $\frac{|\partial B|}{|B|} = 2 \not\leq \mu = 1$. For sufficiently small ϵ, it is possible to prove that

$$D_\mu \subseteq B_{e-\epsilon}(0,0) \cup B_{2e-\epsilon}(4e,0). \tag{2.17}$$

Here we will just give a brief sketch of the proof of (2.17). Suppose that there exists a point $p \in \partial D_\mu$ in the ϵ-neighborhood of $\partial\Omega$, $distance(p, \partial\Omega) \leq \epsilon$; then $|\nabla u_\mu^+(p)| > \frac{C}{\epsilon}$ for some fixed constant C and $|\nabla u_\mu^-(p)| > \frac{C}{\epsilon} - \mu^2$ from the interface condition (1.3). Therefore, for $\epsilon \leq \frac{C}{\mu^2}$, p must lie on the boundary of the negative part of the plasma region, $p \in \partial\{u_\mu < 0\}$. Recall that D_μ has exactly one connected negative set and the negative set has finite measure bounded below since the smallest eigenvalue of the Laplacian operator on the negative set is bounded, $\lambda(\{u_\mu < 0\}) < J^\Omega(\mu) + \mu^2|\Omega|$. So there exists a nonzero measure connected negative set near the ϵ-neighborhood of the boundary $\partial\Omega$. It draws a contradiction to the fact that $J^\Omega(\mu)$ is bounded since

$|\nabla u^+|$ is of order $\frac{1}{\epsilon}$ along the boundary of the negative set with nonzero length and the harmonic function defined on $\Omega \setminus \bar{D}$ generates unbounded energy near the point p as ϵ approaches 0.

Hence the plasma region D_μ is away from ϵ-neighbor of $\partial\Omega$, that is, $D_\mu = D_L \cup D_R$ where $D_L = D_\mu \cap B_L$ and $D_R = D_\mu \cap B_R$. Also, $u_\mu < 0$ on either D_L or D_R and $u_\mu \equiv 0$ in the other set, if it is not empty. Let u_μ^ϵ be defined as follows: $u_\mu^\epsilon = u_\mu$ in D_μ, $u_\mu^\epsilon = 1$ in $\Omega \setminus B_R \cup B_L$, and u_μ^ϵ is the harmonic function in $B_R \cup B_L \setminus D_\mu$ with boundary data $u_\mu^\epsilon = 1$ on $\partial(B_L \cup B_R)$ and $u_\mu^\epsilon = 0$ on ∂D_μ. Then the energy difference is quite small:

$$J_\mu^\Omega(u_\mu^\epsilon) = J_\mu^\Omega(u_\mu) + O(\epsilon) \tag{2.18}$$

and we can view $J_\mu^\Omega(u_\mu^\epsilon)$ as a sum of contributions from B_L and from B_R, separately. Two possible cases exist. First, B_L contains $u_\mu < 0$ set and B_R does not. Second, B_R does and B_L does not.

Case 1. $J_\mu^\Omega(u_\mu^\epsilon) \geq J_\mu^{B_L}(u_\mu^{B_L}) + \tilde{J}_\mu^{B_R}(\tilde{u}_\mu^{B_R}) \approx 6.045 - 31.300 = -25.255.$

Case 2. $J_\mu^\Omega(u_\mu^\epsilon) \geq \tilde{J}_\mu^{B_L}(\tilde{u}_\mu^{B_L}) + J_\mu^{B_R}(u_\mu^{B_R}) \approx 0.000 - 30.989 = -30.989.$

Since $J_\mu^\Omega(u_\mu^\epsilon) \geq J_\mu^\Omega(u_\mu) = J_\mu^\Omega(u_\mu^\epsilon) - O(\epsilon)$, we can conclude that the second case gives the minimal energy. Thus, $\{u_\mu < 0\} \subset B_R$ and

$$\tilde{J}_\mu^{B_L}(u_\mu) \leq O(\epsilon). \tag{2.19}$$

Now we want to prove $D_L = \emptyset$. At a glance over (2.19), one may guess that it must be $u_\mu \approx 1$ in B_L; however, there exists a counterexample of such a conclusion. To avoid such a mistake, the radius of B_L and μ should be taken into account. Suppose D_L is not an empty set; then $u_\mu \equiv 0$ in D_L and the interface condition (1.3) gives $|\nabla u_\mu| = \mu = 1$ on ∂D_L. Since $u_\mu = 1 + O(\epsilon)$ on ∂B_L, we have

$$\begin{aligned}
\tilde{J}_\mu^{B_L}(u_\mu) &= \int_{B_L} |\nabla u_\mu|^2 - |D_L| \\
&= \int_{\partial B_L} \frac{\partial u_\mu}{\partial \nu} u_\mu - |D_L| \\
&= \int_{\partial B_L} \frac{\partial u_\mu}{\partial \nu} - |D_L| + O(\epsilon) \\
&= \int_{\partial D_L} |\nabla u_\mu| - |D_L| + O(\epsilon) \\
&= |\partial D_L| - |D_L| + O(\epsilon).
\end{aligned}$$

Therefore it follows from (2.19) that

$$|\partial D_L| - |D_L| \leq O(\epsilon),$$

that is,

$$\frac{|\partial D_L|}{|D_L|} \leq 1 + O(\epsilon).$$

It follows from an elementary geometry that $|D_L| \geq 4\pi + O(\epsilon)$ to satisfy the above perimeter to area ratio. Let $r_0 = \sup\{|x| : x \in D_L\}$ and x_0 be a point on ∂D_L such that $|x_0| = r_0$. Then it must be $r_0 \geq 2 + O(\epsilon)$. Let H be the harmonic function in $B_L \setminus \bar{B}_{r_0}$ with the boundary condition $H = u_\mu$ on ∂B_L and $H = 0$ on ∂B_{r_0}. By maximum principle,

$$1 = \mu = |\nabla u_\mu(x_0)| \geq |\nabla H(x_0)| = \frac{1}{r_0(1 - \log r_0)} + O(\epsilon)$$
$$\geq \frac{1}{2(1 - \log 2)} + O(\epsilon) > 1.629 + O(\epsilon),$$

which is not possible. This proves $D_L = \emptyset$. □

This example shows that the condition $\frac{|\partial B|}{|B|} \leq \mu$ is indispensable in the disk covering theorem. D_μ can be exactly found if the disk covering technique method is able to check whether a point (or a disk with arbitrary small radius) is included inside of D_μ; however, this is not possible. As μ gets larger, the radius of the test disk can be chosen smaller, so the technique allows us to find a better approximation of the shape of the free boundary. It seems reasonable that detection of the interface with large jump of the normal derivatives along the interface is easier than that with small jump corresponding to small μ.

3. The mirror image covering properties. In this section we want to find geometric properties of the plasma region D in some class of domains Ω. Our basic motivation is derived from the idea of the moving plane method used by Serrin [11]. We now introduce a new concept of mirror covering domain to describe our results. For a real number $t \in \mathbf{R}$ and a unit vector $\xi \in \mathbf{R}^2$, let us denote the hyperplane with normal vector ξ passing through $t\xi$ by $T_\xi(t) = \{x : x \cdot \xi = t\}$, the right-hand side portion of the domain by $\Sigma_\xi^\Omega(a) = \{x \in \Omega : x \cdot \xi > a\} = \Omega \cap \cup_{t>a} T_\xi(t)$, and the reflected image of $\Sigma_\xi^\Omega(a)$ with respect to $T_\xi(a)$ by $\tilde{\Sigma}_\xi^\Omega(a) = \{x' : x' = x + 2(\xi \cdot x - a)\xi, x \in \Sigma_\xi^\Omega(a)\}$. Then a domain Ω is called a mirror covering domain with respect to a line $T_\xi(t_0)$ if $\tilde{\Sigma}_\xi^\Omega(t) \subset \Omega$ for all $t > t_0$.

THEOREM 3.1. *For given $\mu \geq 0$, let u be a minimizer of $\mathcal{M}_\mu^\Omega$ and D be the corresponding plasma region. Suppose Ω is a mirror covering with respect to $T_\xi(a)$,*

$$\tilde{\Sigma}_\xi^\Omega(t) \subset \Omega \text{ for all } t > a. \tag{3.1}$$

Then so is the plasma region D,

$$\tilde{\Sigma}_\xi^D(a) \subset D. \tag{3.2}$$

Proof. Let

$$t_0 := \inf\{t \geq a : \tilde{\Sigma}_\xi^D(t) \subset D\}.$$

It suffices to prove $t_0 = a$. Suppose not, that is, $t_0 > a$. Then, as in the proof of Serrin [11], the following two events may occur: (i) $\tilde{\Sigma}_\xi^D(t_0)$ becomes internally tangent to the boundary of D at some point P not on $T_\xi(t_0)$. (ii) $T_\xi(t_0)$ is orthogonal to the boundary of D.

Let us introduce the reflected function v defined as follows:

$$v(x') := u(x) \quad \text{for } x' \in \tilde{\Sigma}_\xi^\Omega(t_0), \tag{3.3}$$

where x' is the reflected point of x across $T_\xi(t_0)$. Let $w := v - u$ in $\tilde{\Sigma}_\xi^\Omega(t_0)$. Then w satisfies

$$\begin{aligned}(\Delta+\lambda)w &= 0 \quad \text{in } \tilde{\Sigma}_\xi^D(t_0),\\ \Delta w &= 0 \quad \text{in } \tilde{\Sigma}_\xi^\Omega(t_0)\setminus \bar{D},\\ w &\geq 0 \quad \text{on } \partial\tilde{\Sigma}_\xi^D(t_0) \text{ and } \partial\tilde{\Sigma}_\xi^\Omega(t_0).\end{aligned}$$

Hence, using the monotonicity property of the first eigenvalue λ of the domain D and the maximum principle,

$$w > 0 \text{ in } \tilde{\Sigma}_\xi^D(t_0), \tag{3.4}$$

$$w > 0 \text{ in } \tilde{\Sigma}_\xi^\Omega(t_0)\setminus \bar{D}. \tag{3.5}$$

Using these inequality properties, we want to draw a contradiction to case (i) and case (ii), respectively. The proof is rather technical and the proof for the second case requires quite tedious computation.

Case (i). Since $w = 0$ at the contact point $P \in \partial\tilde{\Sigma}_\xi^D(t_0)$, by the Hopf lemma

$$\nabla w^+(P)\cdot\nu(P) > 0 \quad \text{and} \quad \nabla w^-(P)\cdot\nu(P) < 0,$$

where ∇w^+ and ∇w^- denote the gradients of w from outside and inside of $\tilde{\Sigma}_\xi^D(t_0)$, respectively, and $\nu(P)$ denotes the outer normal vector of $\partial\tilde{\Sigma}_\xi^D(t_0)$. Therefore,

$$|\nabla v^+(P)| = \nabla v^+(P)\cdot\nu(P) > \nabla u^+(P)\cdot\nu(P) = |\nabla u^+(P)|$$

and similarly,

$$|\nabla v^-(P)| < |\nabla u^-(P)|.$$

Hence by the inequalities above and the transmission condition (1.3) on the free boundary, we obtain a contradiction,

$$\mu^2 = |\nabla v^+(P)|^2 - |\nabla v^-(P)|^2 > |\nabla u^+(P)|^2 - |\nabla u^-(P)|^2 = \mu^2.$$

Case (ii). Let P be an orthogonal intersection point of $T_\xi(t_0)$ and D. We may assume $\nu(P) = e_2$ without loss of generality. Let $X_{\pm h} := P \pm he_1 + he_2$ and $Y_{\pm h} := P \pm he_1 - he_2$. For sufficiently small $h > 0$, $X_{\pm h}$ are in the vacuum region and $Y_{\pm h}$ are in the plasma region since the free boundary in two dimensions is smooth. Thus, we can derive two Taylor expansions for u near $X_{\pm h}$ and $Y_{\pm h}$, separately. Using the fact that u^+ is harmonic in $\Omega\setminus\overline{D}$,

$$\begin{aligned}u(X_{\pm h}) &= \partial_2 u^+(P)h \pm \partial_{12}u^+(P)h^2 + \frac{1}{2}(\partial_1^2+\partial_2^2)u^+(P)h^2 + O(h^3)\\ &= \partial_2 u^+(P)h \pm \partial_{12}u^+(P)h^2 + O(h^3).\end{aligned} \tag{3.6}$$

Similarly, since $\Delta u^- + \lambda u^- = 0$ in D, we obtain

$$\begin{aligned}u(Y_{\pm h}) &= -\partial_2 u^-(P)h \mp \partial_{12}u^-(P)h^2 + \frac{1}{2}(\partial_1^2+\partial_2^2)u^-(P)h^2 + O(h^3)\\ &= -\partial_2 u^-(P)h \mp \partial_{12}u^-(P)h^2 + O(h^3).\end{aligned} \tag{3.7}$$

The transmission condition (1.3) gives

$$(\partial_2 u^+(P))^2 - (\partial_2 u^-(P))^2 = \mu^2 \tag{3.8}$$

and

$$\partial_2 u^+(P)\partial_{12} u^+(P) = \partial_2 u^-(P)\partial_{12} u^-(P). \tag{3.9}$$

From (3.6), (3.7), (3.8), and (3.9), we obtain

$$|u(X_{\pm h})|^2 - |u(Y_{\pm h})|^2 = \mu^2 h^2 + O(h^4).$$

Therefore,

$$|u(X_h)|^2 - |u(Y_h)|^2 - (|u(X_{-h})|^2 - |u(Y_{-h})|^2) = O(h^4). \tag{3.10}$$

On the other hand, it follows from the maximum principle on w in (3.4), (3.5) that, for sufficiently small $h > 0$,

$$0 < |u(X_h)|^2 - |u(X_{-h})|^2 \quad \text{and} \quad 0 < |u(Y_{-h})|^2 - |u(Y_h)|^2. \tag{3.11}$$

Therefore, (3.10), (3.11) implies

$$|u(X_h)|^2 - |u(X_{-h})|^2 = O(h^4).$$

The same computation using (3.6) gives

$$|u(X_h)|^2 - |u(X_{-h})|^2 = 4h^3 \partial_2 u^+(P)\partial_{12} u^+(P) + O(h^4)$$

and we obtain $\partial_{12} u^+(P) = 0$ by comparing these two expressions. Therefore,

$$w(X_{-h}) = u(X_h) - u(X_{-h}) = O(h^3).$$

Hence, to derive a contradiction, it suffices to prove that

$$\lim_{h \to 0^+} \frac{w(X_{-h})}{h^{3-\delta}} = \infty \text{ for some } \delta > 0.$$

To do this, let us estimate w near P in a different way. We may assume $P = 0$ without loss of generality. Let us start with a truncated cone A with vertex P:

$$A := \left\{ (r, \theta) : 0 < r < r_0\,,\ \frac{1.1}{2}\pi < \theta < \frac{1.9}{2}\pi \right\},$$

where $r_0 > 0$ is chosen so small that A is contained in $\tilde{\Sigma}_\xi^\Omega(t_0) \setminus \bar{D}$.

Define a bounded harmonic function ϕ on A as follows:

$$\phi(r, \theta) := r^{\frac{2}{0.8}} \sin\left(\frac{2}{0.8}\theta - \frac{1.1}{0.8}\pi \right).$$

Then

$$\phi(r, \frac{1.1}{2}\pi) = \phi(r, \frac{1.9}{2}\pi) = 0 \quad (0 < r < r_0).$$

Since $w > 0$ in $\tilde{\Sigma}_\xi^\Omega(t_0) \setminus \bar{D}$, $w > 0$ on $\partial A \setminus \{P\}$. Therefore, there is a positive constant C by the maximum principle so that

$$\phi(r,\theta) \le Cw(r,\theta) \quad \text{for all } x \in A.$$

Hence, if $0 < \delta < 3 - \frac{2}{0.8}$, then

$$\infty = \lim_{r\to 0} \frac{\phi(r,\theta)}{|r|^{3-\delta}} \le C \lim_{r\to 0} \frac{w(r,\theta)}{|r|^{3-\delta}}.$$

This completes the proof. □

An open set Ω is said to be convex in ξ-direction if the intersection of Ω and any straight line with the direction ξ is an interval.

Corollary 3.2. *Let Ω be symmetric with respect to x_2-axis and convex in e_1-direction where e_i refers to a unit vector in the positive x_i direction. Then D is also symmetric with respect to x_2-axis and convex in e_1-direction.*

The mirror covering theorem tells us that the plasma region on each of two disks in a dumbbell shaped domain with thin and long connecting bridge is connected using mirrors passing the center points of the disks. However, the theorem does not tell us whether the number of plasma components is one or more. In fact, in some dumbbell shaped domain case presented in the paper [6], the plasma region consists of two or more components. Therefore, it is interesting to know the condition under which D is assured to be connected. We still do not know whether D is connected even in convex domain Ω; however, the following corollary provides a sufficient condition for a plasma region D to be connected.

Corollary 3.3. *Let B be a disk satisfying the requirements in Theorem* 2.2. *Suppose $\bigcup_{t>a} \tilde{\Sigma}_\xi^\Omega(t) \subset \Omega$ whenever $T_\xi(a)$ is a tangent line of ∂B. Then D is connected.*

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